Appendix A: Table of Notations

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tr>
<td>$X$</td>
<td>market size (total number of potential customers)</td>
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<tr>
<td>$F(\cdot)$</td>
<td>distribution function of $X$</td>
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<tr>
<td>$X_1$</td>
<td>realized demand in period 1</td>
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<tr>
<td>$X_2$</td>
<td>market size of new customers in period 2</td>
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<tr>
<td>$X'_2$</td>
<td>market size of repeat customers in period 2</td>
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<tr>
<td>$V$</td>
<td>customer valuation for 1st-generation product</td>
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<td>$G(\cdot)$</td>
<td>distribution function of $V$, $G(\cdot) = 1 - G(\cdot)$</td>
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<tr>
<td>$h(\cdot)$</td>
<td>hazard rate function of $V$, i.e., $h(v) = g(v)/G(v)$</td>
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<tr>
<td>$\alpha$</td>
<td>innovation level of 2nd-generation product</td>
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<td>$k$</td>
<td>production cost for remanufacturing</td>
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<td>$c_1$</td>
<td>unit production cost of 1st-generation product</td>
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<td>$\kappa_1$</td>
<td>unit environmental impact of 1st-generation product</td>
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<tr>
<td>$\ell_2$</td>
<td>unit environmental benefit of remanufacturing</td>
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<td>$\delta_1$</td>
<td>discount factor for customers</td>
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<td>$\delta_2$</td>
<td>discount factor for firm</td>
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Appendix B: Auxiliary Results

In this section, we present some auxiliary results in the NTR model and the model of social optimum. These results are building blocks of our subsequent analysis. The proofs of these results are available from the authors upon request. To begin with, we characterize the second-period equilibrium pricing and production strategy in the NTR model. Let $Q_2^a(X_2',X_2^a)$ and $Q_2^r(X_2',X_2^r)$ be the equilibrium production quantities for new and repeat customers, respectively.

**Lemma 2.** (a) For any $(X_2',X_2^a)$, $p_2^a (X_2',X_2^a) = \max_{p_2^a > 0} \Pi_2^a (p_2^a | X_2', X_2^a)$, where $\Pi_2^a (p_2^a | X_2', X_2^a) := X_2^a (p_2^a - c_2) \bar{G} \left( \frac{p_2^a}{1 + \alpha} \right) + X_2' (p_2^a - c_2) \bar{G} \left( \frac{p_2^a}{1 + \alpha} \right)$.

(b) For any $(X_2',X_2^a)$, $Q_2^a (X_2',X_2^a) = \bar{G} \left( \frac{p_2^a (X_2',X_2^a)}{1 + \alpha} \right) X_2^a$, and $Q_2^r (X_2',X_2^a) = \bar{G} \left( \frac{p_2^r (X_2',X_2^r)}{1 + \alpha} \right) X_2^a$.

(c) $p_2^a (X_2',X_2^a)$ is increasing in $X_2'$ and decreasing in $X_2^a$. Moreover, for any $(X_2',X_2^a)$, $p_2^a \leq p_2^r (X_2',X_2^a) \leq p_2^a$, where the inequalities are strict if $X_2',X_2^a > 0$.

Let $\Pi_1^a (Q_1 | \delta_1) (p_1^a (Q_1 | \delta_1))$ be the expected profit (equilibrium first-period price) of the firm to produce $Q_1$ products in period 1 in the NTR model with customer discount factor $\delta_1$. We compute $\Pi_1^a (\cdot | \cdot)$ in the following lemma.

**Lemma 3.** In the NTR model, we have $p_1^a (Q_1 | \delta_1) = \mu + \delta_1 (\sigma_1^a (Q_1) - \sigma_1^a (Q_1^a))$ and $\Pi_1^a (Q_1 | \delta_1) = (m_1^a (Q_1 | \delta_1) - s) \mathbb{E} (X \wedge Q_1) - (c_1 - s) Q_1 + \delta R_2^a (Q_1)$, where $m_2^a (Q_1 | \delta_1) = \mu + \delta_2 (\beta_2^a (Q_1) - \beta_2^a (Q_1^a)) + \delta_1 (\sigma_1^a (Q_1) - \sigma_1^a (Q_1^a))$, $\beta_2^a (Q_1) := \mathbb{E} [\hat{v}_2^a (p_2^a (X_2',X_2^a)) | X_2 = X \wedge Q_1^a]$, $\beta_2^a (Q_1) := \mathbb{E} [\hat{v}_2^a (p_2^a (X_2',X_2^a)) | X_2 = X \wedge Q_1]$, and $R_2^a (Q_1) = \mathbb{E} [\hat{v}_2^a (p_2^a (X_2',X_2^a)) | X_2 = X \wedge Q_1^a]$. Moreover, $\beta_2^a (\cdot)$ is increasing, whereas $\sigma_1^a (\cdot), \sigma_1^a (\cdot), \beta_2^a (\cdot)$ and $R_2^a (\cdot)$ are decreasing in $Q_1$, respectively.

It is clear that $\beta_2^a (Q_1)$ and $\beta_2^a (Q_1)$ are the expected second-period unit profit from new and repeat customers in the NTR model, respectively, whereas $m_2^a (Q_1 | \delta_1)$ is the effective first-period marginal revenue. $\beta_2^a (\cdot), \beta_2^a (\cdot),$ and $m_1^a (\cdot)$ are the counterparts of $\beta_2^a, \beta_2^a,$ and $m_1^a$ in the NTR model. The following theorem summarizes the equilibrium price and production quantity $(p_1^a (\delta_1), Q_1^a (\delta_1))$ in the NTR model.
THEOREM 11. In the NTR model, for any customer discount factor $\delta_c$, a unique RE equilibrium exists with
(a) $Q_i^*(\delta_c) = \arg \max_{Q_i \geq 0} \Pi_i(Q_i|\delta_c)$; (b) $p_i^*(\delta_c) = \mu + \delta_c(\sigma_i^* (Q_i^*(\delta_c)) - \sigma_i^*(Q_i^*(\delta_c)))$; and (c) the expected profit of the firm $\Pi_i^*(\delta_c) = (m_i(Q_i^*(\delta_c))|\delta_c) - s)E(X \wedge Q_i^*(\delta_c)) - (c_1 - s)Q_i^*(\delta_c) + \delta R_2^*(Q_i^*(\delta_c))$.

