Our Knowledge of Mathematical Objects

I have recently been attempting to provide a new approach to the philosophy of mathematics, which I call ‘procedural postulationism’. It shares with the traditional form of postulationism, advocated by Hilbert and Poincare, the belief that the existence of mathematical objects and the truth of mathematical propositions are to be seen as the product of postulation. But it takes a very different view of what postulation is. For it takes the postulates from which mathematics is derived to be imperatival, rather than indicative, in character; what is postulated are not propositions true in a given mathematical domain, but procedures for the construction of that domain.

This difference over the cognitive status of postulates has enormous repercussions for the development and significance of the postulational view. The philosophy of mathematics is faced with certain fundamental problems. How are we capable of acquiring an understanding of mathematical terms? How do we secure reference to mathematical objects? What is the nature of these objects? Do they exist independently of us or are they somehow the products of our minds? What accounts for the possibility of applying mathematics to the real world? And how are we capable of acquiring knowledge of mathematical truths? The procedural version of postulationism, in contrast to the propositional version, appears to be capable of providing plausible answers to each of these questions. By going procedural, we convert a view that has appeared completely untenable to one that is worthy of serious consideration.

In what follows I shall focus on the last question concerning our knowledge of mathematics (although this will inevitably involve the other questions). I do this, not because this question is the most interesting or even because it provides the most convincing illustration of the value of our approach, but because it helps to bring out what is most distinctive - and also most problematic - about the approach. If one can go along with what it recommends in this particular case, then one is well on the way to accepting the view in its entirety.

As with the ‘big three’ traditional approaches to the philosophy of mathematics - logicism, formalism, and intuitionism - the present one rests upon a certain technical program within the foundation of mathematics. It attempts to derive the whole of mathematics - or a significant part thereof - within the limitations imposed by its underlying philosophy. Since the interest of the underlying philosophy largely depends upon the possibility of carrying out such a program, it will be helpful to give a sketch - if only in the barest form - of what the program is and of how it is to be executed. In this way, one may acquire a more concrete understanding of what the philosophical issues are and of why they might matter.

§1 The Language and Logic of Postulation

According to the standard forms of postulationism, a postulate is taken to be an indicative sentence, expressing a proposition that is capable of being either true or false. A stock example would be ‘every number has a successor’. The way one then gets at a mathematical theory is through postulating some set of propositional axioms for the theory. Thus in the case of number theory one might postulate the sentence above plus various other axioms governing the notions of number and successor.

Under our approach to postulation, on the other hand, a postulate is an imperatival sentence, which expresses a procedure that may or may not be implemented but certainly cannot properly be said to be true or false. We might compare postulates, as so conceived, to computer programs. Just as a computer program specifies a set of instructions that govern the state of a machine, so a postulate for us will specify a set of instructions that govern the composition of the mathematical domain; and just as the instructions specified by a computer program will tell us how to go from one state of a machine to another, so the instructions specified by a postulate will tell us how to go from one 'state' or composition of the mathematical domain to another (one that, in fact, is always an expansion of the initial state). Indeed, so arresting is this analogy that it will be helpful to pretend that we have a genie at our disposal who automatically attempts to execute any postulate that we might lay down. Postulates are then simply the means by which we 'program' the genie.

If we are to make the above idea precise then we need to specify a 'programming language' within which the instructions to the genie might be formulated. This language, at least in its most basic incarnation, is very simply described. The postulates are of two kinds, *simple* and *complex*; simple postulates are not built up from other postulates, while complex postulates are. There is only one form of simple postulate:

(i) <u>Introduction</u>. !x.C(x);

and it may be read:

introduce an object x conforming to the condition C(x).

In response to the stipulation of such a postulate, the genie will introduce an object into the domain that conforms to the condition C(x) if there is not already such an object in the domain and otherwise he will do nothing. For example, in response to the stipulation:

$!x.\forall y(x > y)$,

he will introduce an object that stands in the >-relation to the pre-existing objects in the domain (unless such an object already exists).

There are four kinds of complex postulate:

(ii) <u>Composition</u>. Where β and γ are postulates, then so is β;γ. We may read β;γ as: do β and then do γ; and β;γ is to be executed by first executing β and then executing γ.

