

Equilibrium Corporate Finance and Intermediation*

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Abstract

This paper analyzes a class of competitive economies with production, incomplete financial markets, and agency frictions. Since markets are incomplete, firms choose their capital structure – and, to some extent, their investment – to satisfy investors’ hedging needs. This class of economies generates interesting implications for investment in physical capital, leverage, corporate bond spreads, as well as excess equity returns.

Even though markets are incomplete, we show that an objective function of the firm is generally defined such that shareholders always unanimously support firms’ choices. Furthermore, with no agency frictions competitive equilibria are constrained efficient.

Keywords: capital structure, competitive equilibria, incomplete markets, general equilibrium, asymmetric information

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1 Introduction

The notion of competitive equilibrium in incomplete market economies with production is considered problematic in economics. This is because, when financial markets are incomplete and equity is traded in asset markets, firms' production decisions may affect the set of insurance possibilities available to consumers, the asset span of the economy.¹ As a consequence, macro models with production and incomplete markets typically assume that firms' equity is not traded, or that firms operate with a backyard technology and are managed by households.²

Similarly, while agency frictions are at the core of corporate finance, they have been hardly studied in the context of equilibrium models. This is also arguably due to the conceptual difficulties involved in the definition of competitive equilibria with asymmetric information.³

But when markets are incomplete and agents face hedging needs, the production and financial decisions of firms are linked (the Modigliani-Miller theorem does not hold). Firms then choose their capital structure – and, to some extent, their investment – to satisfy investors' hedging needs. Similarly, when firms face agency frictions, these account in part for capital structure and investment choices. All these opens up new interesting interacting forces which contribute to the determination of physical capital, leverage, corporate bond spreads, as well as excess equity returns.

In this paper we show that the analysis of production economies with incomplete markets and agency frictions can be grounded on solid theoretical foundations in general equilibrium, providing some foundations to the integrated study of macroeconomics and corporate finance. We show that this is the case if, at a competitive equilibrium, price-taking firms take their production, financing, and contractual decisions so as to maximize their value defined on the basis of *rational conjectures*, as in Makowski (1983a,b).⁴

¹It is only in rather special environments, as pointed out by Diamond (1967), that the spanning condition holds and such issue does not arise; see also the more recent contribution by Carceles-Poveda and Coen-Pirani (2009).

²This is the case, for instance, in Bewley economies, the workhorse of macroeconomic model with incomplete markets; see e.g., Ljungqvist and Sargent (2004) and Heathcoate, Storesletten, and Violante (2010) for recent surveys.

³See, e.g., Bolton and Dewatripont (2005) and Tirole (2006). A few notable exceptions include Dow, Gorton, and Krishnamurthy (2005), Acharya and Bisin (2009), and Parlour and Walde (2011).

⁴These conjectures guide firms' decisions when the value of production plans lies outside the asset span of the economy and the rationality condition can be interpreted as a consistency condition on firms' out-of-

We show that competitive equilibria, when firms possess *rational conjectures*, exist and have theoretically appealing properties.⁵ First of all, in the absence of agency frictions (or when such frictions satisfy appropriate conditions) equilibrium allocations are constrained efficient. In addition, shareholders unanimously support value maximization and hence firms' choices, even when allocations are not efficient at equilibrium. Furthermore, we extend the analysis to financial intermediation under frictions, permitting short-sales on equity and general financial intermediation.

In Section 2, we employ a numerical example to explore the potential our framework to generate appealing implications for capital structure and investment choices. In particular, we show that firms choose their capital structure – and, some extent, their investment – in order to cater to the hedging needs of households whose endowment is riskier. We also show that when hedging demand is higher, ex-ante identical firms end up choosing different strategies. Some opt for a particularly safe technology, which provide risky investors with a new hedging instrument. In Section 3 we establish existence, welfare properties, and unanimity of equilibria. In this section we also discuss more in detail various aspects of the notion of equilibrium we adopt and place it in the previous literature. In Section 4 we extend the analysis to allow for financial intermediation and short sales. Section 5 concludes. Proofs are collected in the Appendix.

2 Incomplete markets and capital structure

In this section we introduce a simple economy to showcase the role of the incompleteness of markets in an otherwise standard general equilibrium competitive economy with production.⁶ The economy lasts two periods, $t = 0, 1$, and at each date a single commodity is available for production and consumption. It is populated by I representative households and H representative firms. For simplicity we restrict the production sector to a single representative

equilibrium beliefs.

⁵To highlight the foundational aspect of our analysis, we restrict attention to simple two period economies along the lines of classical general equilibrium theory. In future work we will consider Bewley economies, that is, infinite horizon economies with incomplete markets but no agency frictions.

⁶While we shall anticipate here how we deal with the issue of the objective function of the firm discussed in the Introduction, we refer the reader to Section 3 for a proper discussion and the derivation of the unanimity and welfare properties of the competitive equilibrium notion that is considered.

firm, $H = 1$.⁷ The economy is perfectly competitive and both firms and households take prices as given.

The representative firm produces at date 1 using as physical input the single commodity invested as capital at time 0. Its production function, $f(k)$, is random and displays decreasing returns to scale. The firm takes both production and financial decisions. Production decisions are simply given by the choice of the investment level k and financial decision by the amount B of (non contingent) bonds that are issued. Hence the firm's capital structure is simply described by the amount B of bonds issued.

At time $t = 0$, the (representative) household of type i has deterministic endowment $w_0^i \geq 0$. At time $t = 1$ the household's endowment, w_1^i is random. Each household is also initially endowment with an amount θ_0 of equity. Let c_0^i and c_1^i denote consumption of household i at time 0 and at time 1, respectively. Households' preferences are described by strongly increasing, strictly quasi-concave, Von Neumann-Morgerstern utility function:

$$u(c_0^i) + \beta E [u(c_1^i)], \quad \beta > 0.$$

The total payment due to bondholders at $t = 1$ equals B , but the actual payment may be smaller if the resources available for such payment, at most equal to the firm's net output $f(k)$, are insufficient. In this case the firm defaults and these resources are divided pro-rata among all bondholders. As a consequence, the payoff of the bonds issued by the firm is a random variable depending on its production and financing choices. The payoff of equity is the residual, what is left of the firm's net output after paying off bonds. Formally, the unit payoffs of the firm's bonds and equity, respectively, are

$$d^b(k, B) = \min \left\{ 1, \frac{f(k)}{B} \right\}, \quad (1)$$

$$d^e(k, B) = \max \{ f(k) - B, 0 \}. \quad (2)$$

The firm's equity and debt are the only assets in the economy. Let θ^i and b^i denote the holdings of equity and bonds, respectively, of a household of type i .⁸ Short sales are not

⁷Formally, a continuum of households of 2 different types, each of them of unit mass, and a continuum of identical firms, of unit mass.

⁸We state here the consumers' choice problem for the case where all firms make the same production and financial choice, hence only one type of equity and corporate debt is available for trade to consumers. As we will see in what follows, in equilibrium firms may also end up making different choices, in which case different types of equities and bonds are available for trade.

allowed, $(\theta^i, b^i) \geq 0$. The household makes his portfolio and consumption decisions taking as given the price of equity and bonds, q and p , as well as their payoffs, d^e and d^b , and the firm's market value, V . His optimization problem is then

$$\begin{aligned} & \max_{c_0^i, \theta_1^i, b^i, c_1^i} u(c_0^i) + \beta \mathbb{E} [u(c_1^i)] \\ \text{s.t.} \quad & c_0^i = w_0^i + \theta_0 V - q\theta_1^i - pb^i \\ & c_1^i = w_1^i + \theta_1^i d^e + b^i d^b \\ & \theta_1^i \geq 0, \quad b^i \geq 0 \end{aligned}$$

In evaluating alternative production and financing plans k, B , the firm operates on the basis of price conjectures $q(k, B)$ and $p(k, B)$, which specify the market valuation at $t = 0$ of the future yields of equity and debt for any possible choice of production and financing plan. The firm's decision problem consists then in the choice of k, B that maximizes its initial market value, at date $t = 0$:

$$\max_{k, B} -k + q(k, B) + p(k, B)B, \tag{3}$$

At equilibrium we shall require conjectures to be *rational*, that is:⁹

$$\begin{aligned} q(k, B) &= \max_i \mathbb{E} \left[\frac{\beta u'(c_1^i)}{u'(c_0^i)} d^e(k, B) \right] \\ p(k, B) &= \max_i \mathbb{E} \left[\frac{\beta u'(c_1^i)}{u'(c_0^i)} d^b(k, B) \right] \end{aligned}$$

This requirement prescribes that, for any production and financing plan of the firm k, B , the conjectured price of equity and bonds, $q(k, B)$ and $p(k, B)$, equals the highest marginal valuation, across all households in the economy¹⁰, of the return on equity and bonds associated to k, B . Notice that for the firm's equilibrium choice of k, B such requirement coincides with the agents' Euler equations, provided conjectures are correct. The *rationality* of conjectures prescribes that an analogous condition holds also for all other production and financing plans, not chosen in equilibrium.

We will show in Section 3 that when firms operate on the basis of *rational* conjectures the objective of the firm of maximizing its market value is well defined, since shareholders always

⁹The *rationality* of conjectures is, more precisely, the *Makowski criterion for rational conjectures* (after Makowski (1983a), (1983b)).

¹⁰The households' marginal rates of substitution are taken as given, independent of the firm's decisions.

unanimously support firms' choices, even when financial markets are incomplete. With complete financial markets marginal rates of substitutions are equalized across households and hence a unique stochastic discount factor prices all possible payoffs of existing assets. This is no longer true when markets are incomplete, in which case the *rationality* of conjectures implies that the stochastic discount factor pricing equity might be different than the one pricing bonds, and both might be different for different plans k, B .

At a competitive equilibrium k, B solves the firm's problem, with conjectures satisfying the *rationality* condition, and $\theta^i, b^i, c_0^i, c_1^i$ solves the household problem for each i . Furthermore, the following consistency conditions hold, ensuring that for k, B corresponding to the choices made by firms in equilibrium, the prices conjectured by firms coincide with the actual prices faced by households and the same is true for asset returns:

$$q = q(k, B), \quad p = p(k, B), \quad V = -k + q(k, B) + p(k, B)B, \quad d^e = d^e(k, B), \quad d^b = d^b(k, B).$$

Finally, markets clear:

$$\begin{aligned} \sum_i b^i &\leq B \\ \sum_i \theta^i &\leq 1 \end{aligned}$$

Notice that the *rationality* of conjectures implies that the markets for all other 'types' of firms' debt and equity, corresponding to different choices of k, B , clear with zero trades.

2.1 Baseline parameterization

In what follows, we will describe the properties of the investment and leverage decisions made by firms and hence of equilibrium asset returns and consumption levels for the economy described in the previous section. We will see that when markets are incomplete an important determinant of firms' decisions is the difference in the demand for hedging across households, since firms' decisions contribute to determine the returns of the assets available for trade. We will report the value of equilibrium returns and firms' decisions for a variety of parameter values capturing both the extent by which market incompleteness matters and some comparative statics exercises.

Specialize the economy introduced in the previous section by considering the case of two representative agents, $I = 2$. Let the fundamental uncertain be Normally distributed,

$$\varepsilon \sim N(\mu, \sigma^2), \quad \sigma > 0.$$

The production function of the single representative firm is:

$$f(k) = Ae^{z_1}k^\alpha, \quad \alpha \in (0, 1)$$

with z_1 an AR(1) stochastic process describing a productivity shock realized at $t = 1$,

$$z_1 = \rho z_0 + \varepsilon$$

Let each agent preferences be represented by a CRRA utility, $u(c) = \frac{c^{1-\psi}}{1-\psi}$. Furthermore, let each agent $i = 1, 2$ time 1 endowment have the form:

$$w_1^i = e^{\rho z_0 - \chi_i \mu - \frac{1}{2} \chi_i^2 \sigma^2 + \chi_i \varepsilon}, \quad \chi_i \in [0, 1].$$

This implies the following moments for the endowment process:

$$E[w_1^i] = e^{\rho z_0}, \quad Var[w_1^i] = e^{2\rho z_0} [e^{\chi_i^2 \sigma^2} - 1], \quad Cov(w_1^i, e^{z_1}) = e^{2\rho z_0 + \mu + \frac{1}{2} \sigma^2} (e^{\chi_i \sigma^2} - 1).$$

Hence households may differ in their wealth and in their exposure to risk. For $\chi_i = 0$, the endowment is riskless. As χ_i increases, so do the variance of the endowment and its covariance with the common shock to firms' productivity – the aggregate shock. Initial equity ownership is uniformly distributed across household and the outstanding amount of equity is normalized to 1. Hence each household is initially endowment with an amount $\theta_0 = 0.5$ of equity.

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Some key parameters in our analysis are then the individual loadings on the aggregate shock χ_i , $i = 1, 2$, together with the initial wealth distribution as described by w_0^i , $i = 1, 2$. To ensure that households are sufficiently different in their hedging needs, we select $\chi_1 = 0$ and $\chi_2 = 0.9$, respectively. Hence the endowment profile of type-1 households is riskless, while that of type-2 households is risky so that the latter have a relatively high demand for hedging and hence for assets with relatively safe returns, while the first ones are more willing to hold risky assets. Figure 1 illustrates the variation of endowments and firm productivity

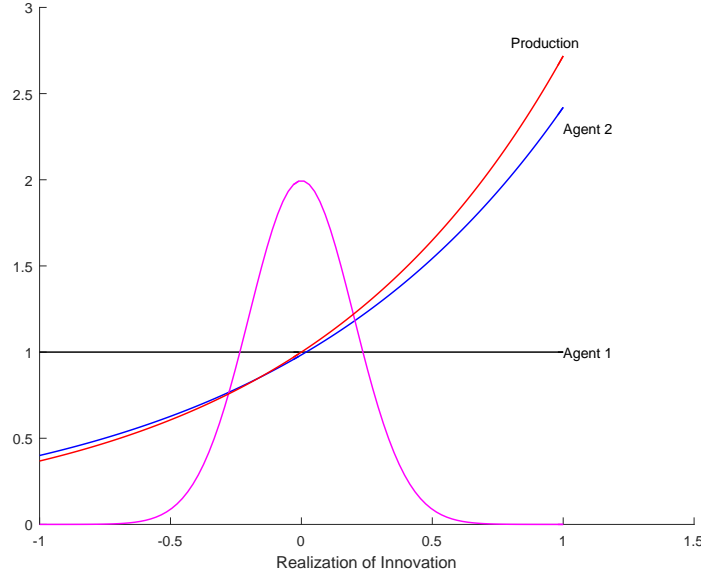


Figure 1: Household Endowments and Firm Productivity.

at $t = 1$ over the (truncated) set of realizations for the innovation ε , along with the density function for the latter. Also, the higher is the initial endowment of type-2 households w_0^2 relative to that of type-1, w_0^1 , the higher the demand for hedging and hence for safe assets in the economy.