Finally, we have the following lemma that characterizes the equilibrium second-period pricing strategy in the model of social optimum.

**Lemma 4.** (a) $p_i^*(X_2^*, X_3^*) \equiv p_i^{\ast\ast}$ and $p_i^*(X_2^*, X_3^*) \equiv p_i^{\ast\ast}$, where $p_i^{\ast\ast} = c_2 + \kappa_2$ and $p_i^{\ast\ast} = c_2 - r_2 + \kappa_2 - \nu_2$.

Hence, $p_i^{\ast\ast} > p_i^{\ast\ast}$ if and only if $r_2 > 0$ or $\nu_2 > 0$.

(b) $w_2(X_2^*, X_3^*) = \sigma_i^{\ast\ast} X_2^* + \sigma_i^{\ast\ast} X_3^*$, where $\sigma_i^{\ast\ast} = \mathbb{E}((1 + \alpha) V - p_i^{\ast\ast})$ and $\sigma_i^{\ast\ast} = \mathbb{E}((k + \alpha) V - p_i^{\ast\ast})$.

**Appendix C: Proofs of Statements**

**Proof of Lemma 1: Part (a).** Given $(p_2^*, p_3^*)$ with $p_2^* \leq p_3^*$, the ex-ante probability that a new customer will purchase the second-generation product is $G \left( \frac{p_2^*}{1 + \alpha} \right)$, whereas the probability that a repeat customer will join the trade-in program is $G \left( \frac{p_2^*}{k + \alpha} \right)$. Therefore, conditioned on the realized market size $(X_2^*, X_3^*)$, the expected profit of the firm in period 2 is given by: $\Pi_2(p_2^*, p_3^*, X_2^*, X_3^*) := X_2^*(p_2^* - c_2) G \left( \frac{p_2^*}{1 + \alpha} \right) + X_3^*(p_2^* - c_2 + r_2) G \left( \frac{p_2^*}{k + \alpha} \right) - X_2^* v_2(p_2^*) - X_3^* v_3(p_2^*)$, where $v_2(p_2^*) := (p_2^* - c_2) G \left( \frac{p_2^*}{1 + \alpha} \right)$ and $v_3(p_2^*) := (p_2^* - c_2 + r_2) G \left( \frac{p_2^*}{k + \alpha} \right)$.

We now show that $v_3(\cdot)$ is quasiconcave in $p_2^*$, and $v_3(\cdot)$ is quasiconcave in $p_2^*$. Note that $\frac{\partial v_3(v_2^*)}{\partial v_2^*} = 0$ has a unique solution $p_2^{\ast\ast}$ and $\frac{\partial v_3(v_2^*)}{\partial v_2^*} = 0$ has a unique solution $p_2^{\ast\ast}$. Clearly, for $i = n, r$, $v_3(\cdot)$ is strictly increasing on $[0, p_2^{\ast\ast}]$ and strictly decreasing on $(p_2^{\ast\ast}, +\infty)$.

Therefore, $\Pi_2(\cdot, X_2^*, X_3^*)$ is quasiconcave in $(p_2^*, p_2^*)$, and $(p_2^*(X_2^*, X_3^*), p_2^*(X_2^*, X_3^*)) = (p_2^{\ast\ast}, p_2^{\ast\ast})$.

It remains to show that $p_2^{\ast\ast} > p_2^{\ast\ast}$. Note that $p_2^{\ast\ast}$ satisfies $\left( \frac{p_2^{\ast}\alpha - c_2}{1 + \alpha} \right) / G \left( \frac{p_2^{\ast}}{1 + \alpha} \right) = 1$, and $p_2^{\ast\ast}$ satisfies $\left( \frac{p_2^{\ast\ast} c_2 + r_2}{1 + \alpha} \right) / G \left( \frac{p_2^{\ast\ast}}{1 + \alpha} \right) = 1$. Since $k < 1$, $\frac{p_2^{\ast\ast} c_2 + r_2}{1 + \alpha} > \frac{p_2^{\ast\ast} c_2}{1 + \alpha}$, and the increasing failure rate condition implies $\frac{p_2^{\ast\ast}}{1 + \alpha} / G \left( \frac{p_2^{\ast\ast}}{1 + \alpha} \right) = 1$, and, hence, $\frac{\partial v_3(v_3^*)}{\partial v_3^*} < 0$. Since $v_3(\cdot)$ is quasiconcave, $p_2^{\ast\ast} < p_2^{\ast\ast}$.

**Part (b).** Because all new customers with willingness-to-pay $(1 + \alpha) V$ greater than $p_2^*(X_2^*, X_3^*) \equiv p_2^*$ would make a purchase. Hence, $Q_2^*(X_2^*, X_3^*) = \mathbb{E}[X_2^* 1_{(1 + \alpha) V < p_2^*} | X_3^*] = G \left( \frac{p_2^*}{1 + \alpha} \right) X_2^*$. Analogously, all repeat customers with willingness-to-pay $(k + \alpha) V$ greater than $p_2^*(X_2^*, X_3^*) \equiv p_2^*$ would make a purchase. Hence, $Q_2^*(X_2^*, X_3^*) = \mathbb{E}[X_2^* 1_{(k + \alpha) V < p_2^*} | X_3^*] = G \left( \frac{p_2^*}{k + \alpha} \right) X_2^*$.

**Part (c).** Since $\pi_3(X_2^*, X_3^*) := \max \{ \Pi_3(p_2^*, p_2^*, X_2^*, X_3^*) : 0 \leq p_2^* \leq p_2^* \} = \max \{ v_2(p_2^*) | X_2^* + \max_v v_2(p_2^*) | X_2^* \}$. To complete the proof, it remains to show that $\beta_n^* = \max_v v_2(p_2^*) > 0$ and $\beta_r^* = \max_v v_2(p_2^*) > 0$. It is straightforward to check that $p_2^* - c_2 > 0$, $G \left( \frac{p_2^*}{1 + \alpha} \right) > 0$, $p_2^* - c_2 + r_2 > 0$, and $G \left( \frac{p_2^*}{k + \alpha} \right) > 0$. Hence, $\beta_n^* = (p_2^* - c_2) G \left( \frac{p_2^*}{1 + \alpha} \right) > 0$ and $\beta_r^* = (p_2^* - c_2 + r_2) G \left( \frac{p_2^*}{k + \alpha} \right) > 0$. Q.E.D.