(iii) <u>Conditional.</u> Where β is a postulate and A an indicative sentence, then $A \rightarrow \beta$ is a postulate. We may read $A \rightarrow \beta$ as: if A then do β. How $A \rightarrow \beta$ is executed depends upon whether or not A is true: if A is true, $A \rightarrow \beta$ is executed by executing β; if A is false, then $A \rightarrow \beta$ is executed by doing nothing.

(iv) <u>Universal</u>. Where β(x) is a postulate, then so is ∀xβ(x). We may read ∀xβ(x) as: do β(x) for each x; and ∀xβ(x) is executed by simultaneously executing each of $\beta(x_1)$, $\beta(x_2)$, $\beta(x_3)$, ..., where x_1, x_2, x_3,...are the values of x (within the current domain). Similarly for the postulate ∀Fβ(F), where F is a second-order variable.

(v) <u>Iterative Postulates.</u> Where β is a postulate, then so is β*. We may read β* as: iterate β; and β* is executed by executing β, then executing β again, and so on ad infinitum.

All postulates may be obtained by starting with the simple postulates and then applying the various rules specified above for forming complex postulates. A simple postulate specifies a procedure for introducing a single new object into the domain, suitably related to itself and to pre-existing objects. A complex postulate therefore specifies multiple applications of such procedures; it requires that we successively, or simultaneously, apply the simple procedures to yield more and more complex extensions of the given domain. The only simple procedure the genie ever performs is to add a new object to the domain suitably related to pre-existing objects in the domain (and perhaps also to itself). Everything else the genie then does is a vast iteration, either sequential or simultaneous, of these simple procedures.

Let us see how we might use this simple language of postulation to state postulates for various mathematical domains. We consider two examples: number theory; and a version of set theory (to be exact: cumulative type theory).

Arithmetic. Read Nx as 'x is a number' and Syx as 'y is the successor of x'. Postulates for arithmetic are then given by:

ZERO: $!x.Nx$
SUCCESSOR: $\forall x(Nx \rightarrow !y.(Ny \,\&\, Syx))$
NUMBER: ZERO; SUCCESSOR*.

ZERO says: introduce an object x that is a number. In application to a domain that does not contain a number, it therefore introduces a new object that is a number. We may take this new object to be 0. (Its uniqueness will be guaranteed, under our theory of postulation, by the fact that it is the first number to be introduced through postulation. But the general question of the identity of objects of postulation is not one that we shall pursue). SUCCESSOR says: for each object x in the domain that is a number, introduce a number y that is the successor of x (unless such an object already exists). NUMBER (which I also pronounce 'Let there be numbers!') says: first perform ZERO, i.e. introduce 0, and then keep on introducing the successor of numbers that do not already have a successor. It should be intuitively clear that, in application to a domain that contains no numbers, NUMBER will introduce an ω-progression of objects 0, 1, 2, ..., with each but the first standing in the successor-relation to its immediate predecessor.

Set Theory. We now read Sx as 'x is a set' and x∊y as 'x belongs to y'. We consider two postulates:

POWER: $\forall F!y.(Sy \,\&\, \forall x(x \in y \equiv Fx))$; and
SET: POWER* .

POWER says: for any 'concept' F of pre-existing objects, introduce an object that is a set and has exactly the objects falling under the concept as members. SET (also pronounced 'Let there be sets!') says: keep on performing POWER, i.e. adding sets corresponding to any given concept of objects. It should be clear that, in application to a domain that contains no sets, POWER will introduce all sets (of finite rank) that may be constructed from the given objects.

I would argue that we can obtain all other specific mathematical domains in a similar way. These include the cumulative hierararchy of ZF, the various extensions of the number system to integers, rationals, reals, and complex numbers, and the various different forms of geometry. For some of the set-theoretic cases, we need to make use of a stronger form of iteration in which the iteration of a postulate β* can proceed into the transfinite. But, with this difference aside, the basic forms of postulation can remain the same.