With regard to the other parameter values, we will assume throughout that households have a relative risk aversion coefficient $\psi = 3$. The span of control parameter is $\alpha = 0.6$, while the parameters of the normal distribution are $\mu = -0.025$ and $\sigma = 0.3$; also $z_0 = 0$ and $A = 2$.

In this environment we find that in equilibrium both households hold equity, but type-2 households hold all the debt issued by firms. The reason is rather intuitive, since debt is relatively safer compared to equity and, as argued above, type-2 agents have a significantly higher hedging demand. Also, we see from the expression of $d^b(k, B)$ that, for given k , the yield of debt is safer the lower is firm's leverage B . On the other hand, the lower B is, the lower is the supply of safer assets in the economy. As we said above, in equilibrium firms

choose their investment and leverage so as to maximize their market value, operating on the basis of rational conjectures regarding the price of equity and debt that reflect the demand for hedging in the economy.

2.2 Leverage and Investment choices

To better understanding the properties of the levels of investment and debt issue chosen by firms it is useful to derive first the (necessary) conditions for a solution of the firms' choice problem (3).

Using the property stated above that, in the environment considered, only type-2 households hold firms' debt while both types hold equity, the necessary condition for the optimal choice of capital k is as follows:

$$1 = \beta A e^{\rho z_0} \alpha A k^{\alpha-1} \left[\int_{\varepsilon^*(k,B)}^{+\infty} \frac{u'(c_1^2)}{u'(c_0^2)} e^\varepsilon g(\varepsilon) d\varepsilon + \int_{-\infty}^{\varepsilon^*(k)} \frac{\beta u'(c_1^2)}{u'(c_0^2)} e^\varepsilon g(\varepsilon) d\varepsilon \right],$$

where $g(\cdot)$ denotes the density of the normal distribution and $\varepsilon^*(k, B) \equiv \log\left(\frac{B}{k^\alpha}\right) - \rho z_0$ the lowest realization of the innovation ε for which the firm is solvent. A little algebra allows to rewrite the above condition as

$$1 = \beta A e^{\rho z_0} \alpha A k^{\alpha-1} \left[\text{cov} \left(\frac{u'(c_1^2)}{u'(c_0^2)}, e^\varepsilon \right) + \mathbb{E} \left(\frac{u'(c_1^2)}{u'(c_0^2)} \right) \mathbb{E} (e^\varepsilon) \right]. \quad (4)$$

In the square brackets on the right-hand side we have two terms familiar in asset pricing models. The first is the covariance between household 2's marginal rate of substitution and the innovation. The second is the inverse of the risk-free rate, given by the reciprocal of the market value of the risk-free asset¹¹. The fact that the level of k is increasing in both these terms makes it transparent that the investment policy is also shaped by household 2's hedging needs. We further elaborate on this issue below, when we consider a comparative statics exercise with respect to the distribution of the endowment at $t = 0$.

The necessary condition for debt optimization is then

$$\beta \int_{\varepsilon^*(k,B)}^{+\infty} \frac{u'(c_1^1)}{u'(c_0^1)} g(\varepsilon) d\varepsilon = \beta \int_{-\infty}^{\varepsilon^*(k)} \frac{u'(c_1^2)}{u'(c_0^2)} e^\varepsilon g(\varepsilon) d\varepsilon \quad (5)$$

At the margin, raising debt issuance transfers resources from shareholders to bondholders in the states of nature where the firm is solvent. Given the incompleteness of the market, the

¹¹This is in turn given by the price conjecture $p(k, B)$ in correspondence of a level of debt B sufficiently low that debt is riskless.

Modigliani-Miller indeterminacy result does not hold: firms' leverage is determinate because the marginal rates of substitution of the two types of investors are not equal. In other words, the choice of leverage is dictated by the desire to cater to agent 2's hedging needs.

2.3 Comparative statics with respect to the wealth distribution

In this section, we assess how the distribution of wealth at time $t = 0$ affects the equilibrium allocation. To this end, we compute equilibria for several values of the ratio $w_0^2/(w_0^1 + w_0^2)$, for given total endowment. An increase in such ratio leads to a larger demand for hedging, since the share of savings and assets' demand coming from type-2 households' savings increase.

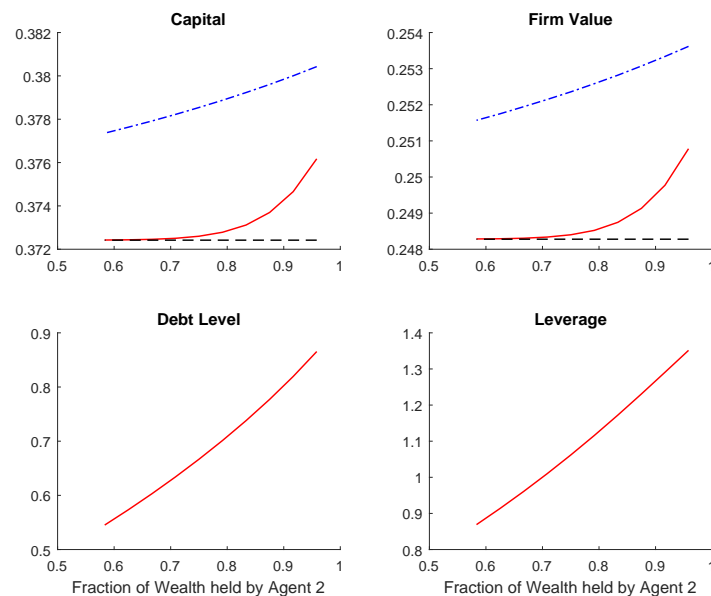


Figure 2: Firms' Choices.

The red (solid) lines in Figure 2 describe how the equilibrium levels of capital, debt issuance, leverage, and firm's market value vary, as a greater fraction of total endowment accrues to type-2 agents. Leverage is computed as $p_b B/k$. The black (dashed) and blue (dash-dot) lines refer instead to alternative scenarios where markets are complete and equity

is the only asset, respectively. Debt or leverage are not reported in the complete market scenario since in that case firms' capital structure is indeterminate.

As type-2 households grow wealthier, their share in asset demand increases. Since their earnings are positively correlated with equity payoffs, the demand for hedging in the economy and hence for firms' debt will also increase. This will drive up the price of corporate debt and so firms will find it optimal to issue more debt and cater so to the consumers' higher hedging needs as we see in Figure 2.

We also see that in the scenario where debt issuance is not allowed, equilibrium capital is uniformly higher. This is the case because the supply of hedging instruments is very limited and hence, faced with higher risk, type-2 agents engage in larger precautionary savings. The firm caters to such need by increasing its size, in line with our comments to equation (4). In the complete market scenario the level of investment is instead lower, and the reason is similar. Moreover we observe that, as the fraction of wealth held by agent 2 declines towards $1/2$, investment in the incomplete market economy considered converges towards the complete-market level. This is not surprising, since a larger fraction of wealth in the hands of agents with safe endowments implies that aggregate demand for hedging declines and so the available supply from firms may become sufficient to satisfy it.

Figure 3 further corroborates our narrative. When equity is the only asset, type-2 agents' consumption growth is uniformly higher, again reflecting the greater precautionary saving in this case. When hedging demand is relatively low (i.e. the fraction of wealth held by type-2 agents is low), the addition of debt is enough to bring consumption profiles close to those of the complete market scenario, in line with what we saw for capital. In contrast, when the fraction of initial endowment held by type-2 households approaches 100%, both the mean and variance of consumption growth for such agents exceed the values obtained with complete markets.

Figure 4 illustrates the implications for assets returns. The risk-free rate is determined by the marginal valuation of type-2 agents for all values of the wealth distribution, since type-1 agents have always a strictly lower valuation for the risk-free asset. In turn, this means that any marginal increase (from zero) in the supply of that asset would go to the benefit of type-2 agents only. We see that the risk-free rate declines as the type-2 agents becomes richer, since hedging demand increases driving up the price of debt.

The behavior of excess equity returns (relative to the risk free rate) under incomplete markets can be rationalized by appealing to the role of leverage. In the scenario without

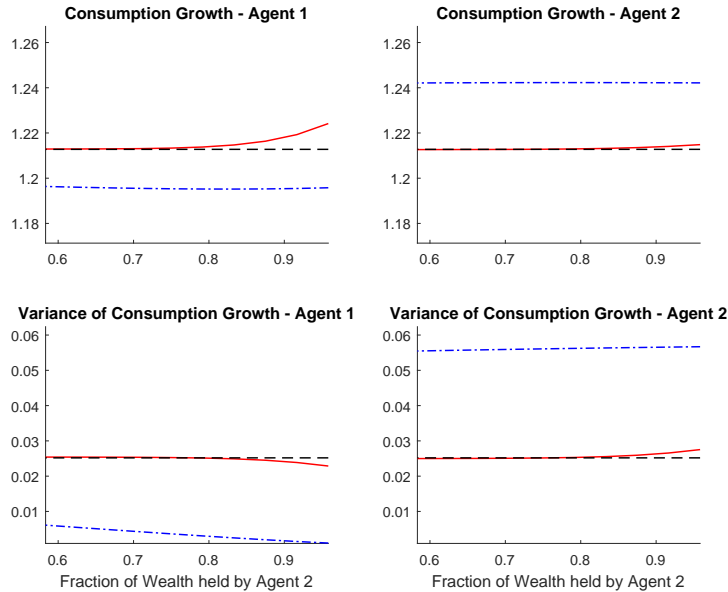


Figure 3: Consumption profiles.

debt – the blue, dash-dot line – equity is less risky simply because it is unlevered. Recall also that leverage increases as type-2 agents’ wealth increases, which contributes to widening the difference between excess returns with incomplete market and the alternative scenario without debt as well as the one with complete markets¹². Finally, corporate bond spreads increase, as the probability of default rises with leverage.

2.4 Business Cycles and Public Debt

Next we investigate how firms’ choices vary with business cycle fluctuations. In particular we examine the case where z_0 takes a lower value. As a consequence both the productivity of the firms’ technology at date 1 and consumers’ endowment at date 1 are lower.

In Figures 5 and 6 we show how the firms’ equilibrium choices and asset returns vary with the size of the (negative) aggregate shock to the economy. We see that investment declines

¹²In the latter scenario, as we noticed, firms’ debt level is indeterminate. In Figure 4 excess returns are reported for the case where debt is zero, in line with the first scenario, without debt.

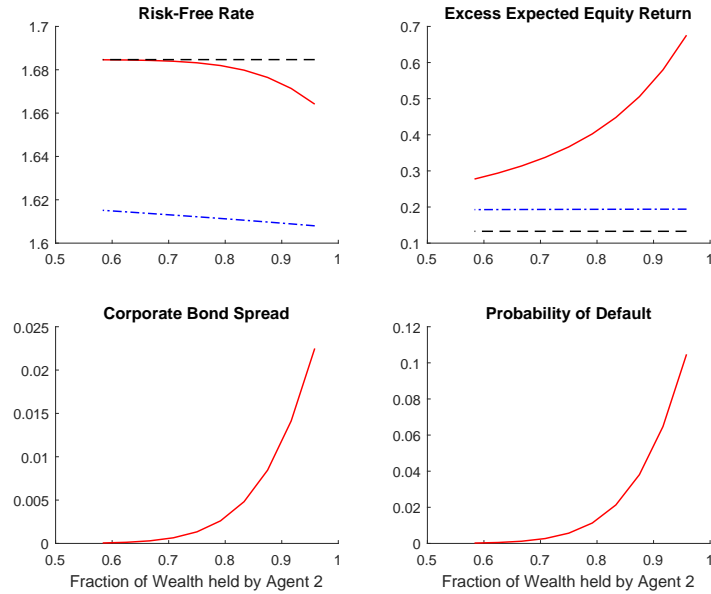


Figure 4: Financial Assets.

(both in the incomplete market economy and the two alternative scenarios). This shows that the effect on savings of the decline in the expected productivity of capital prevails over the opposite effect of the decrease in households' future endowments. Also, firms' leverage increases while the risk-free rate decreases, suggesting that consumers' demand for hedging increases in a recession. This can be attributed to the fact that the lower households' income at date 1 makes them (in particular type 2 agents, whose income is risky) more risk averse, and this increases their demand for hedging.