**Proof of Theorem 1: Part (a).** This part has already been shown by the discussions before the theorem.

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1 Due to the page limit requirement, we only provide a sketch of the proof. The complete proof is available from the authors upon request.
Part (b,c). Plugging $p_1^*(\cdot)$ into $\Pi_f(\cdot)$ and, with some algebraic manipulations, we have $\Pi_f(Q_1|\delta_u) = (m_1^*(\delta_u) - s)E(X \wedge Q_1) - (c_1 - s)Q_1 + \beta^*_uE(X)$. Therefore, $Q_1^*(\delta_u)$ is the solution to a newsvendor problem with marginal revenue $m_1^*(\delta_u) - s$, marginal cost $c_1 - s$, and demand distribution $F(\cdot)$. Hence, $Q_1^*(\delta_u) = F^{-1}(\frac{c_1 - s}{m_1^*(\delta_u) - s})$ and $\Pi_f^*(\delta_u) = \Pi_f(Q_1^*(\delta_u)|\delta_u) = (m_1^*(\delta_u) - s)E(X \wedge Q_1^*(\delta_u)) - (c_1 - s)Q_1^*(\delta_u) + \beta^*_uE(X)$. Q.E.D.

Proof of Theorem 2: Part (a). It follows from Theorem 1(a) that $p_1^*(\delta_u) = \mu + \delta_u(\sigma^*_r - \sigma^*_n)$ and $m_1^*(\delta_u) = \mu + \delta(\beta^*_r - \beta^*_u) + \delta(\sigma^*_r - \sigma^*_n)$ are strictly increasing (decreasing) in $\delta_u$ if $\sigma^*_r > \sigma^*_n$ ($\sigma^*_n < \sigma^*_r$). By Theorem 1(b), $Q_1^*(\delta_u) = F^{-1}(\frac{c_1 - s}{m_1^*(\delta_u) - s})$ is increasing (decreasing) in $\delta_u$ if and only if $\sigma^*_r > \sigma^*_n$ ($\sigma^*_n < \sigma^*_r$). Moreover, for any $Q_1$ and any $\delta_u > \delta_c$, $\Pi_f(Q_1|\delta_u) - \Pi_f(Q_1|\delta_c) = (\delta_u - \delta_c)(\sigma^*_r - \sigma^*_n)E(X \wedge Q_1) > 0$ if and only if $\sigma^*_r > \sigma^*_n$. Therefore, $\Pi_f^*(\delta_u) = \max\Pi_f(Q_1|\delta_u) > \max\Pi_f(Q_1|\delta_c) = \Pi_f^*(\delta_c)$ if and only if $\sigma^*_r > \sigma^*_n$. If, on the other hand, $\sigma^*_n < \sigma^*_r$, it follows immediately from the same argument that $\Pi_f^*(\delta_u) < \Pi_f^*(\delta_c).

To show that $\sigma^*_r > \sigma^*_n$ (resp. $\sigma^*_n < \sigma^*_r$) if $k \in (\bar{k}, \tilde{k})$ (resp. $k < \bar{k}$ or $k > \tilde{k}$), it suffices to prove that if $\sigma^*_r$ is increasing in $k$ at $k = \bar{k}$, it is increasing in $k$ when $k < \bar{k}$, $\sigma^*_r$ is increasing in $k$ at $k = \tilde{k}$, and $\sigma^*_n < \sigma^*_r$ if and only if $k < \tilde{k}$ or $k > \tilde{k}$.

Part (b). We first show (b-iii). By definition, $\sigma^*_n(Q_1) - \sigma^*_n(Q_1) = E[(k + \alpha)V - p_2^*(X_2, X_2)]^+ + E[(1 + \alpha)V - p_2^*(X_2, X_2)]^+$. Since $k < 1$ and $p_2^*(X_2, X_2) < (p_2^{*n}, p_2^{*r})$ (Lemma 2), $\sigma^*_n(Q_1) - \sigma^*_n(Q_1) < 0$ for all $Q_1 \geq 0$.

By Theorem 11, $p_1^{*n}(\delta_u) = \mu + \delta_u(\sigma^*_n(Q_1^*(\delta_u)) - \sigma^*_n(Q_1^*(\delta_u)))$ is continuously differentiable in $\delta_u$. Since the right derivative of $p_1^{*n}(\cdot)$ at 0 is $\partial^+_\delta_u p_1^{*n}(0) = \sigma^*_n(Q_1^*(\delta_u)) - \sigma^*_n(Q_1^*(\delta_u)) < 0$, there exists a positive threshold $\delta_0 > 0$ such that $p_1^{*n}(\cdot)$ is strictly decreasing on $[0, \delta_0]$.

To show that $Q_1^*(\delta_u)$ is strictly decreasing in $\delta_u$, it suffices to show that $\Pi_f^*(Q_1|\delta_u)$ is strictly submodular on a neighborhood of $(Q_1^*(\delta_u), \delta_u)$. Direct computation yields $\partial_{k_u} \Pi_f^*(Q_1|\delta_u) = (\sigma^*_n(Q_1^*(\delta_u)) - \sigma^*_n(Q_1^*(\delta_u)))E(X \wedge Q_1)$. Note that $\sigma^*_n(Q_1) - \sigma^*_n(Q_1) < 0$ and is decreasing in $Q_1$, whereas $E(X \wedge Q_1) > 0$ and is strictly increasing in $Q_1$ in a neighborhood of $Q_1^*(\delta_u)$. It follows immediately that $\partial_{k_u} \Pi_f^*(Q_1|\delta_u)$ is strictly decreasing in $Q_1$ on a neighborhood of $Q_1^*(\delta_u)$. Therefore, $\Pi_f^*(Q_1|\delta_u)$ is strictly submodular on a neighborhood of $(Q_1^*(\delta_u), \delta_u)$ and, thus, $Q_1^*(\delta_u)$ is strictly decreasing in $\delta_u$. By the envelope theorem, $\partial_{\delta_u} \Pi_f^*(\delta_u) = (\sigma^*_n(Q_1^*(\delta_u)) - \sigma^*_n(Q_1^*(\delta_u)))E(X \wedge Q_1^*(\delta_u)) > 0$. Hence, $\Pi_f^*(\delta_u)$ is strictly increasing in $\delta_u$. Q.E.D.