For epistemological purposes, we not only need a characterization of the mathematical domain in terms of a procedural postulate, we also need to show that the characteristic axioms for the domain can be derived from that postulate. We therefore need to develop a logic within which this derivation can be carried out. This cannot be a logic of a standard sort, since it infers propositions from procedures rather than propositions from propositions. The characteristic form of inference is as follows:

$$\frac{A_1, A_2, ..., A_n; \alpha}{B} \quad \text{(from } A_1, A_2, ..., A_n\text{, given } \alpha\text{, we may infer B)},$$

and such an inference is valid if the execution of α converts a domain in which $A_1, A_2, ..., A_n$ are true to one in which B is true.

Although I shall not go into details, it is possible to develop such a logic - I call it 'the logic of postulation' - and it is then possible to show that the standard axioms for a given domain can indeed be derived from the postulate for that domain. From NUMBER above, for example, we can derive the standard (second-order) axioms for the theory of number and from SET we can derive a standard set of axioms for set theory.

We obtain in this way a kind of axiom-free foundation for mathematics. The various axioms for the different branches of mathematics are derived, not from other more basic axioms, but from (procedural) postulates. Thus axioms, which describe the composition of a given mathematical domain, give way to the stipulation of procedures for the construction of that domain. We therefore obtain a form of logicism, though with a procedural twist. The axioms of mathematics are derived from definitions and logic, as in the standard version of logicism, but under a very different conception of definition and of logic, since the definitions are procedural and the logic postulational. Moreover, in contrast to the logic required by the standard forms of logicism, our logic is ontologically neutral. We do not assume that there are any objects. Indeed, the logic itself, which is simply concerned with what would follow under the stipulation of various postulates, is compatible with there being no objects whatever.

§2 The Problem of Consistency

After this very brief sketch of the underlying technical program, we return to the original question. How are we capable of achieving mathematical knowledge and how, in particular, are we capable of acquiring knowledge of mathematical objects? No current epistemology of mathematics is altogether satisfactory; and so we would do well to consider to what extent the procedural approach is able to shed any additional light on this question.

The prospects might look encouraging. Consider the case of numbers. We may stipulate the postulate NUMBER above. From this, using postulational logic, we may derive the standard axioms of arithmetic. We thereby obtain the axioms of arithmetic without any apparent epistemic cost. Number theory is derived on the basis of logic, that is itself without existential import, and sheer stipulation.

Unfortunately, things are not so simple. There are two main problems - one concerning consistency and the other existence. I shall deal with the first here and with the second, more important, problem in the two remaining sections. The first of the problems is that we are not free to stipulate anything we like - at least, if we think of this as entitling us to assert what would thereby be rendered true under the stipulation. Suppose I introduce an object x that is both a number and not a number (!x.(Nx & ~Nx)). It would then follow, within postulational logic, that something was both a number and not a number, which is clearly not something that we should be willing to assert. Or, to take a less obvious example, suppose I postulate the *indefinite* iteration of POWER (i.e., POWER* under the strong reading of *). Then it follows, at least under the version of the postulational logic that we have adopted, that unrestricted comprehension will hold ($\forall F \exists y \forall x(x \in y \equiv Fx)$) - and so, by the reasoning of Russell's paradox, we are again saddled with a contradiction.

It should be clear that a necessary condition for us to be entitled to stipulate - or lay down - a postulate is that we should show it to be consistent. But how is this to be done? The problem is already familiar from the propositional form of postulationism and *there* appears to lack any satisfactory solution. Consider the example of number theory. We wish to show that some standard set of axioms for number theory - say those of Dedekind - are consistent. But how? We cannot appeal to the existence of a model for the axioms, since that is a mathematical fact which requires a mathematical proof of a sort whose justification is already in question. For the same reason, we cannot appeal to the consistency of the axioms within some formal system, since this requires reference to formulas and formal proofs. Nor can we appeal to the truth of the axioms under their standard interpretation, since it is this that we hoped to establish. It seems that the best we can do is appeal to the fact that we have so far failed to derive a contradiction from the axioms. The evidence, in other words, is inductive.

But such inductive evidence hardly does justice to the degree of confidence that we feel entitled to place in the consistency of number theory and the various other theories of mathematics. For it provides the consistency of those theories with no better credibility than that of a well-confirmed mathematical conjecture. We have inductive evidence in favour of the consistency of Quine's New Foundations. So why do we feel entitled to place such greater confidence in the consistency of number theory? The propositional postulationist is unable to say.