Next, in Figures 7 and 8 we analyze the effects of the introduction of a fixed, exogenous supply of risk-free public debt. Households can now invest their savings also in this asset, in addition to firms' equity and debt. We see in Figures xx and xx that both firms' investment and leverage decrease as the supply of public debt increases. This is due to a crowding out effect, which affects primarily corporate debt, whose level goes down by around 0.8 for each unit increase of the supply of public debt. Also, the risk-free rate increases and the excess return on equity declines, as a consequence of the decrease in leverage and hence in the

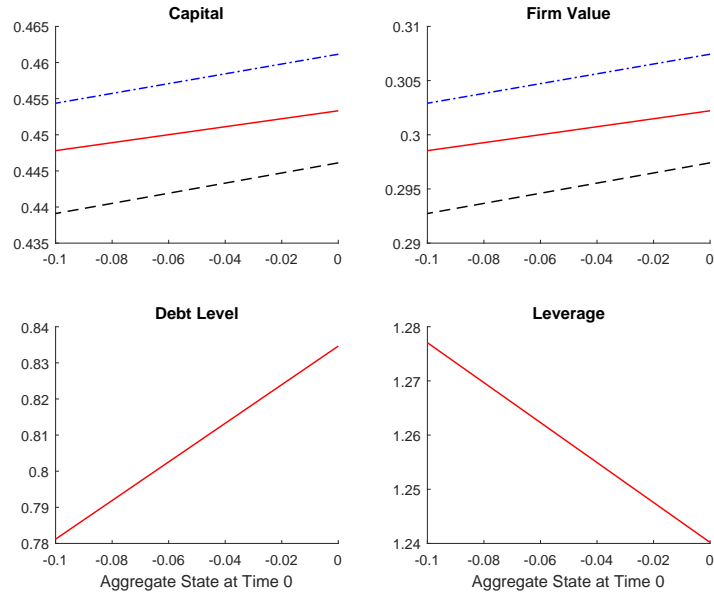


Figure 5: Equilibrium choices and aggregate shock

volatility of equity returns. The effects obtained in this case can be viewed as primarily the result of an increase in the supply of hedging instruments, in contrast to the previous case where the main driver was a change in the demand for hedging.

2.5 Specialization

When markets are incomplete, *rational conjectures* imply that different production and financing plans are possibly evaluated by different stochastic discount factors, given by the marginal rates of substitutions of different types of households. As a consequence, the firm's problem is not convex when markets are incomplete and it is then possible that the equilibrium features firms specializing in different production and financial plans. This is not just a technical issue but reflects a fundamental implication of the rationality of firms' conjectures: firms have an incentive to specialize their plans so as to better cater to the different demand of different types of households. By so doing, different types of equities and bonds with distinct returns are made available for trade to consumers. Of course there

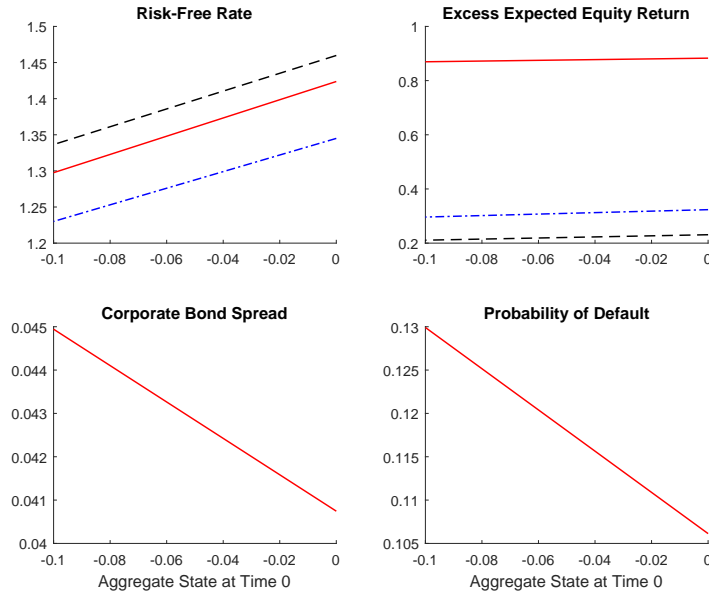


Figure 6: Asset returns and aggregate shock

are also costs for a firm to specialize since it means the market for the specific assets issued by the firm will be smaller, thus specialization does not always occur (as we saw indeed happened in all the situations considered so far)

To illustrate the possibility of separation in a clear way, we consider now an economy in which firms can choose between the risky technology considered so far, $f(k) = Ae^{z_1}k^\alpha$, and another, safer, technology. For simplicity, let the safer technology be entirely deterministic: $h(k) = A_w e^{\rho z_0} k^\alpha$, and less productive on average than $f(k)$: $Ae^{\rho z_0} = 2 > e^{\rho z_0} A_w = 1.888$. It is convenient to write the production function of the (representative) firm as

$$F(k, \phi) = \phi f(k) + (1 - \phi)h(k), \quad \phi \in \{0, 1\}.$$

Replacing $f(k)$ with $F(k, \phi)$ in (1) and (2) yields expressions of the payoffs of debt and equity $d^b(k, B, \phi)$ and $d^e(k, B, \phi)$ and of the corresponding *rational* conjectures $q(k, B, \phi)$ and $p(k, B, \phi)$, which now depend on k, B, ϕ .

Though firms remain ex-ante identical we show that, for some initial endowment distributions, in equilibrium firms specialize, making different choices so as to cater to the different

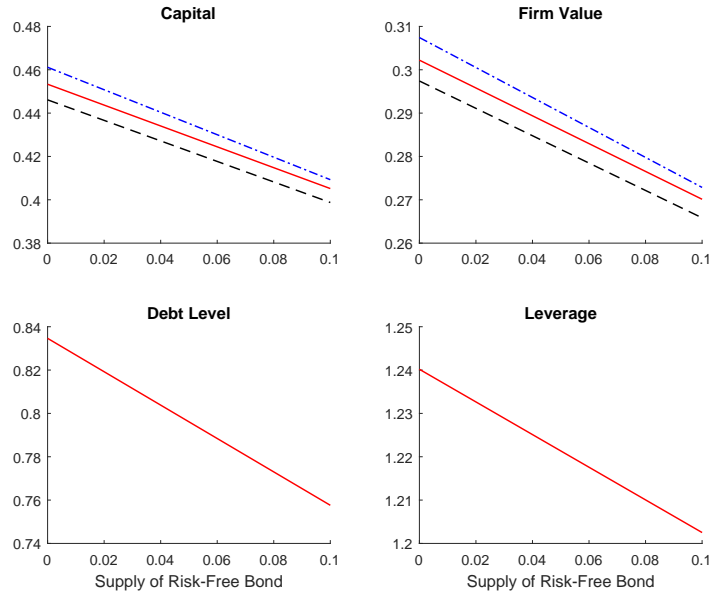


Figure 7: Equilibrium choices and riskfree public debt

demands of different households.¹³ In particular specialization takes here the clearcut form of choosing a different technology.

Figure 9 indicates that specialization occurs only when the demand for hedging is high. The red, solid line reproduces the choices made by firms in the equilibrium under incomplete markets obtained in Section 2.3, while the blue dash-dot line shows the choices in the current situation, where the production function is $F(k, \phi)$, whenever they are different. When the fraction of wealth held by type-2 agents is relatively low all firms choose the risky technology and hence the equilibrium allocation is the same as the one obtained earlier. [HOW CAN

In contrast, as wealth is sufficiently redistributed towards type-2 households in equilibrium a non-degenerate fraction of firms chooses the safe technology. In such an environment, as we already noticed, the demand for hedging by type-2 households is high. A non-zero fraction of firms find then optimal to cater to this demand by choosing the safe, though less

¹³Though the above specification of the firms' production function in $F(k, \phi)$ generates an additional element of nonconvexity in the firm's choice problem, we should stress this is not crucial. The fundamental nonconvexity comes, as argued above, from rational conjectures with incomplete markets.

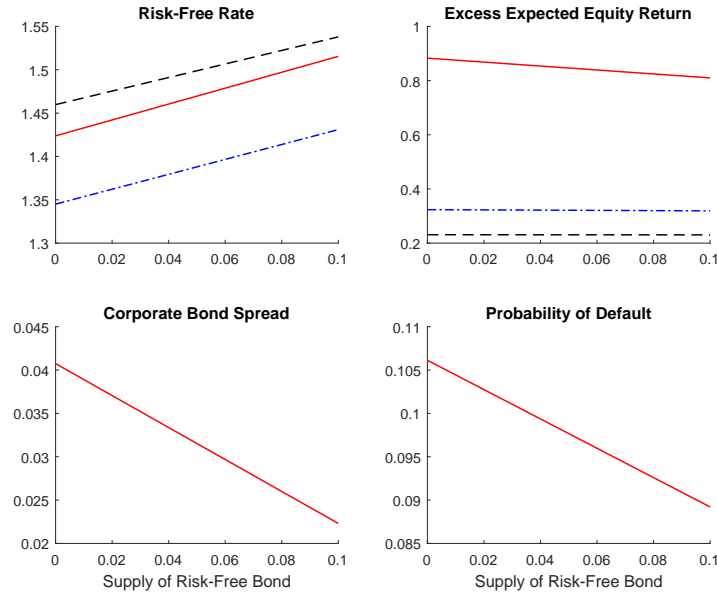


Figure 8: Asset returns and riskfree public debt

productive, technology which allows them to issue a risk-free asset. A new asset becomes so available to households, making markets endogenously more complete. The top-right panel of Figure 9 indicates that, as a consequence, type-2 agents decrease their holdings of equity of the other firms, which choose the risky technology. Also, Figure 10 shows that in such scenarios risky firms shrink in size and value, and reduce their leverage.

Figure 11 illustrates then the impact that specialization and the introduction of a new security has on the returns of the assets issued by risky firms. The supply of risk-free assets by firms choosing the safe technology implies that the demand for hedging faced by firms which still select the risky technology decreases, and so does the households' precautionary motive. As a result, the risk-free rate increases. The excess return of equity, default probability and corporate bond spreads of risky firms all decline, since leverage is now lower.

To better understand the circumstances under which specialization does and does not occur in equilibrium, we should point out that the safe technology is less productive, hence its choice by a subset of firms entails a cost. The benefit of specialization is that it allows

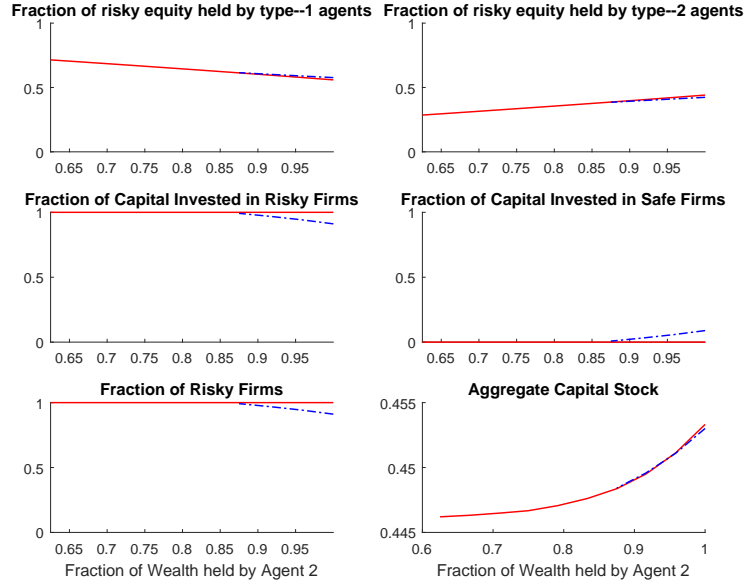


Figure 9: Specialization

to satisfy more effectively the different hedging demands of different types of households. Our findings clearly show that specialization arises when the demand for hedging by type 2 households is sufficiently large that firms cannot properly accommodate it by choosing the risky technology.

2.6 Agency frictions

We extend now the specification of the environment to introduce an agency friction in the firms' choice problem, by assuming that some components of firms' choices are not observable to outside investors. In particular, the specification of the production function of the (representative) firm is still $F(k, \phi) = \phi f(k) + (1 - \phi)h(k)$, as in the previous Section 2.5, but ϕ can now take any value in $[0, 1]$ and represents so more generally the loading on risk chosen by the firm. Also, the choice of ϕ entails a cost $C(\phi) = a_0\phi + \frac{a_1}{1-\phi}$, $a_0, a_1 > 0$, incurred at date 0.

While the choice of capital k and debt B remains observable to households who can then

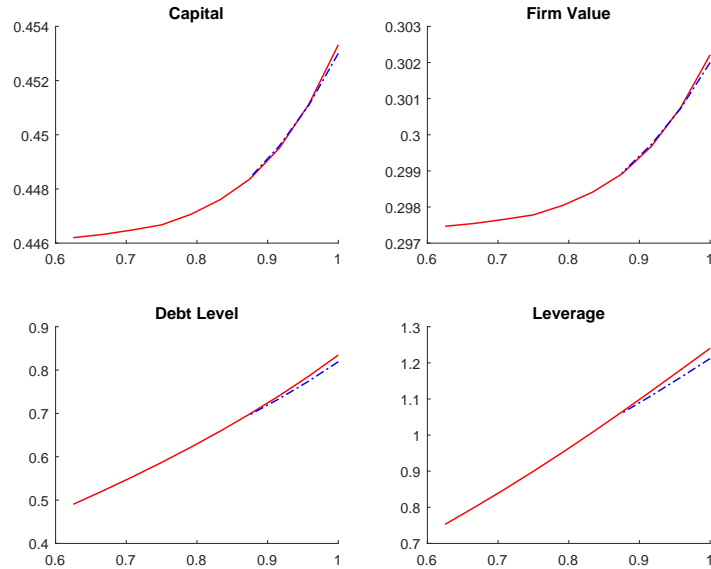


Figure 10: Specialization and the risky firm.

choose their purchases of debt and equity in the market knowing the levels of k, B this is no longer true for the factor loading choice ϕ and can only have expectations about the level of ϕ chosen by the firm for any given k, B . Hence, while k, B will still be chosen by the firm so as to maximize its market value, the same is no longer true for ϕ . We consider here the case where ϕ is chosen by shareholders to maximize their benefits from holding equity, that is the value of equity rather than the value of the firm. Hence, even though ϕ affects both the the yield of equity and debt, it is chosen to maximize only the shareholders' valuation of the return on equity. This induces an agency problem between the firm's equityholders and its bondholders as their objectives are generally not aligned: the first ones have in fact an incentive to choose factor loadings for which the yield of equity is highest, though this may mean the yield of debt is low. This is the *asset substitution* problem, as in Jensen and Meckling (1976). As a consequence, the firm's valuation is lower than if shatreholders could commit to a choice of ϕ , hence the agency problem.