Proof of Theorem 3: Part (a). By Lemma 2, $p_2^* < p_2^*(X_2, X_2) < p_2^{*n}$ with probability 1. Thus, if $Q_1 > 0$, $\sigma^*_r = E[(k + \alpha)V - p_2^*]^+ \geq E[(k + \alpha)V - p_2^*(X_2, X_2)^+, X \wedge Q_1)]^+ = \sigma^*_n(Q_1)$, and $\sigma^*_n = E[(1 + \alpha)V - p_2^*]^+ < E[(1 + \alpha)V - p_2^*(X_2, X_2)^+, X \wedge Q_1)]^+ = \sigma^*_n(Q_1)$.

Part (b). By Theorem 1(b) and Theorem 11(b), for all $\delta_u > 0$, $p_1^*(\delta_u) - p_1^{*n}(\delta_u) = \delta_u[\sigma^*_n - \sigma^*_n] - \delta_u[\sigma^*_n(Q_1^*(\delta_u)) - \sigma^*_n(Q_1^*(\delta_u))] = \delta_u[\sigma^*_n - \sigma^*_n(Q_1^*(\delta_u))] + \delta_u[\sigma^*_n(Q_1^*(\delta_u)) - \sigma^*_n] > 0$. Since $\partial_{\delta_u} E[(k + \alpha)V - p_2^*]^+$
Before showing Theorem 4, we first prove Theorem 5 and Theorem 6.

Proof of Theorem 5: Part (a). Since $Q_1^*(\cdot)$ and $Q^*_n(\cdot)$ are continuous in $\delta$, it suffices to show that $Q_1^*(\delta) > Q^*_n(\delta)$. We first show that $m_1^*(Q_1^*(\delta))$ is decreasing in $Q_1$. Observe that $m_1^*(Q_1^*(\delta)) = \mu + \delta[U_r(Q_1) - U_n(Q_1)]$, where $U_r(Q_1) := \mathbb{E}\left[\left(p_2^r(X_2^2, X_2^2) - c_2^G\left(\frac{p_2^r(X_2^2, X_2^2)}{1 + \alpha}\right)\right) + \mathbb{E}\left[\left((k + \alpha)V - p_2^r(X_2^2, X_2^2)\right)\right]\right]$ and $U_n(Q_1) := \mathbb{E}\left[\left(p_2^n(X_2^2, X_2^2) - c_2\right)\mathbb{G}\left(\frac{c_2(p_2^n(X_2^2, X_2^2))}{1 + \alpha}\right) + \mathbb{E}\left[\left((k + \alpha)V - p_2^n(X_2^2, X_2^2)\right)\right]\right]$. Let $u_r(p) := (p - c_2)^G\left(\frac{1}{1 + \alpha}\right) + \mathbb{E}\left[(k + \alpha)V - p\right]$ and $u_n(p) := (p - c_2)^G\left(\frac{1}{1 + \alpha}\right) + \mathbb{E}\left[(k + \alpha)V\right]$. It’s clear that $u_r(\cdot)$ and $u_n(\cdot)$ are continuously decreasing in $p$. Moreover, $U_r(Q_1) = \mathbb{E}[u_r(p_2^r(X_2^2, X_2^2))]$ and $U_n(Q_1) = \mathbb{E}[u_n(p_2^r(X_2^2, X_2^2))]$, where $X_2^a = (X - Q)^a$ and $X_2^a = X \land Q_1$. Since $p_2^r(X_2^2, X_2^2)$ is increasing in $X_2^2$ and decreasing in $X_2^2$, it is stochastically decreasing in $Q_1$. Hence, it suffices to show that $u_r(p) - u_n(p)$ is increasing in $p$. Observe that $u_r(p) - u_n(p) = -\left[\int_{(1 + \alpha)}^{\infty}(1 + \alpha)V - \max(p, (k + \alpha)V) g(V) dV\right]$. We now show that $m_1^*(Q_1^*(\delta)) < m_1^*(\delta)$ for all $Q_1$. Observe that $m_1^*(Q_1^*(\delta)) = \delta[u_r(p_2^r(X_2^2, X_2^2)) - u_n(p_2^r(X_2^2, X_2^2))]$. Because $p_2^r \leq p_2^n(X_2^2, X_2^2) \leq p_2^n$ and $u_r(\cdot)$ and $u_n(\cdot)$ are continuously decreasing in $p$, $\delta[u_r(p_2^r(X_2^2, X_2^2)) - u_n(p_2^r(X_2^2, X_2^2))] \leq 0$ and $\delta[u_n(p_2^r(X_2^2, X_2^2)) - u_n(p_2^r(X_2^2, X_2^2))] \geq 0$. Hence, $m_1^*(Q_1^*(\delta)) \leq m_1^*(\delta)$. Since $k < 1$, $p_2^* < p_2^*$, one of the inequalities $\mathbb{E}[u_r(p_2^r(X_2^2, X_2^2)) - u_n(p_2^r(X_2^2, X_2^2))] \leq 0$ and $\mathbb{E}[u_n(p_2^r(X_2^2, X_2^2)) - u_n(p_2^r(X_2^2, X_2^2))] \geq 0$ must be strict. Therefore, $m_1^*(Q_1^*(\delta)) < m_1^*(\delta)$ for all $Q_1 \geq 0$.
Next, we show that \(Q_1^*(\delta) > Q_1^{**}(\delta)\). Observe that \(\Pi^*_f(Q_1|\delta) - \Pi_f(Q_1|\delta) = (m_1^*(Q_1|\delta) - m_1^*(\delta))\mathbb{E}(X \wedge Q_1) + \delta \mathbb{E}\left[p_2^*(X^+_2, X_2) - c_2\bar{G}\left(\frac{\mathbb{E}[X^+_2 \wedge X_2]}{1 + \alpha}\right)\right] X\). Let \(\Pi(Q_1, 1) = \Pi_f(Q_1|\delta)\) and \(\Pi(Q_1, 0) = \Pi_f(Q_1|\delta)\). Then, \(\Pi(Q_1, 1) - \Pi(Q_1, 0) = (m_1^*(\delta) - m_1^*(Q_1|\delta))\mathbb{E}(X \wedge Q_1) + \delta \mathbb{E}[\beta^*_n - (p_2^*(X^+_2, X_2) - c_2\bar{G}\left(\frac{\mathbb{E}[X^+_2 \wedge X_2]}{1 + \alpha}\right))] X\). Since \(m_1^*(\delta) \geq m_1^*(Q_1|\delta)\) and \(m_1^*(Q_1|\delta)\) is decreasing in \(Q_1\), \((m_1^*(\delta) - m_1^*(Q_1|\delta))\mathbb{E}(X \wedge Q_1)\) is increasing in \(Q_1\). Also note that \(p_2^*(X^+_2, X_2)\) and thus \(p_2^*(X^+_2, X_2) - c_2\bar{G}\left(\frac{\mathbb{E}[X^+_2 \wedge X_2]}{1 + \alpha}\right)\) is decreasing in \(Q_1\). Therefore, \(\Pi(Q_1, 1) - \Pi(Q_1, 0)\) is increasing in \(Q_1\). Hence, \((\cdot, \cdot)\) is supermodular on the lattice \([0, +\infty) \times \{0, 1\}\) and \(Q_1^{**}(\delta) = \arg\max_{Q_1 \geq 0} \Pi^*_f(Q_1|\delta) \leq \arg\max_{Q_1 \geq 0} \Pi_f(Q_1|\delta) = Q_1^*(\delta)\). Since \(m_1^*(\delta) > m_1^*(Q_1^{**}(\delta))\), \(\partial_\delta \Pi_f(Q_1^{**}(\delta)) > \partial_\delta \Pi_f(Q_1^*(\delta)) = 0\). Since \(\Pi_f(\cdot, \cdot)\) is concave in \(Q_1\), \(Q_1^*(\delta) > Q_1^{**}(\delta)\). Due to the continuity of \(Q_1^{**}(\cdot)\) in \(\delta_n\), there exists a threshold \(\delta_0 \leq \delta\) such that \(Q_1^*(\delta_0) > Q_1^{**}(\delta_0)\) for all \(\delta > \delta_0\).