One might well think that the procedural postulationist is in no better a position to answer this question than his propositional rival. But before we jump to this conclusion, let us consider more carefully how, for him, the issue of consistency is to be construed. For he will take a postulate to specify a procedure rather than a proposition. So what can it mean to say that a procedure is consistent? Just as a proposition is consistent if it can be true, we can take a procedure to be consistent if it can be executed. Thus the issue is whether the procedures specified by NUMBER and like can be executed.

Given a procedural postulate α and an indicative statement A, let us use the indexed modal claim:

$$\Diamond_\alpha A$$

to indicate that it is possible to execute α in such a way that A is then true. We shall also use the unindexed modal claim:

$$\Diamond A$$

to indicate that A is true under the possible execution of *some* postulational procedure. Let $\top$ be any tautology (such as $p \supset p$). The consistency of α might then be symbolized as:

$$\Diamond_\alpha \top,$$

since this indicates that it is possible to execute α in such a way that $\top$ is then true, i.e. that it is possible to execute α. Thus the issue of consistency is the issue of establishing modal claims of the form $\Diamond_\alpha \top$.

Now what is remarkable is that, once consistency claims are formulated in this way, it is possible to provide convincing demonstrations of their truth that are purely modal in character and make no appeal either to models or proofs or to any other kind of abstract object. Consider the postulate NUMBER as an example. This is of the form ZERO; SUCCESSOR*, where ZERO is the postulate !x.Nx and SUCCESSOR the postulate $\forall x(Nx \rightarrow !y.(Ny \& ySx))$. Say that a postulate is *conservative* if it is necessarily consistent ($\Box\Diamond_\alpha\top$), i.e. consistent no matter what the domain. Then it should be clear that the simple postulate ZERO is conservative and that the simple postulate !y.(Ny & ySx) is conservative whatever the object x. For these postulates introduce a single new object into the domain that is evidently related in a consistent manner to the pre-existing objects. It should also be clear that each of the operations for forming complex postulates will preserve conservativity. For example, if α and β are conservative then so is α;β, for α will be executable on any given domain and, whatever domain it thereby induces, will be one upon which β is executable. But it then follows that NUMBER is consistent, as is any other postulate that is formed from conservative simple postulates by means of the operations for forming complex postulates. (Consistency can also be demonstrated for the higher reaches of set theory, though not in such a straightforward way).

The contrast with the propositional approach is striking. There is nothing in the axiomatic characterization of a basic mathematical domain that would enable us to determine its consistency and, in particular, the consistency of a conjunction of axioms cannot be inferred from the consistency of its conjuncts. But once a procedural approach is adopted, the consistency (and, indeed, the conservativity) of a postulate can be read off 'compositionally' from its very formulation; for the conservativity of the constituent simple procedures will guarantee that the postulate itself is conservative.

Moreover, the method of proof can be extended to show that the standard propositional axioms for a theory are also consistent. Suppose that α is a procedural postulate and Ax a corresponding propositional postulate (when α is NUMBER, for example, Ax might be taken to be the conjunction of axioms for the second-order theory of numbers). We may demonstrate the consistency of α, i.e. $\Diamond_\alpha\top$, as above. Within postulational logic, we can derive Ax from α, i.e. we can show:

$$\frac{;\ \alpha}{Ax}$$

This translates into a proof of $\Box_\alpha Ax$. From $\Box_\alpha Ax$ and $\Diamond_\alpha\top$, we can derive $\Diamond_\alpha Ax$ by ordinary modal reasoning; and from this follows the consistency of Ax. (We might think of the axioms Ax as constituting a 'specification' for a program α. A proof of the above sort then constitutes a 'verification' that the program α does indeed meet the specification.)

I also believe, though this is not something I shall pursue, that a postulate for a mathematical domain directly represents our intuitive grasp of that domain and that a proof of the above sort represents the role that intuition is able to play in establishing the truth or consistency of the axioms for that domain. In any case, it is remarkable that the procedural approach is able to solve the consistency problem in such a simple and satisfying way.