The decision problem of the firm is then

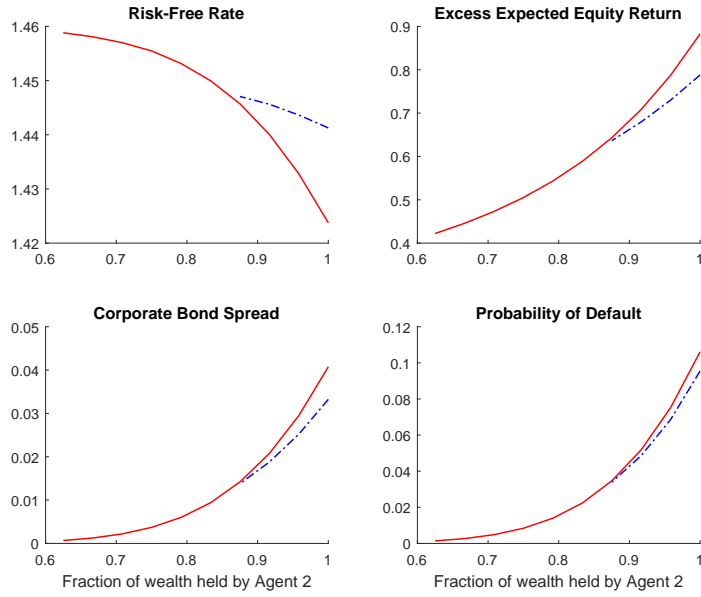


Figure 11: Specialization and asset returns.

$$\max_{k, B, \phi} -k + q(k, B, \phi) + p(k, B, \phi)B, \quad (6)$$

s.t.

$$\phi \in \arg \max q(k, B, \phi) - C(\phi)$$

with price conjectures $q(k, B, \phi), p(k, B, \phi)$ satisfying the same rationality condition as in the previous section. The difference in the firm's problem induced by agency frictions is the fact that the admissible choices of ϕ are restricted by a constraint that captures the fact that, as argued above, factor loading is chosen so as to maximize the value of equity. Note that the admissible values of ϕ obtained as solutions of the problem

$$\phi \in \arg \max q(k, B, \phi) - C(\phi)$$

depend on the levels of k, B which also affect the yield of equity $d^e(k, B, \phi)$ and hence its value $q(k, B, \phi)$. Let $\phi(k, B)$ denote the mapping from (k, B) into ϕ describing the solutions

of this problem. When the firm chooses its levels of debt and capital, it takes then also into account the effect this choice has on the risk loading chosen by equityholders, as described by $\phi(k, B)$. If we substitute this map into $q(k, B, \phi)$ and $p(k, B, \phi)$ we obtain an expression of price conjectures that reflects, for any given k, B the correct anticipation of the level of ϕ induced' by k, B .

To see how the presence of agency frictions affects firms' choices we report first the necessary conditions for the optimal choice of k :

$$\begin{aligned}
1 = & \beta e^{\rho z_0} \phi \alpha A k^{\alpha-1} \left[\max_i \int_{\varepsilon^*(k, B, \phi)}^{+\infty} \frac{u'(c_1^i)}{u'(c_0^i)} e^\varepsilon g(\varepsilon) d\varepsilon + \int_{-\infty}^{\varepsilon^*(k, B, \phi)} \frac{\beta u'(c_1^2)}{u'(c_0^2)} e^\varepsilon g(\varepsilon) d\varepsilon \right] \\
& + \beta e^{\rho z_0} (1 - \phi) \alpha A_w k^{\alpha-1} \left[\max_i \int_{\varepsilon^*(k, B, \phi)}^{+\infty} \frac{u'(c_1^i)}{u'(c_0^i)} g(\varepsilon) d\varepsilon + \int_{-\infty}^{\varepsilon^*(k, B, \phi)} \frac{\beta u'(c_1^2)}{u'(c_0^2)} e^\varepsilon g(\varepsilon) d\varepsilon \right] \\
& + \frac{\partial \phi}{\partial k} \beta e^{\rho z_0} k^\alpha \left[\max_i \int_{\varepsilon^*(k, B, \phi)}^{+\infty} \frac{u'(c_1^i)}{u'(c_0^i)} (A e^\varepsilon - A_w) g(\varepsilon) d\varepsilon + \int_{-\infty}^{\varepsilon^*(k, B, \phi)} \frac{\beta u'(c_1^2)}{u'(c_0^2)} (A e^\varepsilon - A_w) g(\varepsilon) d\varepsilon \right] \\
& - \frac{\partial \phi}{\partial k} C'(\phi),
\end{aligned}$$

and of B :

$$\begin{aligned}
& - \beta \max_i \int_{\varepsilon^*(k, B, \phi)}^{+\infty} \frac{u'(c_1^i)}{u'(c_0^i)} g(\varepsilon) d\varepsilon + \beta \int_{-\infty}^{\varepsilon^*(k)} \frac{u'(c_1^2)}{u'(c_0^2)} e^\varepsilon g(\varepsilon) d\varepsilon + \\
& + \frac{\partial \phi}{\partial B} \beta e^{\rho z_0} k^\alpha \left[\max_i \int_{\varepsilon^*(k, B, \phi)}^{+\infty} \frac{u'(c_1^i)}{u'(c_0^i)} (A e^\varepsilon - A_w) g(\varepsilon) d\varepsilon + \int_{-\infty}^{\varepsilon^*(k, B, \phi)} \frac{\beta u'(c_1^2)}{u'(c_0^2)} (A e^\varepsilon - A_w) g(\varepsilon) d\varepsilon \right] \\
& - \frac{\partial \phi}{\partial B} C'(\phi)
\end{aligned}$$

The term in the last line of each of the two expression is the new term due to the presence of agency frictions, and reflects the effect that the chosen level of k and B , respectively, has on the choice of ϕ . The sign of this additional effect then depends on the sign of $\frac{\partial \phi}{\partial k}$ and $\frac{\partial \phi}{\partial B}$.

We turn then to characterizing the properties of the map $\phi(k, B)$. In the environment considered, since as we saw both types of households hold equity in equilibrium, problem (2.6) becomes

$$\max_\phi \left[\max_i \beta \int_{\varepsilon^*(k, B, \phi)}^{+\infty} \frac{u'(c_1^i)}{u'(c_0^i)} [e^{\rho z_0} k^\alpha (\phi A e^\varepsilon + (1 - \phi) A_w) - B] g(\varepsilon) d\varepsilon \right] - C(\phi).$$

The factor loading ϕ then obtains as a solution of the following first order condition:¹⁴

$$\beta e^{\rho z_0} k^\alpha \left[\max_i \int_{\varepsilon^*(k, B, \phi)}^{+\infty} \frac{u'(c_1^i)}{u'(c_0^i)} [Ae^\varepsilon - A_w] g(\varepsilon) d\varepsilon \right] - C'(\phi) = 0. \quad (7)$$

Differentiating further this expression we then see that the sign of $\frac{\partial \phi}{\partial B}$ depends on the level of B : when B is not too high, that is $B < A_w e^{\rho z_0} k^\alpha$ (so that the firm would be solvent with full loading on the safe technology) we have $\frac{\partial \phi}{\partial B} > 0$, that is increasing leverage increases the loading on the risky factor. This is the case that corresponds to the standard asset substitution effect. However, when B is high the sign of the effect is reversed. In equilibrium we find that B is never too high, so that $\frac{\partial \phi}{\partial B} > 0$.

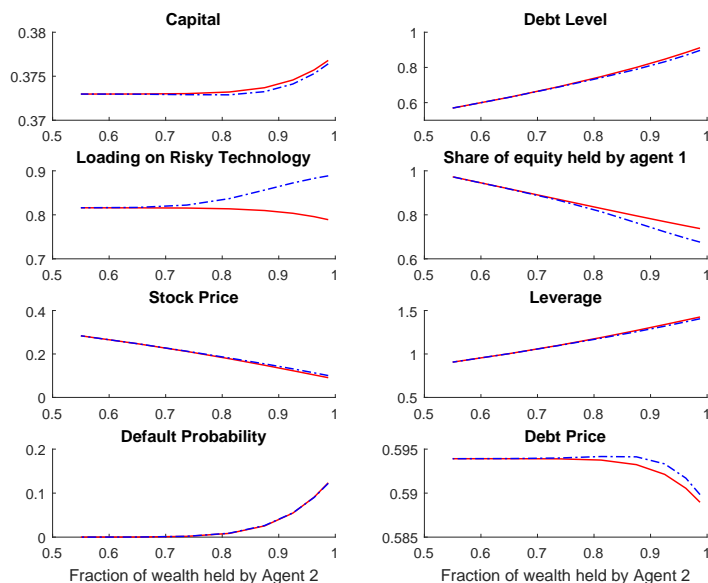


Figure 12: Agency

In Figures 12 we report the values of firms' choices in the equilibrium with agency (blue dash-dot line) as a function of the initial endowment distribution between the two types of

¹⁴It is easy to verify that the terms $[Ae^{\varepsilon^*} - A_w]$ and $\frac{\partial \varepsilon^*}{\partial \phi}$ have always the opposite sign and hence the first order conditions are also sufficient for an optimum provided the second derivative of the cost function $C''(\phi)$ is large enough.

households. These values are then compared to those of the equilibrium without agency, where ϕ is observable and firms choose ϕ, k, B so as to maximize their market value, without constraints (red, solid line). We then see that the loading on the risky factor is higher with agency. This is the most significant effect of agency, and is larger the bigger is the endowment share of type 2 households. This finding, that the effect of the presence of agency on equilibrium outcomes varies with the distribution of wealth is novel. It can be understood in the light of the previous discussion, where we argued that the extent to which market incomplete markets matter depends on wealth distribution.

We also see that both investment and debt are smaller with agency. These findings are in line with what we have seen in the previous paragraphs: with agency corporate debt induces a higher loading on the risky factor, this ends up making debt more 'costly'. In equilibrium we see that firms respond to the presence of agency by lowering their debt issuance, but this is not enough to offset the effect and hence the loading on the risky factor is higher. As a consequence the supply of hedging instruments declines.

Another way to look at the findings in the figure is that at the equilibrium allocation without agency the incentive compatibility constraint, as described in Problem (2.6) is violated. Hence firms need to face the presence of agency frictions by modifying their choices, concerning risk loading, leverage and investment. We see that in the situation considered they primarily respond by increasing their risk loading choice, but also in part by lowering their debt issuance.

The environment considered in this section allows to see how the analysis of competitive equilibria with incomplete markets may yield interesting implications for important issues in corporate finance. In the corporate finance literature it is often the case that the capital structure of firms is fully determined by its role in providing incentives to shareholders or managers, typically in partial equilibrium. But in a general equilibrium, incomplete market framework the capital structure is also determined by the consumers' willingness to pay for the different forms of borrowing instruments issued by firms. In the previous sections we referred to this as the hedging demand by households and saw it was the only determinant of firms' capital structure decisions. This is important because, as we saw, demand forces link the capital structure to fundamental macroeconomic factors as the allocation of risk in the economy, asset prices and business fluctuations.

When agency is introduced a new factor comes into play, the effect that firms' leverage has on the agency problem, in line with what shown by the literature on corporate finance

recalled above. This affects the supply of hedging instruments by firms, by introducing an agency cost of debt. In the present competitive equilibrium with incomplete markets and agency frictions, firms' decisions are then pinned down by the interaction between the demand for hedging, coming from consumers, and the restriction on the supply of hedging instruments, induced by agency frictions.

A final comment is important, to better appreciate the role of market incompleteness in the determination of the capital structure. With complete financial markets the financial structure would only be determined by supply considerations, that is by the need to provide incentives. In particular, in the economy considered above where the agency friction is of the asset substitution type, setting $B = 0$ ensures that the incentive constraint in Problem (2.6) is always satisfied. Hence incentives can be provided to shareholders without any need to affect firms' production choices and the risk allocation among agents, the equilibrium is so fully Pareto efficient.

3 Production economies with incomplete markets and agency frictions

In this section we study the general economy introduced in Section 2, extended to account for agency frictions; that is, where Problem 2.6 substitutes 3 as the problem of the firm. For this economy we shall prove existence, welfare, and characterization theorems.

In particular, the firm's choice of $\phi \in \Phi$ is subject to an abstract constraint described by $\phi \in \phi(k, B; c(s))$, where $c(s) = \{c^i(s)\}_{i=1}^I$. Thus the level of ϕ depends on the other decisions of the firm k, B and, possibly, also on other variables external to the firm, as the consumption allocation $c(s)$.¹⁵

In the analysis of competitive equilibria the map $\phi(\cdot)$ is taken as exogenously given. The specific form of the map $\phi(\cdot)$ depends on the nature of the agency frictions faced by firms and hence of the choice problem determining ϕ . When ϕ is univocally determined by the constraint, that is the map $\phi(k, B; c(s))$ is single valued, we could write *rational* conjectures

¹⁵This is without loss of generality in our environment: other variables, as equilibrium prices for instance, can be written in terms of the consumption allocation $c(s)$.

as

$$q(k, B) = \max_i \mathbb{E} [MRS^i(c^i(s))d^e(k, \phi(k, B; c(s)), B; s)]$$

$$p(k, B) = \max_i \mathbb{E} [MRS^i(c^i(s))d^b(k, \phi(k, B; c(s)), B; s)], \quad \forall k, B.$$

The presence of the map $\phi(\cdot)$ in the specification of the price conjectures and the fact that B appears among its arguments generate an additional link between production and financing decisions, due to the agency frictions.¹⁶

3.1 A few remarks on the equilibrium concept

The key feature of the competitive equilibrium notion we propose consists in the formulation of the restriction imposed on firms' price conjectures.

Note that *rational* price conjectures are consistent with competitive (indeed Walrasian) markets: the consumers' marginal rate of substitution $MRS^i(c^i(s))$ used to determine the conjectures over the market valuation of debt and equity are taken as given, evaluated at the equilibrium consumption values and unaffected by the firm's choice of k, B, ϕ . In this sense each firm is price taker, is "small" relative to the market, and we can think of each consumer as holding a negligible amount of shares of any given firm.

We claim this equilibrium notion is natural in competitive production economies. Before discussing the properties of equilibria, we argue here that this notion is equivalent to two others adopted in the literature (in different environments).