**Part (b).** We first show that \(m_1^*(Q_1|\delta)\) is increasing whereas \(\beta^*_n(\cdot)\) is decreasing in \(Q_1\). Therefore, \(m_1^*(Q_1|\delta)\) is increasing in \(Q_1\).

We then show that there exists a threshold \(Q_1\) such that \(m_1^*(Q_1|\delta) > m_1^*(\delta)\) \(m_1^*(\delta) < m_1^*(Q_1|\delta)\) if \(Q_1 > Q_1\) \((Q_1 < Q_1)\). Let \(\beta^*_n = \max_{r \geq 0} \beta^*_n(p) = \lim_{r \rightarrow +\infty} \beta^*_n(Q_1)\). Since \(k < 1\), \(\beta^*_n = v_1^r(p_2^*) < \beta^*_n\). It is clear that \(\beta^*_n - \beta^*_n\) is increasing in \(r_2\) with \(\beta^*_n = \beta^*_n\) if \(r_2 = 0\). Let \(r_2 > 0\) be the threshold such that \(\beta^*_n - \beta^*_n = \beta^*_n - \beta^*_n\). Hence, \(\beta^*_n - \beta^*_n = \beta^*_n - \beta^*_n\) for all \(r_2 > r_2\). Moreover, by the monotone convergence theorem, \(\lim_{r_1 \rightarrow +\infty} m_1^*(Q_1|\delta) = m_1^*(Q_1|\delta)\) \((Q_1 < Q_1)\). Let \(\hat{\pi}_2(Q_1) := \mathbb{E}[v_2^*(p_2^*(X^+_2, X_2))|X]\), where \(X^+_2 = (X - Q_1)\) \((X - Q_1)\) and \(X^+_2 = X \wedge Q_1\). It’s clear that \(\hat{\pi}_2(\cdot)\) is differentiable and, by the chain rule \(\hat{\pi}'_2(Q_1) = \mathbb{E}[\partial_\delta v_2^*(p_2^*(X^+_2, X_2))|\partial X^+_2 p_2^*(X^+_2, X_2) + \partial X^+_2 p_2^*(X^+_2, X_2)]X_{(X \geq Q_1)}\).

As \(Q_1 \rightarrow \bar{X}\), for any realization of \(X \leq \bar{X}\), \(\partial X^+_2 p_2^*(X^+_2, X_2)\) and \(\partial X^+_2 p_2^*(X^+_2, X_2)\) converges to \(0\). Hence, by the dominated convergence theorem, there exists a threshold \(\bar{Q} \in \{\bar{Q}, \bar{X}\}\), such that \(\hat{\pi}_2(\bar{Q}) < -c \mathbb{E}(X \geq \bar{Q})\) for all \(Q_1 > \bar{Q}\), where \(\epsilon = m_1^*(Q_1|\delta) - m_1^*(\delta)/2 > 0\). Let \(\epsilon_1(r_2) \in (0, \epsilon(r_2))\) be the threshold such that, if \(c_1 < c_1(r_1)\), we have \(Q_1^* > Q_1 \geq \bar{Q}\) \((c_1 < c_1(r_1)\). Therefore, \(\partial_\delta \Pi_f(Q_1^{**}(\delta))|0) = (m_1^*(\delta) - r_1)\mathbb{E}(X \geq Q_1^{**}(\delta)) - (c_1 - r_1) < (m_1^*(Q_1^{**}(\delta)|0) - r_1)\mathbb{E}(X \geq Q_1^{**}(\delta)) - (c_1 - r_1) \leq (m_1^*(Q_1^{**}(\delta)|0) - r_1)\mathbb{E}(X \geq Q_1^{**}(\delta)) + \hat{\pi}'_2(Q_1^{**}(\delta)) - (c_1 - r_1) \leq \partial_\delta \Pi_f(Q_1^{**}(\delta)|0) = 0\), where the first inequality follows from \(m_1^*(Q_1^{**}(\delta)|0) - m_1^*(\delta) \geq (m_1^*(Q_1^{**}(\delta) - m_1^*(\delta)) = 2\epsilon > c\), the second from \(\hat{\pi}_2(Q_1^{**}(\delta)) \in [-c \mathbb{E}(X \geq Q_1^{**}(\delta))|0\), and the last from the monotonicity that \(m_1^*(\delta)\) is increasing in \(Q_1\). Because \(\Pi_f(\cdot, \cdot)\) is concave in \(Q_1\), \(Q_1^{**}(\delta) = \arg\max_{Q_1 \geq 0} \Pi_f(Q_1|\delta) < Q_1^{**}(\delta)\) follows immediately. Since \(Q_1^*(\delta)\) and \(Q_1^{**}(\delta)\) are continuous in \(\delta_n\), there exists a threshold \(\delta_0\) such that \(Q_1^*(\delta_0) < Q_1^{**}(\delta_0)\) for all \(\delta_0 \in [0, \delta_0]\).