§3 The Problem of Existence

We wish to establish the axioms $Ax = (A_1 \& A_2 \& ... \& A_n)$ of some mathematical theory in the following schematic way:

$$\Diamond_\alpha \top$$

$$;\ \alpha$$

Ax

We first establish the consistency of an appropriate postulate α; this entitles us to stipulate α; and, on the basis of the stipulation of α, we then establish the axioms Ax. Thus the whole of mathematics might be derived with the help of postulational logic and an underlying theory of consistency. Such an approach is not without its difficulties and, in particular, we need to know the basis for the modal principles from which the underlying claims of consistency are derived. But what appears to give it the epistemic edge over other approaches is that the consistency claims can be stated and demonstrated without appeal to mathematical or objects. Thus, if the basis is indeed sufficient to derive the whole of mathematics, the problem of our access to an infinite realm of mathematical objects is avoided.

This brings us to the second problem. As we have seen, the consistency of α is necessary to license the stipulation of α. Our present question is whether it is sufficient (given, of course, that the stipulation of α is then taken to warrant the subsequent assertion of the axioms Ax). We might dub the inference from $\Diamond_\alpha \top$ to Ax, via the stipulation of α, the 'the magical inference', since it takes us from a mere claim of consistency to an assertion of existence. We seem to put far less into the inference than we get out. The question is: what could possible justify an inference of this sort?

Before we will be in a position to provide an answer, we must enquire more deeply into the nature of postulation. A postulate is meant to provide us with an understanding of the domain of discourse - which, for present purposes, we may identify with the domain of quantification; and it provides us with that understanding on the basis of a prior understanding of the domain of discourse. The postulate NUMBER, for example, take us from an understanding of a domain of discourse which does not include numbers to one that does. So let us ask how in general we might be capable of specifying or otherwise understanding a domain of discourse.

One method is familiar and unproblematic. We understand one domain of discourse as a restriction of another. If a given domain ranges over all animals, for example, we may restrict it to the sub-domain of dogs. Now a postulate, as we understand it, is meant to result in an expansion of the domain and so, if this method is to have application to postulation, we must suppose that a postulate effects an expansion in the domain by relaxing a restriction on the domain that is already in force. It could do this by lifting the current restriction - going from *male professor*, for example, to *professor* - or by loosening the current restriction - going from *male professor* to *male or female professor* - or perhaps in some other way.

If this is how postulation works, then I see no way in which it might plausibly be taken to provide us with a new form of justification. For there is no reason to suppose that the relaxation of a restriction will in general result in an expansion of the domain, even if it might consistently be supposed to result in such an expansion.

However, I believe that there may be a radically different way of understanding how a postulate might be capable of defining an expansion in the domain. Call a domain of discourse *unrestricted* if it is not to be understood as the restriction of some other domain. The quantifiers in an unrestricted domain of discourse must be taken to range over everything whatever for, if they were taken to range over X's or Y's, then we could understand the domain as the restriction of an unrestricted domain to the X's or the Y's. Now if there were a method of expansion that applied to an unrestricted domain, then it could not be understood as operating through the relaxation of some restriction on that domain, since there is no restriction to relax.

One might suspect that in any case in which we succeed in expanding a domain, the domain *must* have been explicitly or implicitly restricted. And so the most convincing way of demonstrating the possibility of expansion in the case of an unrestricted domain is to show that there are methods of expansion that work *whatever* the domain might be. But there do appear to be such methods. Consider the following postulate:

RUSSELL: $!y.\forall x(x \in y \equiv \sim x \in x)$,

which introduces the 'set' whose members are exactly those pre-existing objects that are not members of themselves. It seems reasonable to suppose that, whatever the domain of discourse might be, it is possible to introduce this object into the domain; and yet we cannot, on pain of contradiction, suppose that it already is in the domain.

We may take the way postulation works in this case as a model for how it works in general. Of course, in many cases the postulate may not *always* result in an expansion of the given domain and so the alternative view of postulation is not forced upon us. Once NUMBER has been applied to yield the finite numbers, for example, there is no scope for further application of the postulate. But even in those case we may suppose, on general grounds of resemblance to the set-theoretic case, that the same underlying mechanism for expanding the domain is at work.