All markets open at market clearing prices. Consider a specification where markets for all possible 'types' of equity and bonds are open: that is, equity and bonds corresponding to any possible value of k', B', ϕ' are available for trade to consumers at the prices $q(k', B', \phi')$, $p(k', B', \phi')$. It is immediate to see that all such markets - except the one corresponding to the firms' equilibrium choice k', B', ϕ' - clear at zero trades. As a consequence, $q(k', B', \phi')$ and $p(k', B', \phi')$ correspond to the equilibrium prices of equity and bonds of a firm who were to "deviate" from the equilibrium choice and choose k', B', ϕ' instead. In this sense, we can say that rational conjectures impose a consistency condition on the out of equilibrium values

¹⁶Rational conjectures, when markets are incomplete, introduce fundamental non-convexities in the objective problem of the firm. The possibility of specialization, which we discussed in Section 2.5, is just a consequence of this. To simplify notation, however, the above definition and most of the presentation refers to the case of symmetric equilibria, where all firms choose the same production and financial plan.

of the equity and bonds price conjectures, that corresponds to a “refinement” somewhat analogous to subgame perfection.

Prescott and Townsend equilibria. Consider the equilibrium concept adopted by Prescott and Townsend (1984) for exchange economies with asymmetric information. In this concept prices depend both on unobservable as well as observable choices and this is sustained, drawing a parallel with mechanism design formulations of related problems relying on the Revelation Principle, by restricting admissible choices to those which are incentive compatible. In contrast, the equilibrium concept we propose relies on price conjectures that reflect the correct anticipation of unobservable choices. It is however straightforward to show that these two approaches are equivalent. The equilibrium notion proposed by Prescott and Townsend (1984), once extended to the environment under consideration, and hence to production economies and incomplete markets, features markets and prices for any possible value of k, B, ϕ and the presence of the constraint in the firm’s problem (Pb F). In light also of the equivalence result established in the previous paragraph, it is then easy to verify that these Prescott Townsend competitive equilibria are equivalent to competitive equilibria as defined in the previous section.

3.2 Equilibrium properties

The equilibrium notion we propose has several desirable properties: i) existence of an equilibrium is ensured, ii) equilibrium allocations have well-defined welfare properties, and iii) shareholders unanimously support firms’ decisions. We present and discuss these properties in turn.

Existence. As we noted in the previous section, the firms’ choice problem is not convex and to ensure the existence of an equilibrium we have to allow for asymmetric equilibria. The existence proof (in the Appendix) exploits the presence of a continuum of firms of the same type to convexify the firms’ choice problem.¹⁷

Proposition 1 *A competitive equilibrium always exist.*

¹⁷Also, the existence proof requires for simplicity that Φ is a discrete set and a natural regularity condition for the implementability constraints $\phi \in \phi(k, m, B; c(s))$ (spelled out in the Appendix). But existence is also guaranteed when Φ is more generally a compact set if the *first order approach* is satisfied, that is, if the problem whose solution yields the map $\phi(k, m, B; c(s))$ has a unique solution, described by a continuous function.

Efficiency. The appropriate efficiency notion for our economy is constrained: attainable allocations are restricted not only by the limited set of financial assets that are available but also by the presence of agency frictions. More formally, a consumption allocation $c(s)$ is *admissible* if:¹⁸

1. it is *feasible*: there exists a production plan k, m, ϕ of firms such that

$$\begin{aligned} \sum_i c_0^i + k &\leq \sum_i w_0^i \\ \sum_i c_1^i(s) &\leq \sum_i w_1^i(s) + F(k, \phi; \epsilon); \end{aligned} \tag{8}$$

2. it is *attainable with the existing asset structure*: that is, there exists B and, for each consumer's type i , a pair θ^i, b^i such that

$$c_1^i(s) = w_1^i(s) + d^e(k, B, \phi)\theta^i + d^b(k, B, \phi)b^i; \tag{9}$$

3. it is *incentive compatible*: given the observable component of the production plan k , the financing plan B , and the consumption allocation $c(s)$, the unobservable component satisfies

$$\phi \in \phi(k, B; c(s)) \tag{10}$$

We then say that a competitive equilibrium allocation is *constrained Pareto efficient* if we cannot find another admissible allocation which is Pareto improving.

Proposition 2 *Competitive equilibria are constrained Pareto efficient when no agency frictions are present or whenever the incentive compatibility map $\phi(\cdot)$ only depends on the firm's choice variables k, B .*

In an economy with no agency frictions, where ϕ is observable and its choice is unrestricted in Φ , constrained efficiency always holds. With agency frictions, constrained efficiency obtains when the friction is such that ϕ is determined independently of $c(s)$. [examples] Note that a key feature for the specification of the incentive constraint in these examples, and thus also for the efficiency result, is that the manager's trades are observable, so that

¹⁸Again, we restrict notation for simplicity to symmetric allocations.

the manager cannot trade his way out of his compensation package. In other words, it is crucial that the manager's compensation contract is exclusive.¹⁹

On the other hand constrained efficiency may fail when the incentive constraint depends also on variables not directly chosen by the firm, like the consumption allocation $c(s)$, as we showed it happens in the shareholders/bondholders problem considered in Section 2.6.

Unanimity. In both the economies described in Sections ?? and ?? shareholders unanimously agree on the firm's production and financing decisions, that is on the choice of k, B, ϕ which maximizes the firm's market value, defined by rational conjectures (subject, when ϕ is unobservable, to the implementability constraint (??)):

Proposition 3 (Unanimity) *Let k, B, ϕ be the firms' choice at a competitive equilibrium and $c(s)$ be the consumption allocation. Then every agent i holding a positive initial amount θ_0^i of equity of a firm will be made - weakly - worse off by any other possible incentive compatible choice of the firm ($k', B', \phi' \in \phi(k, B; c(s))$).*

The result follows from the fact that, as noticed in Section 3.1, the equilibrium allocation is the same as the one which would obtain if markets for all possible types of equity and bonds were open. Consequently, the unanimity result holds by the same argument as the one used to establish this property for Arrow-Debreu economies.

3.3 A few remarks on the relationship with the literature

The problems found in the literature concerning the specification of the firms' objective function, do not arise for the equilibrium notion we propose. As shown in the previous section, in the set-up typically considered in this literature (that is, with no agency frictions), both unanimity and constrained efficiency hold. The key difference between this paper and this literature lies in the specification of the firms' price conjectures. It is useful then to compare (the *Makowski criterion* for) *rational conjectures* to the two main alternative specifications in the literature, the *Dreze* and the *Grossman-Hart criteria*, in the context of an economy without agency friction.

¹⁹The inefficiency of economies where this assumption is not satisfied have been studied in the literature; see, Arnott and Stiglitz (1993) and, more recently, e.g., Acharya and Bisin (2009) and Bisin, Gottardi, and Rampini (2008).

Starting with the contributions of Dreze (1974), Grossman and Hart (1979) and Duffie and Shafer (1986), a large literature has dealt with the question of what is the appropriate objective function of the firm in economies with incomplete markets (under symmetric information, that is, with no agency frictions). Different objective functions have been proposed and results generally display unappealing theoretical properties, in particular the lack of unanimity of shareholders on the firms' decisions. This literature however seems to have somewhat overlooked an important contributions by Louis Makowski (1983a).²⁰ Indeed, Makowski showed that if firms operate on the basis of rational conjectures, under the condition that agents cannot short-sell equity and under symmetric information, value maximization is unanimously supported by shareholders as the firm's objective.²¹

Dreze (1974) proposes the following criterion for equity price conjectures (a similar condition holds for bond prices):

$$q(k, B, \phi) = \mathbb{E} \sum_i \theta^i MRS^i(c^i(s)) d^e(k, B, \phi), \quad \forall k, B, \phi \quad (\text{Dreze})$$

It requires the conjectured price of equity for any plan k, B, ϕ to equal - pro rata - the marginal valuation of the agents who in equilibrium are shareholders of the firm (that is, the agents who value the most the plan chosen by the firm in equilibrium and hence choose to buy equity). It does not however require that the firm's shareholders are those who value the most any possible plan of the firm. Intuitively, the choice of a plan which maximizes the firm's value with $q(k, B, \phi)$ as in (Dreze) corresponds to a situation in which the firm's shareholders choose the plan which is optimal for them without contemplating the possibility of selling the firm in the market, to allow the buyers of equity to operate the plan they instead prefer. Equivalently, the value of equity for out of equilibrium production and financial plans is determined using the - possibly incorrect - conjecture that the agents who in equilibrium own the equity of a firm remain the firm's shareholders also for any alternative production and financial plan.²²

²⁰For instance, Makowski is not cited in Dreze (1985) nor in the main later contributions to this literature, like DeMarzo (1993), Kelsey and Milne (1996), Dierker, Dierker and Grodal (2002), Bonnisseau and Lachiri (2004), Dreze, Lachiri and Minelli (2007), Carceles-Poveda and Coen-Pirani (2009). When it is cited, as in Duffie and Shafer (1986), it is to a large extent disregarded. Makowski is not even cited in the main surveys of the GEI literature, as Geanakoplos (1990) and Magill and Shafer (1991).

²¹Under the same conditions, Makowski (1983b) shows that competitive equilibria are constrained Pareto optimal.

²²It is then easy to see that any allocation constituting an equilibrium with rational conjectures according

Grossman and Hart (1979) propose an alternative criterion for price conjectures which, when applied to the price of equity, requires:

$$q(k, B, \phi) = \mathbb{E} \sum_i \theta_0^i MRS^i(c^i(s)) d^e(k, B, \phi), \quad \forall k, B, \phi$$

We can interpret this specification as describing a situation where the firm's plan is chosen by the initial shareholders (i.e., those with some predetermined equity holdings at the beginning of date 0) so as to maximize their welfare, again without contemplating the possibility of selling the equity to other consumers who value it more. Equivalently, the value of equity for out of equilibrium production and financial plans is derived using the conjecture that the firm's initial shareholders stay in control of the firm whatever is the plan that is chosen.

In summary, according to the Makowski criterion for rational conjectures each firm evaluates different production and financial plans using possibly different marginal valuations (that is, possibly different pricing kernels, but all still consistent with the consumers' marginal rate of substitution at the equilibrium allocation). This is essential to ensure the unanimity of shareholders' decisions and is a key difference with respect to Dreze (1974) and Grossman and Hart (1979), both of whom rely on the use of a single pricing kernel.²³

With regards to agency frictions and asymmetric information, most of the competitive equilibrium concepts which have been proposed build on the concept proposed by Prescott and Townsend (1984) for exchange economies, therefore exhibiting no traded equity.²⁴ While Prescott and Townsend's approach, rooted in mechanism design, is quite different from ours, which instead relies on the extension of *rational conjectures* to economies with asymmetric information, we show that our equilibrium concept is indeed equivalent to the one of Prescott and Townsend once this is extended to economies with incomplete markets where firms rather

to the criterion is also an equilibrium under the Dreze criterion: all shareholders of a firm have in fact the same valuation for the firm's production and financial plan and their marginal utility for any other possible plan is lower, hence a fortiori the chosen plan maximizes the weighted average of the shareholders' valuations. But the reverse implication is not true, i.e., an equilibrium under the Dreze criterion is not in general an equilibrium under rational conjectures.

²³This feature distinguishes also the equilibrium notion based on the Makowski rationality criterion from the several others proposed in the literature, including those applying elements from the theory of social choice and voting to model the control of shareholders over the firm's decisions; see for instance DeMarzo (1993), Boyarchenko (2004), Cres and Tvede (2005).

²⁴See, e.g., Magill and Quinzii (2002), Prescott and Townsend (2006), and Zame (2007).

than consumers face agency frictions.²⁵ Nonetheless, interesting and important conceptual differences emerge when the analysis is extended from exchange to production economies, since we show there are natural environments where informational asymmetries in firms' decisions give rise to externalities while in consumers' problems they do not.

On a different note, our analysis also highlights an interesting and important difference between the properties of equilibria when agency frictions are faced by consumers, as e.g., in Prescott and Townsend's analysis of exchange economies with asymmetric information, and when instead such frictions are faced by firms. While competitive equilibria are always constrained efficient in the exchange economies considered by Prescott and Townsend, this is not necessarily the case in production economies, as we have shown in Proposition 2. The nature of the equilibrium concept adopted plays no role in this: as we discussed in Section 3.1, our equilibrium concept is equivalent to the one of Prescott and Townsend once this is extended to production economies. Rather, agency frictions and production may naturally interact to generate an externality.²⁶

An important implication of the welfare properties of production economies with agency frictions is that in economies where equilibrium allocations are constrained inefficient, e.g. when the agency friction is between shareholders and bondholders, a Pareto improvement may be achieved with different types of agents owning equity than the ones who do in equilibrium. Since the unanimity result in Proposition 3 always holds, even when equilibrium allocations are not constrained efficient, this misallocation of equity ownership is not a consequence of lack of unanimity, as it might instead be the case in equilibrium concepts adopting the Dreze or the Grossman-Hart criterion. It is rather a consequence of the externality affecting firms' incentive constraints, which may turn out to be more severe when some types of agents are shareholders than when others are.

²⁵We do not discuss economies with adverse selection in this paper. We conjecture that the equilibrium concepts studied by Bisin and Gottardi (2006) have an equivalent reformulation in terms of equilibria with *rational conjectures* in economies with production along similar lines to those considered in the present paper.

²⁶Prescott and Townsend also assume that markets are complete, while we do not. But whether markets are complete or not, and hence whether $MRS^i(c^i(s))$ are equalized or not across i , is not crucial for the welfare result. What is crucial is that the agents' marginal rates of substitution enter the incentive constraint, so that a change in the consumption allocation may relax this constraint.

4 Intermediation and short sales

In this section we introduce markets for derivatives on the firms' financial assets. This is important obviously because derivatives markets exist and we claim their modeling fits naturally into our set-up with interesting implications for financial economics. They are also a natural way to model short sales of the existing assets. Indeed in this section we focus our attention on the case where derivatives are simply given by short and long positions on the firms' equity.²⁷ A short position on a firm's equity is in fact, both conceptually and in the practice of financial markets, different from a simple negative holding of equity: it is a loan contract with a promise to repay an amount equal to the future value of equity. To model short sales it is then natural to introduce financial intermediaries who can issue claims corresponding to short positions on the firm's equity.