**Part (c).** By Theorem 2, \(Q_1^*(\delta)\) is strictly increasing in \(\delta_n\) if \(\sigma^*_n > \sigma^*_n\), whereas \(Q_1^{**}(\delta)\) is strictly decreasing in \(\delta_n\). Therefore, \(\delta_0 = \delta_0\) if \(\sigma^*_n > \sigma^*_n\) \(Q.E.D.

**Proof of Theorem 6:** **Part (a).** A straightforward algebraic manipulation yields \(I_1^*(\delta) = I_1(Q_1^*(\delta))\), where \(I_1(Q_1) := \kappa_1 Q_1 + \left[\delta \kappa_2 \bar{G}\left(\frac{p_2^*}{1 + \alpha}\right) - \delta \kappa_2 \bar{G}\left(\frac{p_2^*}{1 + \alpha}\right) - \delta \kappa_2 \bar{G}\left(\frac{p_2^*}{1 + \alpha}\right)\right] \mathbb{E}(X \wedge Q_1) + \delta \kappa_2 \bar{G}\left(\frac{p_2^*}{1 + \alpha}\right) \mathbb{E}[X].\) If \(\kappa_1 \geq \delta \kappa_2 \bar{G}\left(\frac{p_2^*}{1 + \alpha}\right)\), it is easy to check that \(I_1^*(Q_1^*(\delta)) > \kappa_1 - \delta \kappa_2 \bar{G}\left(\frac{p_2^*}{1 + \alpha}\right) + \delta \kappa_2 \bar{G}\left(\frac{p_2^*}{1 + \alpha}\right) \mathbb{E}(X \geq \bar{Q})\).
$Q_1^*(\delta_c) > 0$ where the first inequality follows from $\mathbb{P}(X \geq Q_1^*(\delta_c)) < 1$, whereas the second inequality follows from the assumptions that $\kappa_1 \geq \delta \kappa_2 G \left( \frac{p_2^*}{1 + \alpha} \right)$ and $\kappa_2 > \nu_2$. Thus, by Theorem 2(a), if $\sigma_r^* > \sigma^*_r$, $Q_1^*(\delta_c)$ is strictly increasing in $\delta_c$, so is $I_2^*(\delta_c) = I_4(Q_1^*(\delta_c))$; if $\sigma_r^* < \sigma^*_r$, $Q_1^*(\delta_c)$ is strictly decreasing in $\delta_c$, so is $I_2^*(\delta_c) = I_4(Q_1^*(\delta_c))$. Furthermore, by Theorem 2, $\sigma_r^* > \sigma^*_r$ if and only if $k \in (k, \bar{k})$, and $\sigma_r^* < \sigma^*_r$ if and only if $k < k < \bar{k}$.

**Part (b).** As shown in part(a), $I_2^*(\delta_c)$ is strictly increasing in $Q_1^*(\delta_c)$ and, by the proof of Theorem 3(c), $Q_1^*(\delta_c)$ is increasing in $k$ when $k$ is small, and decreasing in $k$ when $k$ is big. Therefore, with the same argument as Theorem 3, we know $I_2^*(\delta_c)$ is quasiconcave in $k$, and thus there exists a threshold $K_*$, such that $I_2^*(\delta_c)$ is increasing (resp. decreasing) in $k$ for $k \leq K_* \ (\text{resp. } k \geq K_*)$. \textit{Q.E.D.}

**Proof of Theorem 4:** **Part (a).** Since $\delta_c > \delta_y$, Theorem 5(a) implies that $Q_1^*(\delta_c) > Q_1^{**}(\delta_c)$. Now we compute $I_2^*(\delta_c)$. Given the market size $(X_2^*, X_2^*)$, the equilibrium total second-period production quantity, $Q_2^*(X_2^*, X_2^*)$, is given by $Q_2^*(X_2^*, X_2^*) = X_2^* G \left( \frac{p_2^*(X_2^*, X_2^*)}{1 + \alpha} \right) + X_2^* G \left( \frac{p_2^*(X_2^*, X_2^*)}{k + \alpha} \right)$. Therefore, following the same argument as in the proof of Theorem 6, we have $I_2^*(\delta_c) = \mathbb{E}[\kappa_2 Q_1^*(\delta_c) + \delta \kappa_2 Q_2^*(X_2^*, X_2^*)]$.

**Part (b).** Since $\delta_c < \delta_y$, Theorem 5(b) implies that $Q_1^*(\delta_c) < Q_1^{**}(\delta_c)$. Lemma 2 implies that $p_2^* < p_2^{(\nu_2, \cdot)} < p_2^{**}$. Hence, $\kappa_1 Q_1^*(\delta_c) + \delta \kappa_2 G \left( \frac{p_2^*(X_2^*, X_2^*)}{1 + \alpha} \right) \{\mathbb{E}[X] - \mathbb{E}(X - Q_1^*(\delta_c))^+\} < \kappa_4 Q_1^{**}(\delta_c) + \delta \kappa_2 [\mathbb{E}G \left( \frac{p_2^*(X_2^*, X_2^*)}{k + \alpha} \right) - (X - Q_1^{**}(\delta_c))^+]].$ Let $\hat{\nu}_2 := \left( \hat{G}(\frac{p_2^*}{1 + \alpha}) - \hat{G}(\frac{p_2^*(X_2^*, X_2^*)}{k + \alpha}) \right) < \kappa_2$. If $\nu_2 > \hat{\nu}_2$, since $Q_1^{**}(\delta_c) > Q_1^*(\delta_c)$ and $p_2^* < p_2^{(\nu_2, \cdot)} < p_2^{**}$, then $\mathbb{E}[\kappa_2 G (\hat{G}(\frac{p_2^*(X_2^*, X_2^*)}{k + \alpha}) (Q_1^{**}(\delta_c) \wedge X)] + \mathbb{E}[(\delta \kappa_2 - \nu_2)G (\frac{p_2^*(X_2^*, X_2^*)}{k + \alpha}) (X - (X - Q_1^{**}(\delta_c))^+)] < \mathbb{E}[\kappa_2 G (\hat{G}(\frac{p_2^*(X_2^*, X_2^*)}{k + \alpha}) (X - Q_1^{**}(\delta_c))^+) + \mathbb{E}[(\delta \kappa_2 - \nu_2)G (\frac{p_2^*(X_2^*, X_2^*)}{k + \alpha}) (Q_1^{**}(\delta_c) \wedge X)] + \mathbb{E}[(\delta \kappa_2 - \nu_2)G (\frac{p_2^*(X_2^*, X_2^*)}{k + \alpha}) (X - Q_1^{**}(\delta_c))^+]].$ This shows part (b). \textit{Q.E.D.}