If this is correct, then there are essentially two different ways in which one may understand one domain in terms of another. One is *classical* or *restrictive*; a wider domain of discourse is presupposed and, in so far as postulation succeeds in expanding a given domain, it will do so by relaxing some restriction that is already implicit in our understanding of what the domain is. The new domain is understood 'from above'. The other method is *creative* or *expansive*; no wider domain is presupposed and postulation works by directly effecting an expansion in the given domain. The new domain is understood 'from below'. It works by means of a genuine expansion of the domain rather than through a de-restriction.

Under the first approach, the predicate (or condition) by which the wider domain is specified must already be understood as having application to the objects of that domain; the predicate will 'seek out' those objects to which it is to apply. But under the creative approach, we have no prior understanding of a wider domain or of how the predicates of interest to us are to apply the objects of such a domain. We understand what the new objects are and how the predicates are to apply to them by directly specifying how the predicates are to apply to those objects. The predicate do not 'find' their objects but are taken to them.

If the present point of view is properly to be understood, it is important to guard against certain possible misconceptions. The present view is not creationist; we do not suppose that the quantifiers range over what does and did exist and that postulation works by literally creating new objects which then enter into the domain of quantification. The objects that are introduced through postulation existed prior to all acts of postulation (if indeed they exist in time at all) and would have existed even if there had been no postulation or people to postulate. By postulation, we incorporate these objects into the domain of the unrestricted quantifier, but through a.reinterpretation of the quantifier rather than a re-invigoration of the ontology.

The present view is also relativistic. Suppose that numbers have not yet been postulated. So numbers are not yet in the unrestricted domain of discourse. Suppose now that I stipulate NUMBER. Numbers *are* then in the unrestricted domain of discourses. But how can the pre-postulational domain of discourse, which excludes numbers, and the post-postulational domain, which includes them, both be unrestricted? Surely, if one is more inclusive than the other, the less inclusive domain must be restricted. Indeed, can it not be understood as the restriction of the more inclusive domain to all those objects that are not numbers?

To deal with this difficulty, we must accept that the status of the domain as unrestricted is one that obtains relative to the postulational context, to what has so far been postulated. Thus relative to a postulational context in which numbers have not been postulated, the unrestricted domain will not include numbers while, relative to a context in which they have been postulated, it will. Of course, once we have expanded the domain with numbers, we may conceive of the old domain as the restriction of the new domain to those objects that are not numbers. But this is not how the old domain *was* conceived, even implicitly; it was taken to be an unrestricted domain (relative to what had so far been postulated). Indeed, far from conceiving the old domain as a restriction of the new domain, we conceive the new domain as an expansion of the old domain.

If it is insisted that the domain is to be unrestricted in an absolute sense, one that is not relative to a postulational context, then it should be disputed that any such understanding of the domain is to be had; for whatever domain we might settle upon is always subject to postulational expansion. Thus in so far as we can make any sense of an unrestricted domain, it must be taken to be relative to a postulational context. But surely, my opponent may respond, we *are* capable of achieving an absolutely unrestricted understanding of the domain. For surely we can form the conception of an object that might be introduced through postulation. And can we not then take the domain to include all such objects? Can the domain not 'reach out' in this way to absolutely all those objects that might be introduced through postulation?

We here have a head-on clash between two intuitions: that the domain is always susceptible of expansion, which is what I previously insisted upon; and that it may be absolutely unrestricted, which is what my opponent currently insists upon. Both have a great deal of plausibility yet, since both cannot be correct, one must be given up.

We might just leave it at that and, in the face of the conflict, simply cleave to the first intuition. But the intuition on the other side is so strong that it would be preferable if we could say what is right about it and why we might have been misled into thinking it was correct. I believe this can be done (and it is far from clear that the proponent of the second intuition is in an equally good position to accommodate *our* intuition).