We assume that intermediation is subject to frictions, e.g. default or transaction costs. This ensures that the notion of competitive equilibrium is well-defined, even if such frictions are arbitrarily small (and hence short sales are "essentially unlimited").²⁸

Consider an environment where intermediaries bear no cost to issue derivative claims, but face the possibility of default by consumers on the short positions they issue (e.g., on the loans induced by the sale of such positions to consumers), this is the friction on intermediation. Assume for simplicity that i) the default rate on the short positions issued is exogenously given and constant in every state, for all consumers;²⁹ ii) a solvency constraint is imposed on intermediaries' portfolio, to ensure they are never insolvent.

More specifically, an intermediary who is intermediating H units of the derivative on the firm's equity (that is, issuing H long and short positions) is repaid only a fraction

²⁷It should be clear that the analysis could be extended to short sales of the bond as well as other forms of derivatives, at only a notational cost.

²⁸The analysis of equilibria with intermediated short sales is also important from a theoretical standpoint. It is evident from our analysis in the previous sections that the unlimited short sales paradigm adopted by the GEI literature cited in the Introduction, while elegant and convenient, is incompatible with competitive equilibrium modeling in economies with production and incomplete markets. With infinite short sales, e.g., of equity, a *small* firm can in fact have a *large* effect on the economy by choosing a production plan with cash flows which, when traded as equity, change the asset span and hence the admissible trades of all consumers, allocations and equilibrium prices. In this section we show how not only limited but also "essentially unlimited" short sales can be consistently introduced in our competitive economy with production.

²⁹In Appendix A we show how the analysis and results extend to the general case where default rates are endogenously chosen by consumers.

$(1 - \delta) \in (0, 1)$ of the amount due on each short position issued.³⁰ To ensure its own solvency, the intermediary must hold an appropriate portfolio of claims, as a form of collateral, whose yield can cover the shortfalls in its revenue from the intermediation activity due to consumers' defaults. The best hedge against the risk of consumers' default on short positions on equity is clearly equity itself. The intermediary's solvency constraint then requires that it holds an amount γ of equity of the firm to ensure its ability to meet all its future obligations:

$$H \leq H(1 - \delta) + \gamma, \quad (11)$$

To cover the cost of this collateral, intermediaries may charge a different price for long and short positions in the derivative issued. Let q^+ (resp. q^-) be the price at which long (resp. short) positions in the derivative issued by the intermediary are traded, while q is still the price at which equity trades in the market. The intermediary chooses then the amount H issued of long and short positions in the derivative and the amount γ of equity held as a hedge, so as to maximize its total revenue at date 0:

$$\max_{H, \gamma \in R_+^2} [(q^+ - q^-)H - q\gamma] \quad (12)$$

subject to the solvency constraint (11).

A solution to the intermediary's choice problem exists provided

$$q \geq \frac{q^+ - q^-}{\delta}; \quad (13)$$

and is characterized by $\gamma = \delta H$ and $H > 0$ only if $q = \frac{q^+ - q^-}{\delta}$.

Let $h_+^i \in R_+$ denote consumer i 's holdings of long positions in the derivative issued by intermediaries, and $h_-^i \in R_+$ his holdings of short positions. The consumer's choice problem consists in maximizing his expected utility subject to the budget constraints

$$c_0^i = w_0^i + V\theta_0^i - q\theta^i - p b^i - q^+ h_+^i + q^- h_-^i \quad (14)$$

$$c_1^i(s) = w_1^i(s) + R^e(s)(\theta^i + h_+^i - h_-^i) + R^b(s)b^i \quad (15)$$

and $(\theta^i, b^i, h_+^i, h_-^i) \geq 0$.

The asset market clearing conditions are now, for equity

³⁰The analysis in this section holds for all $\delta > 0$, even arbitrarily small (hence even when the friction introduced is of negligible amount).

$$\gamma + \sum_{i \in I} \theta^i = 1,$$

and for the derivative security

$$\sum_{i \in I} h_+^i = \sum_{i \in I} h_-^i = H.$$

Furthermore, the firm's choice problem is unchanged, still given - if we consider for simplicity the case where there is no agency friction, as in Section ?? - by (?). However, the condition specifying the criterion for *rational conjectures* for equity, $q(k, \phi, m, B)$, has to be properly adjusted to reflect the fact that now intermediaries may also demand equity in the market:

$$q(k, \phi, m, B) = \max \left\{ \frac{\max_i \mathbb{E} [MRS^i(c^i(s))R^e(k, \phi, m, B; s)]}{\max_i \mathbb{E} [MRS^i(c^i(s))R^e(k, \phi, m, B; s)] - \min_i \mathbb{E} [MRS^i(c^i(s))R^e(k, \phi, m, B; s)]} \delta \right\} \quad (16)$$

for all k, ϕ, m, B .

The above expression states that the conjecture of a firm over the price of its equity when the firm chooses the plan k, ϕ, m, B equals the maximal valuation, at the margin, *among intermediaries as well as consumers*, of the equity's cash flow corresponding to k, ϕ, m, B . The second term on the right hand side of the above expression is in fact the intermediaries' marginal valuation for equity and can be interpreted as the *value of intermediation*. Since an appropriate amount of equity is needed, to be retained as collateral, in order to issue the corresponding derivative claims, the intermediary's willingness to pay for equity with yield $R^e(k, \phi, m, B; s)$ is determined by the consumers' marginal valuation for the corresponding derivative claims which can be issued³¹. Hence the above specification of the firms' equity price conjectures allows firms to take into account the effects of their decisions on the value of intermediation.

In all other respects, a competitive equilibrium of the economy with intermediation and short sales is defined along similar lines to Section ?. By a similar argument as in Propositions 1, 2 and 3 we can show that a competitive equilibrium of an economy with intermediated short sales exists; moreover, any equilibrium allocation is constrained Pareto efficient and shareholders unanimously support the production and financial decisions of the firms.

³¹More precisely, the first term on the numerator of the second expression in (16) equals the consumers' valuation for long positions in the derivative, the second one their valuation for short positions; dividing by δ yields the profits of intermediation, per unit of equity purchased.

The model of intermediation proposed in this section is admittedly quite stylized. We believe however it allows to capture in a simple way the relationship between the financial claims issued by firms and the intermediation process. The key feature is that the derivatives issues by intermediaries are backed by the claims issued by firms in two ways. First, the yields of these derivatives are pegged to the yield of the claims issued by firms; second, the intermediaries must hold some amount of these claims to back the derivatives issued. Hence part of the demand for the firms' claims now also comes from intermediaries (as such claims enter as some sort of input in the intermediation technology).

It is interesting to compare this optimality result with Theorem 5 in Allen and Gale (1991), where it is shown that the competitive equilibria of an economy where consumers face a finite, exogenous bound \bar{K} on short sales are constrained inefficient. In their set-up, long and short positions trade at the same price, i.e., the bid ask spread is zero. The inefficiency result in Allen and Gale (1991) then follows from the fact that firms maximize a conjecture over their market value which ignores the effect of their decisions on the value of intermediation. In other words, a firm does not take into account the possible gains arising from the demand for short positions in the firm's equity. In contrast in our economy, when a firm makes its production and financial decisions the firm considers the value of its equity not only for the consumers but also for the intermediaries who use equity as an input in the intermediation process. The gains from trade due to intermediation are so taken into account by firms.

It is also useful to contrast our findings with the inefficiency result in Pesendorfer (1995). Example 2 in Pesendorfer (1995) shows that a competitive economy where financial intermediaries can introduce complementary innovations in the market may get stuck at an equilibrium in which no intermediary innovates, even though welfare would be higher if all innovations were traded in the market.³² The source of the inefficiency arising in the environment considered by Pesendorfer (1995) is analogous to the one of the result of Allen and Gale (1991) just discussed: each intermediary is implicitly restricted not to trade with other intermediaries. Equivalently, equilibrium prices for non-traded innovations do not include their effect on the value of intermediation. If instead prices for non-traded innovations were specified so as to equal the maximum between the consumers' and the intermediaries' marginal valuation, as in (16) above, constrained efficiency would obtain at equilibrium.

³²This finding is related to similar ones obtained in competitive equilibrium models with differentiated goods; notably Hart (1980) and Makowski (1980).

Finally, we can provide the following simple characterization of the intermediation levels at equilibrium, which follows from (13):

Proposition 4 (Intermediation) *In the economy with financial intermediation and short sales, at an equilibrium, either (i) $q = (q^+ - q^-)/\delta > q^+$ and intermediation is full (the whole amount of outstanding equity is purchased by intermediaries) or (ii) $q = q^+$ and intermediation is partial (some if not all the amount of outstanding equity is held by consumers).*

At an equilibrium where intermediation is full, equity sells at a premium over the long positions on the derivative claim issued by the intermediary, due to its additional value as input in the intermediation technology. Intermediaries in turn recoup the higher cost of equity through a sufficiently high spread $q^+ - q^-$ between the price of long and short positions on the derivative. When on the contrary intermediation is partial, equity and long positions in the derivative trade at the same price, intermediaries may not be active in equilibrium and the bid ask spread $q^+ - q^-$ is low (in particular, less or equal than δq).³³

5 Conclusions

In this paper we have provided an equilibrium foundation to the study of corporate finance by showing how a consistent definition of competitive equilibria can be provided in environments with production, incomplete financial markets, and agency frictions. We have shown that, once firms are postulated to operate under rational conjectures, along the lines of Makowski (1983a,b), equilibria exist and have natural and appealing properties (in terms, e.g. of welfare and unanimity).

We have considered various classes of economies and examples to illustrate how the equilibrium concept we introduced allows to study simple finance and macroeconomic issues, from the firms' capital structure, to firms' specialization, corporate default, and financial intermediation.³⁴

The next step, which we leave for future work, consists in adapting the equilibrium concept and extending the analysis to dynamic economies, e.g., Bewley economies, as the ones typically considered in macroeconomics and finance.

³³Interestingly, we see from (16) that the same two situations arise for equity price conjectures.

³⁴Acharya and Bisin (2013) extend the analysis of this equilibrium concept to a class of financial intermediation economies with strategic default to capture counterparty risk.

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Appendix

Proof of Proposition 1

We only provide here an outline of the main steps. Since the firms' choice problem is non convex, we allow for the possibility that firms undertake different production and financial plans in equilibrium. By Caratheodory's Theorem, given the finite dimensionality of the sets where these variables lie, it is enough to consider the case where firms make at most a finite number N of different choices k^n, ϕ^n, m^n, B^n . As a consequence, we extend the consumers' budget constraints (??)-(??) to allow for the possibility that they trade N different types of equity and bonds, with prices q^n, p^n and returns $R^{e,n}(s), R^{b,n}(s)$. Since short sales are not allowed, the consumers' budget set is non empty, compact and convex for all $p^n, q^n \gg 0$, all $R^{e,n}(s), R^{b,n}(s) \geq 0$ and all $V^n \geq 0$, for $n = 1, \dots, N$. Under the assumptions made on individual preferences, consumers' net demand (for the consumption good and the different types of bonds and equity) are then well behaved, continuous functions.

Let us turn then our attention to the firms' problem (??). Whenever the first order approach is not satisfied and the map $\phi(k, m, B; c(s))$ is not single-valued and continuous, it is convenient to write the implementability constraint (??) in terms of the inverse map:

$$k, m, B \in \phi^{-1}(\phi; c(s)).$$

We also impose here the following regularity condition, requiring that the above inverse map can be described by a set of functions

$$k, m, B \in \phi^{-1}(\phi; c(s)) \Leftrightarrow G(k, m, B; c(s), \phi) \leq 0, \quad (17)$$

with $G(\cdot)$ assumed to be continuous in $k, m, B, c(s)$ for all $\phi \in \Phi$. Note that this condition is satisfied in natural environments, as for instance in the case of (??) and (??).

Let us partition the set $N \equiv \{1, \dots, N\}$ into equal-sized subsets $N(\phi)$ for each $\phi \in \Phi$. The firms' choice problem can then be rewritten as

$$\begin{aligned} \max_{(k^n, m^n, B^n, \gamma^n)_{n \in N(\phi), \phi \in \Phi}} & \left[\sum_{\phi \in \Phi} \sum_{n \in N(\phi)} \gamma^n (-k^n + \mathbb{E} [\max_i MRS^i(c^i(s)) R^e(k^n, \phi, m^n, B^n; s)] \right. \\ & \left. + \mathbb{E} [\max_i MRS^i(c^i(s)) R^b(k^n, \phi, m^n, B^n; s)] B^n \right] \\ \text{s.t.} & \begin{cases} \gamma \in \Delta^{N-1} \\ G(k^n, m^n, B^n; c(s), \phi) \leq 0 \text{ for all } n \in N(\phi) \text{ and all } \phi \end{cases} \end{aligned} \quad (18)$$

where $\gamma \in \Delta^{N-1}$ can be equivalently interpreted as the fraction of firms choosing each of the N plans, or the probability weights of the lottery over production and financial plans describing the choice of each firm³⁵. In the above expression of the firms' problem we have also used condition M) to substitute for the equity and bond price conjectures and used (17) to rewrite the incentive constraint (??).

The objective function and the constraints of the firms' problem (18) are continuous w.r.t. $(k^n, m^n, B^n, \gamma^n)_{n \in N(\phi), \phi \in \Phi}$ and $c(s)$. Since the sets K, M, \mathcal{B} are compact, the correspondence describing the solution of the firm's problem (18) above is then non empty and upper hemicontinuous, for all $c_0^i \in (0, \max \{\sum_i w_0^i\}]$, $c_1^i(s) \in (0, \max \sum_i w^i(s)]$.