**Proof of Theorem 7:** We first derive $S_r^*(\delta_c)$ and $S_r^{**}(\delta_c)$. Let $a_r^*(\delta_c)$ and $a_r^{**}(\delta_c)$ be the in-stock probability in the base model and the NTR model, respectively. The expected surplus of a customer with discount factor $\delta_c$ in the base model is given by: $\frac{a_r^*(\delta_c) \{\mu - p_1^*(\delta_c) + \delta \sigma_r^*\} + (1 - a_r^*(\delta_c)) \delta \sigma_r^*}{a_r^*(\delta_c)(\mu - \delta_c(\sigma_r^* - \sigma_r^*)) + \delta \sigma_r^*} + (1 - a_r^*(\delta_c)) \delta \sigma_r^* = a_r^*(\delta_c)(\mu - \delta_c(\sigma_r^* - \sigma_r^*)) + \delta \sigma_r^* + (1 - a_r^*(\delta_c)) \delta \sigma_r^* = a_r^*(\delta_c)(\delta - \delta_c(\sigma_r^* - \sigma_r^*)) + \delta \sigma_r^*$, where the first equality follows from $p_1^*(\delta_c) = \mu + \delta_c(\sigma_r^* - \sigma_r^*)$. Therefore, the equilibrium total customer surplus is given by
that, it follows from the same argument as the proof of Theorem 1(a).

Proof of Theorem 8: Part (a). It follows from the same argument as the proof of Theorem 1(a) that, \( p_1^*(\delta_s) = \mu + \delta_s(\sigma_v^*-\sigma_u^*) \). Let \( W_s(Q_1|\delta_s) \) denote the expected total social welfare with first-period production quantity \( Q_1 \) and customer discount factor \( \delta_s \). To compute \( W_s(Q_1|\delta_s) \), we have \( W_s(Q_1|\delta_s) = \sigma_v^*X_2 + \sigma_u^*X_2^* \), and demand distribution \( F(\cdot) \). Hence, \( Q_1^*(\delta_s) = \tilde{F}^{-1}(\frac{c_1+s_n-s}{m_1^*+s}) \), and the equilibrium social welfare is \( W_s^*(\delta_s) = (m_1^*-s)E(X \wedge Q_1^*(\delta_s)) - (c_1 + \kappa_1 - s)Q_1^*(\delta_s) + \delta \sigma_u^*E(X) \).

Part (b). It follows immediately from part (a) that \( p_1^*(\delta_s) = \mu + \delta_s(\sigma_v^*-\sigma_u^*) \) is strictly increasing (resp. decreasing) if and only if \( \sigma_v^* > \sigma_u^* \) (resp. \( \sigma_v^* < \sigma_u^* \)). Note that, by part (a), \( \sigma_v^* = E[(k + \alpha)V_2 - c_2 - \kappa_2 + e_2] \), where \( e_2 := r + \nu_2, c_2 \). The same argument as the proof of Theorem 2(a) implies that if \( \sigma_v^* \) is increasing in \( k \) at \( k = k_0 \), then \( \sigma_v^* \) is increasing in \( k \) for all \( k \leq k_0 \). Hence, \( \sigma_v^* \) is quasiconcave in \( k \). Let \( \bar{k}_s := \arg \max_k \sigma_v^* > \sigma_u^* \) and \( \hat{k}_s := \arg \max_k \sigma_v^* > \sigma_u^* \). The quasiconcavity of \( \sigma_v^* \) in \( k \) suggests that \( \sigma_v^* > \sigma_u^* \) if and only if \( k \in (\bar{k}_s, \hat{k}_s) \), and \( \sigma_v^* < \sigma_u^* \) if and only if \( k < \bar{k}_s \) or \( k > \hat{k}_s \). Since \( m_1^* \) is independent of \( \bar{k}_s, Q_1^*(\delta_s) \) is independent of \( \delta_s \) as well. As a result, \( W_s^*(\delta_s) = (m_1^*-s)E(X \wedge Q_1^*(\delta_s)) - (c_1 + \kappa_1 - s)Q_1^*(\delta_s) + \delta \sigma_u^*E(X) \) is independent of \( \delta_s \).

Part (c). The same argument as the proof of Theorem 3(c) demonstrates that \( \frac{c_1+s_n-s}{m_1^*+s} \) is quasiconvex in \( k \). Let \( K_s := \arg \min_k \frac{c_1+s_n-s}{m_1^*+s} \). We have \( Q_1^* \) is increasing in \( k \) for \( k \leq K_s \) and decreasing in \( k \) otherwise. Since \( c_1 \) is convexly decreasing in \( k \), for any realization of \( X \) and production quantity \( Q_1 = Q_1^* = \tilde{F}^{-1}\left( \frac{c_1+s_n-s}{m_1^*+s} \right) \), \( (m_1^*-s)(Q_1 \wedge X) - (c_1 + \kappa_1 - s)Q_1 \) is increasing in \( k \) for \( k \leq K_s \) and decreasing in \( k \) otherwise. Therefore, \( W_s^* = E[(m_1^*-s)(Q_1^* \wedge X) - (c_1 + \kappa_1 - s)Q_1^* + \delta \sigma_u^*E(X)] \) is also increasing in \( k \) if \( k \leq K_s \) and decreasing in \( k \) otherwise.