To this end, we need to distinguish between two quantification-type notions. One is ordinary, or what one might call *actual*, quantification. The other is a modal variant of ordinary quantification - or what we might call *potential* quantification. We say, not '$\exists x$' ('there is an x') but '$\Diamond\exists x$' ('possibly there is an x'), and we say, not '$\forall x$' ('for all x'), but '$\Box\forall x$' ('necessarily for all x'), where the modal operators '$\Diamond$' and '$\Box$' here signify the notions of postulational possibility and necessity that were previously introduced. We can now explain how it might appear to be possible to quantify over absolutely everything. For the potential quantifiers are 'absolute'; in contrast to the actual quantifiers, their 'range' is not relative to the postulational context. There is also a sense in which they are unrestricted, since the inference from $\exists xA$ to $\Diamond\exists xA$ is always justified. However, the potential quantifiers are not strictly quantifiers at all; they do not delimit a domain of discourse; and they are incapable of providing a basis for postulation. The truth in the second intuition can therefore be accommodated without conflict with the first intuition.

We return to the question with which this section began. Suppose we have shown a postulate, say NUMBER, to be consistent. What reason do we then have for supposing that we can lay down the postulate and thereby establish the existence of numbers? My general line of defence is that nothing more than consistency of the postulate can reasonably be required. If my opponent objects that consistency is not enough, then I challenge him to state what else might be required. But to this there might appear to be an obvious response; for, in order to guarantee the existence of the required expansion, we need to know, not merely that objects of the required sort *might* exist, but that they *do* exist.

We now see, given our present understanding of postulation, that this objection is misplaced, since the question of whether the required objects exist cannot even be raised. If we ask ‘are there numbers (objects standing in such and such relationships of successor)?’, then how are we to understand the quantifier ‘there are’? We will want, of course, to understand the quantifier as unrestricted. But on the present view, our understanding of the unrestricted quantifier is relative to a postulation context. So what is the postulational context? If it is the pre-postulational context, the one in which numbers have not yet been postulated, then it is trivial that there are no numbers. If it is the post-postulational context, the one in which numbers have been postulated, then it is trivial that there are numbers.

Given the peculiar nature of postulation, it follows that there can be no *intermediate* understanding of the quantifier, one that allows us to raise the question of whether there are numbers while still leaving open what the answer might be. For the only understanding we can have of a domain that might sensibly be taken to include numbers is one in which the putative numbers are already understood as standing in relationships of successor to one another. Since our understanding of the putative numbers is purely ‘extensional’, there is no possibility of driving apart a conception of what these objects might be from a commitment to their existence.

One wants, of course, to ask whether there are numbers in an absolutely unrestricted sense of ‘there are’. But on the present view, no such sense exists. Nor does it help to appeal to our modal surrogate ($\Diamond\exists x$) for the absolutely unrestricted quantifier, since the question of whether there are numbers in this sense is just equivalent to the question of consistency. One might wonder whether there is some other route to achieving the desired understanding of the quantifier, either of a non-postulational sort or via another form of postulation. But it is plausible to suppose that no such alternative exists, that no object that can be introduced into the domain through postulation might enter into the domain in some other way and that no object that can be introduced via one form of postulation, using certain defining relationships, might be introduced via another form of postulation. Indeed, any coherent conception of the method of postulation demands that this should be so.

We therefore see that, as far as the present objection goes, the magical inference may be allowed to stand.

§4 The Scope of Postulation

I wish, in conclusion, to consider two other epistemological objections to the method of postulation. One is to its generality - that, once the method is allowed, it cannot be properly contained. The other is to its viability - that the method is either incoherent or without significant application. Our answers to both objections will enable us better to understand the nature of postulation and its place within the realm of rational enquiry.

The first objection goes as follows. The postulate:

UNIVERSAL-SET: $!y.\forall x(x \in y)$

is consistent. So, according to the method of postulation, it may legitimately be stipulated; and, from the stipulation of this postulate, we may then infer the existence of a universal set over the given domain. But is not:

UNIVERSAL-LOVER: $!y.\forall x(\text{Person}(x) \supset y \text{ loves } x)$

also consistent? So what is to stop us from stipulating it and thereby establishing the existence of a universal lover? Or again:

GOD: $!y.\text{Divine}(x)$

is presumably consistent, at least under a suitable understanding of 'divine'. So what is to stop us from stipulating it and thereby establishing the existence of God?