By a standard fixed point argument there exists so a value $\bar{\phi}^n, \bar{k}^n, \bar{m}^n, \bar{B}^n, \bar{p}^n, \bar{q}^n, \bar{\gamma}^n, \bar{R}^{e,n}(s), \bar{R}^{b,n}(s)$ for $n = 1, \dots, N$ and $\bar{c}(s)$ such that: (a) $\bar{k}^n, \bar{m}^n, \bar{B}^n, \bar{\gamma}^n$ for $n = 1, \dots, N$ solve the firms' optimal choice problem (18) when the terms MRS^i appearing in the equity and bond price conjecture maps above are evaluated at $\bar{c}(s)$, and $n \in N(\phi)$ implies $\bar{\phi}^n = \phi$, (b) for each $i = 1, \dots, I$, $\bar{c}^i(s)$ is a solution of the choice problem of type i consumers at prices and returns $\bar{p}^n, \bar{q}^n, \bar{V}^n, \bar{R}^{e,n}(s), \bar{R}^{b,n}(s)$, $n = 1, \dots, N$, satisfying the consistency condition C), (c) the market clearing conditions hold (for each type n of equity and bonds, the supply $\bar{\gamma}^n$ equals consumers' demand).■

Proof of Proposition 2

Suppose $\hat{c}(s)$ is admissible and Pareto dominates the competitive equilibrium allocation $\bar{c}(s)$. By the definition of admissibility a collection $\hat{k}, \hat{m}, \hat{\phi}, \hat{B}$ and $(\hat{\theta}^i, \hat{b}^i)_{i=1}^I$ exists such that $\hat{c}(s)$ satisfies (8), (9) and (10). The equilibrium consumption level $\bar{c}^i(s)$ is the optimal choice of a type i consumer at the equilibrium prices \bar{q}, \bar{p} and returns $\bar{R}^e(s) = R^e(\bar{k}, \bar{\phi}, \bar{m}, \bar{B}; s)$, $\bar{R}^b(s) = R^b(\bar{k}, \bar{\phi}, \bar{m}, \bar{B}; s)$. As argued in Section 3.1, the consumer's choice problem is analogous to one where any possible type of equity and bonds are available for trade, at the prices $q(k, \phi, B, m)$, $p(k, \phi, B, m)$ satisfying the Makowski criterion M) with $\phi \in \phi(k, m, B; \bar{c}(s))$. When the map $\phi(\cdot)$ only depends on k, m, B , we have $\hat{\phi} \in \phi(\hat{k}, \hat{m}, \hat{B})$ and so we get:

$$\hat{c}_0^i + \hat{q}\hat{\theta}^i + \hat{p}\hat{b}^i \geq \bar{c}_0^i + \bar{q}\bar{\theta}^i + \bar{p}\bar{b}^i,$$

where $\hat{q} = q(\hat{k}, \hat{\phi}, \hat{m}, \hat{B})$, $\hat{p} = p(\hat{k}, \hat{\phi}, \hat{m}, \hat{B})$. Or, equivalently,

$$\left[-\hat{k} + \hat{q} + \hat{p}\hat{B} \right] \theta_0^i + \tau^i \geq \left[-\bar{k} + \bar{q} + \bar{p}\bar{B} \right] \theta_0^i, \quad (19)$$

³⁵With the realizations of the lottery observed by consumers when choosing their portfolios.

for $\tau^i \equiv \hat{c}_0^i + \hat{q}\hat{\theta}^i + \hat{p}\hat{b}^i - [-\hat{k} + \hat{q} + \hat{p}\hat{B}] \theta_0^i$. Since (19) holds for all i , strictly for some i , summing over i yields:

$$[-\hat{k} + \hat{q} + \hat{p}\hat{B}] + \sum_i \tau^i > [-\bar{k} + \bar{q} + \bar{p}\bar{B}] \quad (20)$$

The fact that $\bar{k}, \bar{m}, \bar{B}$ solves the firms' optimization problem (??) in turn implies that:

$$-\bar{k} + \bar{q} + \bar{p}\bar{B} \geq -\hat{k} + \hat{q} + \bar{p}\hat{B},$$

which, together with (20), yields:

$$\sum_i \tau^i > 0,$$

or equivalently:

$$\sum_i \hat{c}_0^i + \hat{k} > \sum_i w_0^i,$$

a contradiction to (8) at date 0. ■

Proof of Proposition 3

Note that we can always consider a situation where, in equilibrium, each consumer holds at most a negligible amount of equity of any individual firm and so the effects on a consumer's utility of alternative choices by a firm can then be evaluated using the consumer's marginal utility. Let $c(s)$ be the equilibrium consumption allocation. For any possible choice k', ϕ', m', B' by a firm, with $\phi' \in \phi(k', m', B'; c(s))$, the (marginal) utility of a type j consumer if he holds the firm's equity and debt is

$$-k' - W(k', \phi', m', B') + \mathbb{E} [MRS^j(c^j(s))R^e(k', \phi', m', B'; s)] + \mathbb{E} [MRS^j(c^j(s))R^b(k', \phi', m', B'; s)] B',$$

But this is always lower or equal than the agent's utility if instead he sells the firm's equity and bonds at the market price, evaluated on the basis of price conjectures satisfying M),

$$\begin{aligned} & -k' - W(k', \phi', m', B') + \max_i \mathbb{E} [MRS^i(c^i(s))R^e(k', \phi', m', B'; s)] \\ & + \max_i \mathbb{E} [MRS^i(c^i(s))R^b(k', \phi', m', B'; s)] B', \end{aligned}$$

and the latter is in turn lower than the corresponding expression if the firm adopts the equilibrium choice k, ϕ, m, B , since this choice solves problem (??). ■

Further details of the proof of Proposition ??

When (??) holds as equality only for consumer $i = 2$ we have $c_1^2(s) = w_1^2(s) + a_1(s)k^\alpha > c_1^1(s) = w_1^1(s)$, $c_0^2 = w_0 + V0.5 - q < c_0^1 = w_0 + V0.5$. For simplicity we assume here that the following symmetry condition also holds: $\mathbb{E} [MRS(c^2(s))a_1(s)k^\alpha] = \mathbb{E} [MRS(\hat{c}^1(s))a_2(s)k^\alpha]$, for $\hat{c}_0^1 = w_0 + V0.5 - q$, $\hat{c}_1^1(s) = w_1^1(s) + a_2(s)k^\alpha$ for all $k, q, V > 0$.

For $\phi = 0$ to be an optimal choice for the firms, we must have in this case:

$$q = \mathbb{E} [MRS(c^2(s))a_1(s)k^\alpha] \geq \mathbb{E} [MRS(c^1(s))a_2(s)k^\alpha]$$

which contradicts the assumed symmetry condition, since

$$\mathbb{E} [MRS(c^1(s))a_2(s)k^\alpha] > \mathbb{E} [MRS(\hat{c}^1(s))a_2(s)k^\alpha].$$

Consider next the case where $w_1^1(s) + a_2(s)k^\alpha$ and $w_1^2(s) + a_2(s)k^\alpha$ varies comonotonically with $a_1(s)$ for all $k \in K$ (a slightly stronger condition than the comononicity of $w_1^1(s)$, $w_1^2(s)$ and $a_1(s)$). In this case we have

$$\mathbb{E} [MRS(c(s))a_2(s)k^\alpha] > \mathbb{E} [MRS(c(s))a_1(s)k^\alpha]$$

for all $k \in K$, c_0 and $c_1(s) = w_1^i(s) + \theta a_2(s)k^\alpha$, $i = 1, 2$, $\theta \in [0, 1]$, since $Cov(MRS(c(s)), a_2(s)) > 0 > Cov(MRS(c(s)), a_1(s))$. Hence in equilibrium both consumers' types are only willing to buy equity of firms with full loading on factor $a_2(s)$.

Details on the Dierker, Dierker, and Grodal (2002) example

There are two types of consumers, with type 2 having twice the mass of type 1, and (non expected utility) preferences, respectively, $u^1(c_0^1, c_1^1(s_1), c_1^1(s_2)) = c_1^1(s_1) / \left(1 - (c_0^1)^{\frac{9}{10}}\right)^{\frac{10}{9}}$ and $u^2(c_0^2, c_1^2(s_1), c_1^2(s_2)) = c_0^2 + (c_1^2(s_2))^{1/2}$, endowments $w_0^1 = .95$, $w_0^2 = 1$ and $w_1^1(s) = w_1^2(s) = 0$ for all $s \in \mathcal{S}$.

In this economy Dierker, Dierker and Grodal (2002) find a unique, symmetric Dreze equilibrium where all firms choose the same value of k and $\phi \approx 0.7^{36}$ and this equilibrium is constrained inefficient. We show next that a symmetric competitive equilibrium, according to

³⁶The notion of Dreze criterion used by Dierker, Dierker, and Grodal to specify price conjectures differs from the Makowski criterion M) in two main respects: i) only the MRS of the consumers who in equilibrium are shareholders of the firms are considered to evaluate alternative production plans, and ii) these MRS are not constant but vary to take into account the effect of each plan on the agents' consumption.

our definition in Section ??, does not exist. Given the agents' endowments and preferences, both types of consumers buy equity in equilibrium. It is then easy to see that the firms' optimality condition with respect to ϕ can never hold for an interior value of ϕ nor for a corner solution.³⁷ On the other hand, an asymmetric equilibrium exists, where a fraction $1/3$ of the firms choose $\phi^1 = 0.99$ and $k^1 = 0.3513$ and the remaining fraction chooses $\phi^2 = 2/3$ and $k^2 = 0.1667$, type 1 consumers hold only equity of the firms choosing ϕ^1, k^1 and type 2 consumers only equity of the other firms. At this allocation, we have $\frac{\partial u^1/\partial c_1^1(s_1)}{\partial u^1/\partial c_0^1} = 1.0101$, $\frac{\partial u^2/\partial c_1^2(s_2)}{\partial u^2/\partial c_0^2} = 3$. Also, the marginal valuation of type 1 agents for the equity of firms choosing ϕ^2, k^2 is 0.1122, thus smaller than the market value of these firms' equity, equal to 0.1667, while the marginal valuation of type 2 agents for the equity of the firms choosing ϕ^1, k^1 is 0.0105, smaller than the market value of these firms' equity, equal to 0.3513. Therefore, at these values the firms' optimality conditions are satisfied. It can then be easily verified that this constitutes a competitive equilibrium according to our definition and that the equilibrium allocation is constrained optimal.

A Parametric Example

Consumers have identical preferences described by $\mathbb{E}u(c_0, c_1(s)) = u(c_0) + \mathbb{E}u(c_1(s))$, with $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, for $\gamma = 2$. The state space is $\mathcal{S} = \{1, 2, 3\}$ with $\pi(1) = \pi(2) = \pi(3) = \frac{1}{3}$. The production technology is as in (??), with $\alpha = .75$ and productivity shocks $a_1(s)$ and $a_2(s)$ taking values, respectively, $\{1, 2, 3\}$ and $\{1.1, 2, 2.9\}$. The second period endowments of type 1 and type 2 agents take values, respectively, $\{1, 2, 3\}$ and $\{1.1, 2, 2.9\}$, while in the first period they are endowed with $w_0^i = w_1^i(2)$, $i = 1, 2$, units of the good and the same amount $\theta_0 = .5$ units of equity. Also, the utility cost of different choices of ϕ is $v^i(1) = -.006$ and $v^i(0) = 0$, for all i .

The equilibrium values with and without the agency friction are reported in the following table.

In order to implement the same choice $\phi = 0$ the firm modifies its production and financial decisions together with the portfolio of the agent selected as manager (in particular, the manager's compensation exhibits a higher amount of equity, (.6456), a lower one of debt (0)

³⁷Consider for instance $\phi = 0.99$. To have an equilibrium at this value the marginal valuation of equity for both consumers must be the same at $\phi = 0.99$, and higher than at any other values of ϕ , but this second property clearly cannot hold for type 2 consumers.

	Without agency friction	With agency frictions
ϕ	0	0
k	.4888	.4896
B	[.1828,.6431]	.2160
θ^1	.3877	.3544
b^1	[.1828,.3613]	.2160
q	[.5108,.1559]	.4870
p	.7712	.7689
$-k + q + pB - W$.1629	.1633
U^1	-1.0372	-1.0371
U^2	-1.0217	-1.0219

Table 1: Equilibrium values with and without moral hazard.

and also a lower consumption at date 0).

Appendix A: Additional material

Characterization of the firms' optimal capital structure conditions

Let I^e (resp. I^d) denote the set of shareholders (resp. bondholders) of a firm and consider for simplicity the case where capital is the only input, that is the technology is given by $f(k, s)$.

Proposition A. 1 *If the optimal production and financing decisions of a firm are obtained³⁸ at a level B such that bonds are risk free, that is, $f(k; s) \geq B$ with probability 1, then all equity holders are also bond holders (while the reverse may not be true: $I^e \subseteq I^d$):*

$$\max_{i \in I^e} \mathbb{E} MRS^i(c^i(s)) = \min_{i \in I^e} \mathbb{E} MRS^i(c^i(s)) = p = \max_i \mathbb{E} MRS^i(c^i(s)) \quad (21)$$

and

$$\max_{i \in I^e} \mathbb{E} [MRS^i(c^i(s)) f_k(s)] = \min_{i \in I^e} \mathbb{E} [MRS^i(c^i(s)) f_k(s)] = 1; \quad (22)$$

In the situation described above all shareholders value equally the effect on the payoff of equity of an infinitesimal increase in the investment level k , and such value is always equal to the marginal cost of the investment.

Proof of Proposition A. 1 *Note first that*

$$q(k, B + dB) = \max_i \mathbb{E} MRS^i(c^i(s)) [f(k; s) - B - dB].$$

Since for all $i \notin I^e$, $\mathbb{E} MRS^i(c^i(s)) [f(k; s) - B] < q(k, B)$, the max in the above expression is attained for some $i \in I^e$ and hence

$$q(k, B + dB) = q(k, B) + \max_{i \in I^e} \mathbb{E} MRS^i(c^i(s)) [-dB].$$

The right and left derivative of $q(k, B)$ with respect to B are then given by:

$$\frac{\partial q}{\partial B_+} = - \min_{i \in I^e} \mathbb{E} MRS^i(c^i(s)); \quad \frac{\partial q}{\partial B_-} = - \max_{i \in I^e} \mathbb{E} MRS^i(c^i(s)) \quad (23)$$

and may differ. Similarly the derivatives with respect to k are:

$$\frac{\partial q}{\partial k_+} = \max_{i \in I^e} \mathbb{E} [MRS^i(c^i(s)) f_k(s)]; \quad \frac{\partial q}{\partial k_-} = \min_{i \in I^e} \mathbb{E} [MRS^i(c^i(s)) f_k(s)] \quad (24)$$

³⁸We focus here on the conditions concerning the investment level k and capital structure B , ignoring those regarding ϕ , which are straightforward.

where f_k denotes the derivative of f with respect to k .