Proof of Theorem 9: If \( s_2^*(\delta_s) \) is the solution to \( p_2^*(\delta_s) = \arg \max_{p_2^* \geq 0} (p_2^* + s_2 - c_2) \tilde{G} \left( \frac{p_2^*}{1+\alpha} \right) \), it is clear that the subsidy/tax scheme with \( s_2 = s_2^*(\delta_s) \) can induce the equilibrium price \( p_2^* \) for new customers. We now show that \( s_2^*(\delta_s) \) exists. Since \( v_2^*(p_2^*) \) is quasiconcave in \( p_2^* \) for any \( s_2 \), the first-order condition \( \partial_{p_2^*} v_2^*(p_2^*) \) is zero guarantees the optimal price for new customers. Moreover, \( \partial_{p_2^*} v_2^*(p_2^*) = \tilde{G} \left( \frac{p_2^*}{1+\alpha} \right) - \frac{p_2^*+s_2-c_2}{1+\alpha} g \left( \frac{p_2^*}{1+\alpha} \right), \) which is strictly decreasing in \( s_2 \). Hence, there exists a unique \( s_2^*(\delta_s) \), such that \( \partial_{p_2^*} v_2^*(p_2^*) = 0 \), thus inducing the socially optimal equilibrium price \( p_2^* \) for new customers.
If $s^*_s(\delta_s)$ is the solution to $p^*_s = \arg \max_{p_s \geq 0}(p_s^2 + s^*_s(\delta_s) + s_r - c_2 + r_2)G \left( \frac{p_s - c_1}{k + \alpha} \right)$, the subsidy/tax scheme with $s_r = s^*_s(\delta_s)$ can induce the equilibrium trade-in price $p^*_s$ for repeat customers. We now show that $s^*_s(\delta_s)$ exists. Since $v_2^s(p^*_2)$ is quasiconcave in $p^*_2$ for any $(s_2, s_r)$, the first-order condition $\partial_{p_2} v_2^s(p^*_2) = 0$ guarantees the optimal price for new customers. Moreover, if $s_2 = s_2^*(\delta_s)$, $\partial_{p_2} v_2^s(p^*_2) = G \left( \frac{p^*_2}{k + \alpha} \right) - \frac{p^*_2 + s^*_2(\delta_s) + s_r - c_2 + r_2}{k + \alpha} g \left( \frac{p^*_2}{k + \alpha} \right)$, which is strictly decreasing in $s_r$. Hence, there exists a unique $s^*_s(\delta_s)$, such that $\partial_{p_2} v_2^s(p^*_2) = 0$, thus inducing the socially optimal equilibrium trade-in price for repeat customers $p^*_s$.

Given the subsidy/tax scheme $(s_1, s_2^*(\delta_s), s^*_s(\delta_s))$, as shown above, the firm adopts the same second-period pricing strategy as the social welfare maximizing one: $(p^*_1, p^*_2)$. Hence, the first-period price should also be the same as the one that is socially optimal: $p^*_1(\delta_s) = \mu + \delta_s(\sigma^*_r - \sigma^*_n)$. Thus, the expected profit of the firm in period 1 is $\Pi^*_1(Q_1|\delta_s) = (m^*_1(s_1|\delta_s) - s)E(X \land Q_1) - (c_1 - s)Q_1^*(\delta_s) + \delta(E(X \land Q_1^*(\delta_s))(k + \alpha)J \left( \frac{2\alpha + \kappa_2 - k}{k + \alpha} \right) - \delta E[X](1 + \alpha)J \left( \frac{2\alpha + \kappa_2 - k}{k + \alpha} \right) + 

\textbf{Proof of Theorem 10: Part (a).} Under the optimal subsidy/tax policy $s^*_s(\delta_s)$, the firm's profit is $\Pi^*_1(\delta_s) = (p^*_1(\delta_s) + s^*_1(\delta_s) - s)E(Q_1^*(\delta_s) \land X) - (c_1 - s)Q_1^*(\delta_s) + \delta(E(X \land Q_1^*(\delta_s))(k + \alpha)J \left( \frac{2\alpha + \kappa_2 - k}{k + \alpha} \right) - \delta E[X](1 + \alpha)J \left( \frac{2\alpha + \kappa_2 - k}{k + \alpha} \right)$, where we plug in $s^*_1(\delta_s) = \frac{c_1 - s}{c_1 + \kappa_1 - s}(m^*_1 - s) + s - \mu - \delta_s(\sigma^*_r - \sigma^*_n) + \delta(k + \alpha)J \left( \frac{2\alpha + \kappa_2 - k}{k + \alpha} \right) - \delta(1 + \alpha)J \left( \frac{2\alpha + \kappa_2 - k}{k + \alpha} \right)$, with $J(x) := G(x)/h(x)$. It follows immediately from its formula expression that $\Pi^*_1(\delta_s)$ is independent of $\delta_s$.

\textbf{Part (b).} It's clear that $\delta E[X](1 + \alpha)J \left( \frac{2\alpha + \kappa_2 - k}{k + \alpha} \right)$ is independent of the depreciation factor $k$, whereas $\left( \frac{c_1 - s}{c_1 + \kappa_1 - s}(m^*_1 - s) \right)E(X \land Q_1^*(\delta_s)) - (c_1 - s)Q^*_1(\delta_s)$ is a constant proportion of the optimal social welfare that is influenced by the production decision (i.e., $(m^*_1 - s)E[X \land Q_1^*(\delta_s)] - (c_1 - s + \kappa_1)Q^*_1(\delta_s)$). By Theorem 8(c), $(m^*_1 - s)E[X \land Q_1^*(\delta_s)] - (c_1 - s + \kappa_1)Q^*_1(\delta_s)$ is increasing in $k$ for $k \leq K_s$, and decreasing in $k$ for $k \geq K_s$, i.e., $W^*_s$ is maximized at $k = K_s$. Therefore, the firm’s profit under the subsidy/tax scheme $s^*_s(\delta_s)$, $\Pi^*_1(\delta_s)$, is maximized at $k = K_s$ as well. In other words, if the firm has the flexibility to control the depreciation factor $k$ (equivalently, the remanufacturing efficiency), it will set the socially optimal one $K_s$ under the optimal government policy $s^*_s(\delta_s)$.  

\textit{Q.E.D.}