Clearly, UNIVERSAL-LOVER and GOD should not be taken to be postulates or, if they are, then they should be declared inconsistent on the grounds that there is no genuine *postulational* possibility of someone's loving everyone or of something being divine. But why distinguish between the two cases in this way? Intuitively, the difference lies in the predicates. It is legitimate to postulate by means of predicates such as '$\in$' or '$<$' or 'successor', but not by means of predicates such as 'person' or 'loves' or 'divine'. But what is the relevant difference between these predicates?

I should like to suggest that it lies in how the predicates are to be understood. Predicates of the first kind are in a certain sense formal; they are *simply* to be understood in terms of how postulation with respect to them is to be constrained. Thus our understanding of S (set) and $\in$ (membership) is entirely given by the fact that they conform to the following constraints:

<u>Extensionality</u> $\forall x\forall y[Sx \,\&\, Sy \,\&\, \forall z(z\in x \equiv z\in y) \supset x = y]$;

<u>Sethood</u> $\forall y[\exists x(x \in y) \supset Sy]$;

<u>Set-Rigidity</u> $\forall y\forall F[Sy \,\&\, \forall x(x \in y \supset Fx) \supset \Box\forall x(x \in y \supset Fx)]$.

Thus according to the first constraint, no two sets are to be postulated to have the same members; according to the second, anything that is postulated to have members is to be postulated to be a set; and according to the third, no *members* of pre-existing sets are to be postulated (this explains the sense in which sets are formed from their members but not members from their sets). Similarly, our understanding of N (number) and S (successor) is entirely given by their conformity to the following two constraints:

<u>Uniqueness</u> $\forall x\forall y\forall z(Syx \,\&\, Szx \supset y = z)$;

<u>Numberhood</u> $\forall x\forall y(Sxy \supset Nx \,\&\, Ny)$;

<u>Successor-Rigidity</u> $\forall x(Nx \supset (\sim\exists ySxy \supset \Box\sim\exists ySxy) \,\&\, \forall y(Sxy \supset \Box\forall z(Sxz \supset z = y)))$.

(According to the third of these, we cannot postulate *predecessors*, just as we cannot postulate *members*.) Note that these constraints are not axioms of the usual sort; they are entirely without existential import and serve merely to constrain the postulational possibilities.

Say that a predicate is *postulational* if its meaning is entirely given by a set of postulational constraints. Our view is then that we are entitled to postulate by means of postulational predicates, as long as we stay within the constraints by which they are defined. Moreover, we are not free to postulate by means of any other predicates, since there is then an independent question, not to be settled by postulation alone, of what their application should be. Thus it is only when the predicates have no content beyond their role in postulation that they may legitimately be used as vehicles of postulation.

The second objection is more radical. Our whole defense of the method of postulation has been conditional in form: if one accepts postulation as legitimate, then one should accept consistency as a basis upon which it may proceed. But it might be denied that postulation is legitimate - either on the grounds that the fundamental notions of postulation and postulational possibility are incoherent or on the grounds that, even though they are coherent, they have no significant application.

This objection strikes me as being essentially sceptical in spirit; for it is at odds with the commonly accepted epistemic facts. We *do* postulate. And here I do not merely have in mind the somewhat controversial case of sets. The actual practice of mathematics, before it was sanitized by the logicians, contains numerous examples of postulation. The complex number i, for example, was postulated as a number for which $i^2 = -1$; and $+\infty$ was postulated as a number greater than all reals. The philosopher who rejects postulation must reject standard (or, at least, what *was* once standard) postulational practice.

I have no answer to scepticism - in this case or in any other. All I can say is that the sceptic has an unduly narrow conception of how we might come to know what we do. Each kind of object has its own way of being known. It is a peculiarity of perceptible objects that we may get to know of them through perception; it is a peculiarity of the theoretical entities of science that their existence is to be justified by way of inference to the best explanation; and it is a peculiarity of mathematical and other abstract objects that their existence is to be justified by way of postulation. In recent times, many philosophers have been attracted to an ‘assimilationist’ model of mathematical knowledge; they have supposed that we know of mathematical objects in something like the way we know of other objects - either directly through some form of perception or apprehension or indirectly through inference to the best explanation. If the present approach has any value, it lies in its making clear the *distinctive* way in which we may acquire our knowledge of mathematical objects, one that is not reducible to other, more familiar methods and is in keeping with the peculiarly a priori character of mathematical thought.