The first order conditions when $f(k, \phi, m; s) \geq B$ with probability 1 are:

$$\begin{aligned} \frac{\partial V}{\partial B_+} &= \frac{\partial q}{\partial B_+} + p \leq 0, & \frac{\partial V}{\partial k_+} &= \frac{\partial q}{\partial k_+} - 1 \leq 0, \\ \frac{\partial V}{\partial B_-} &= \frac{\partial q}{\partial B_-} + p \geq 0, & \frac{\partial V}{\partial k_-} &= \frac{\partial q}{\partial k_-} - 1 \geq 0; \end{aligned} \quad (25)$$

Since (23) implies that $\frac{\partial q}{\partial B_+} \geq \frac{\partial q}{\partial B_-}$, the above conditions (with respect to B) are equivalent to:

$$\frac{\partial V}{\partial B_+} = \frac{\partial q}{\partial B_+} + p = \frac{\partial V}{\partial B_-} = \frac{\partial q}{\partial B_-} + p = 0,$$

that is:

$$\max_{i \in I^e} \mathbb{E} MRS^i(c^i(s)) = \min_{i \in I^e} \mathbb{E} MRS^i(c^i(s)) = p = \max_i \mathbb{E} MRS^i(c^i(s))$$

or (21) holds. Similarly, from (24) we see that $\frac{\partial q}{\partial k_+} \geq \frac{\partial q}{\partial k_-}$, the above conditions (with respect to k) are equivalent to:

$$\frac{\partial q}{\partial k_+} - 1 = \frac{\partial q}{\partial k_-} - 1 = 0,$$

that is,

$$\max_{i \in I^e} \mathbb{E} [MRS^i(c^i(s)) f_k(s)] = \min_{i \in I^e} \mathbb{E} [MRS^i(c^i(s)) f_k(s)] = 1$$

or (22) holds, thus completing the proof of the proposition. ■

We study next the case where firms can default on their debt obligations, hence corporate debt is risky. Before stating the conditions for an optimum of the firms' decision problem in the presence of risky debt, it is useful to introduce some further notation. Given a face value of debt equal to B , let S^{nd} denote the collection of states in $t = 1$ for which $f(k; s) \geq B$ and by \underline{s}^{nd} the lowest state in S^{nd} , that is the state with the lowest realization of the technology shock for which the firm does not default. Conversely, denote S^d the collection of states in $t = 1$ for which $f(k; s) < B$, i.e. the firm (partially) defaults on its debt.

Proposition A. 2 *If the optimal production and financing decisions of a firm are obtained at a level B such that bonds are risk free, the optimal investment and debt levels obtain either*

at an interior solution, where $f(k; \underline{s}^{nd}) > B$, with:

$$\begin{aligned}
p &= \min_{i \in I^d} \mathbb{E} \left(MRS^i(c^i(s)) \left[\frac{f(k; s)}{B} \right] \mid s \in S^d \right) \Pr\{s \in S^d\} + \\
&\min_{i \in I^e} \mathbb{E}(MRS^i(c^i(s)) \mid s \in S^{nd}) \Pr\{s \in S^{nd}\} = \\
&= \max_{i \in I^d} \mathbb{E} \left(MRS^i(c^i(s)) \left[\frac{f(k; s)}{B} \right] \mid s \in S^d \right) \Pr\{s \in S^d\} + \\
&\max_{i \in I^e} \mathbb{E}(MRS^i(c^i(s)) \mid s \in S^{nd}) \Pr\{s \in S^{nd}\}.
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
1 &= \max_{i \in I^e} \mathbb{E} \{ MRS^i(c^i(s)) f_k(k, s) \mid s \in S^{nd} \} \Pr\{s \in S^{nd}\} + \\
&\max_{i \in I^d} \mathbb{E}(MRS^i(c^i(s)) f_k(k; s) \mid s \in S^d) \Pr\{s \in S^d\} \\
&= \min_{i \in I^e} \mathbb{E} \{ MRS^i(c^i(s_1)) f_k(k, s) \mid s \in S^{nd} \} \Pr\{s \in S^{nd}\} + \\
&\min_{i \in I^d} \mathbb{E}(MRS^i(c^i(s)) f_k(k; s) \mid s \in S^d) \Pr\{s \in S^d\}
\end{aligned} \tag{27}$$

or at a corner solution, $f(k; \underline{s}^{nd}) = B$.

Proof of Proposition A. 2 We first proceed to characterize the conditions for corner solutions.

Claim 1 The conditions for an optimum at a corner, $f(k; \underline{s}^{nd}) = B$, are:

$$\begin{aligned}
&\min_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) \mid s_1 \in S^{nd'} \} \Pr\{s_1 \in S^{nd'}\} + \\
&\min_{i \in I^d} E_{s_0} \left(MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \mid s_1 \in S^{d'} \right) \Pr\{s_1 \in S^{d'}\} \geq p \geq \\
&\geq \max_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} + \\
&\max_{i \in I^d} E_{s_0} \left(MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \mid s_1 \in S^d \right) \Pr\{s_1 \in S^d\} \\
&\min_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) f_k(k, s_1) \mid s_1 \in S^{nd'} \} \Pr\{s_1 \in S^{nd'}\} + \\
&\min_{i \in I^d} E_{s_0} \left(MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \mid s_1 \in S^{d'} \right) \Pr\{s_1 \in S^{d'}\} \geq 1 \geq \\
&\geq \max_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) f_k(k, s_1) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} + \\
&\max_{i \in I^d} E_{s_0} \left(MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \mid s_1 \in S^d \right) \Pr\{s_1 \in S^d\}
\end{aligned} \tag{28}$$

$$\begin{aligned}
&\min_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) f_k(k, s_1) \mid s_1 \in S^{nd'} \} \Pr\{s_1 \in S^{nd'}\} + \\
&\min_{i \in I^d} E_{s_0} \left(MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \mid s_1 \in S^{d'} \right) \Pr\{s_1 \in S^{d'}\} \geq 1 \geq \\
&\geq \max_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) f_k(k, s_1) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} + \\
&\max_{i \in I^d} E_{s_0} \left(MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \mid s_1 \in S^d \right) \Pr\{s_1 \in S^d\}
\end{aligned} \tag{29}$$

$$\begin{aligned}
& 1 - \max_{i \in I^e} E_{s_0} \{ MRS^i(s_1) f_k(k, s_1) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} - \tag{30} \\
& \max_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} = \\
& \left[- \min_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} \right. \\
& \left. - \min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} + p \right] f_k(\underline{s}_1^{nd}) = \\
& \left[- \max_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} \right. \\
& \left. - \max_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} + p \right] f_k(\underline{s}_1^{nd}) = \\
& 1 - \min_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) f_k(k, s_1) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} \\
& - \min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \mid s_1 \in S^d) \Pr\{s_1 \in S^d\}
\end{aligned}$$

Proof of Claim 1 Note first that, in this case, $f(k; \underline{s}_1^{nd}) = B$. Denote by $S_1^{nd'} \subset S_1^{nd}$ the collection of states in $t = 1$ for which the firm does not default, after marginal deviations $dB > 0$ and/or $dk < 0$ (and similarly $S^{d'} \supset S^d$). Evidently, for marginal deviations $dB > 0$ and/or $dk < 0$ the collection of such states is still given by S_1^{nd} .

The partials of the price maps wrt to B are³⁹

$$\begin{aligned}
\frac{\partial q}{\partial B_+} &= - \min_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) \mid s_1 \in S^{nd'} \} \Pr\{s_1 \in S^{nd'}\} \\
\frac{\partial q}{\partial B_-} &= - \max_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial p}{\partial B_+} &= - \min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B^2} \right] \mid s_1 \in S^{d'}) \Pr\{s_1 \in S^{d'}\} \\
\frac{\partial p}{\partial B_-} &= - \max_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B^2} \right] \mid s_1 \in S^d) \Pr\{s_1 \in S^d\}
\end{aligned}$$

Analogously, the partials wrt to k are⁴⁰

$$\begin{aligned}
\frac{\partial q}{\partial k_+} &= \max_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) f_k(k, s_1) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} \\
\frac{\partial q}{\partial k_-} &= \min_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) f_k(k, s_1) \mid s_1 \in S^{nd'} \} \Pr\{s_1 \in S^{nd'}\}
\end{aligned}$$

³⁹Obviously, if S^{nd} is a singleton, the right derivative is equal to 0.

⁴⁰Obviously, if $S^{nd} = \{s_1\}$ - is a singleton - the left derivative is equal to 0.

and

$$\begin{aligned}\frac{\partial p}{\partial k_+} &= \max_{i \in I^d} E_{s_0}(MRS^i(c^i(s_1))) \left[\frac{f_k(k; s_1)}{B} \right] \Big|_{s_1 \in S^d} \Pr\{s_1 \in S^d\} \\ \frac{\partial p}{\partial k_-} &= \min_{i \in I^d} E_{s_0}(MRS^i(c^i(s_1))) \left[\frac{f_k(k; s_1)}{B} \right] \Big|_{s_1 \in S^{d'}} \Pr\{s_1 \in S^{d'}\}\end{aligned}$$

So, if $f(k; \underline{s}_1^{nd}) = B$, the FOCs wrt B are:

$$\begin{aligned}\frac{\partial V}{\partial B_+} &= \frac{\partial q}{\partial B_+} + \left(\frac{\partial p}{\partial B_+} B + p \right) = \\ &- \min_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) \Big|_{s_1 \in S^{nd}} \} \Pr\{s_1 \in S^{nd}\} + \\ &\left(- \min_{i \in I^d} E_{s_0}(MRS^i(c^i(s_1))) \left[\frac{f(k; s_1)}{B^2} \right] \Big|_{s_1 \in S^{d'}} \Pr\{s_1 \in S^{d'}\} B + p \right) \leq 0 \\ \frac{\partial V}{\partial B_-} &= \frac{\partial q}{\partial B_-} + \left(\frac{\partial p}{\partial B_-} B + p \right) = \\ &- \max_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) \Big|_{s_1 \in S^{nd}} \} \Pr\{s_1 \in S^{nd}\} + \\ &\left(- \max_{i \in I^d} E_{s_0}(MRS^i(c^i(s_1))) \left[\frac{f(k; s_1)}{B^2} \right] \Big|_{s_1 \in S^d} \Pr\{s_1 \in S^d\} B + p \right) \geq 0\end{aligned}$$

which implies (28). Finally, the FOCs wrt k are:

$$\begin{aligned}\frac{\partial V}{\partial k_+} &= -1 + \frac{\partial q}{\partial k_+} + \left(\frac{\partial p}{\partial k_+} B \right) = \\ &- 1 + \max_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) f_k(k, s_1) \Big|_{s_1 \in S^{nd}} \} \Pr\{s_1 \in S^{nd}\} + \\ &\left(\max_{i \in I^d} E_{s_0}(MRS^i(c^i(s_1))) \left[\frac{f_k(k; s_1)}{B} \right] \Big|_{s_1 \in S^d} \Pr\{s_1 \in S^d\} B \right) \leq 0 \\ \frac{\partial V}{\partial k_-} &= -1 + \frac{\partial q}{\partial k_-} + \left(\frac{\partial p}{\partial k_-} B \right) = \\ &- 1 + \min_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) f_k(k, s_1) \Big|_{s_1 \in S^{nd'}} \} \Pr\{s_1 \in S^{nd'}\} + \\ &\left(\min_{i \in I^d} E_{s_0}(MRS^i(c^i(s_1))) \left[\frac{f_k(k; s_1)}{B} \right] \Big|_{s_1 \in S^{d'}} \Pr\{s_1 \in S^{d'}\} B \right) \geq 0\end{aligned}$$

which implies (29). Since now expectations in the terms on the two sides of the inequality are taken over different sets, such condition is a little harder to interpret. In particular we can no longer say that all equity holders have the same valuation for the marginal productivity of capital in the no default states. Rather the condition imposes some relationship between the difference among equity holders and bond holders' valuation for the marginal productivity of capital in the two situations (S^d and $S^{d'}$).

We also have to check in this case the optimality of k, B wrt joint deviations of B and k . As before, without loss of generality, we can restrict our attention to changes of B and k such that $f(k; \underline{s}_1^{nd}) = B$ keeps holding (the set of states for which default occurs does not change).

$$\frac{\partial V}{\partial B_+} dB + \frac{\partial V}{\partial k_+} dk = \left[\frac{\partial q}{\partial B_+} + \left(\frac{\partial p}{\partial B_+} B + p \right) \right] dB + \left[-1 + \frac{\partial q}{\partial k_+} + \left(\frac{\partial p}{\partial k_+} B \right) \right] dk \leq 0,$$

for $dB = f_k(\underline{s}_1^{nd})dk > 0$; also,

$$\frac{\partial V}{\partial B_-} dB + \frac{\partial V}{\partial k_-} dk = \left[\frac{\partial q}{\partial B_-} + \left(\frac{\partial p}{\partial B_-} B + p \right) \right] dB + \left[-1 + \frac{\partial q}{\partial k_-} + \left(\frac{\partial p}{\partial k_-} B \right) \right] dk \geq 0$$

for $dB = f_k(\underline{s}_1^{nd})dk < 0$. Substituting the expressions for the partials obtained above, we get

$$\begin{aligned} & \left[-\min_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} \right. \\ & \left. - \min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} + p \right] f_k(\underline{s}_1^{nd}) \\ & - 1 + \max_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) f_k(k, s_1) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} + \\ & \max_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} \leq 0 \end{aligned}$$

or

$$\begin{aligned} & 1 - \max_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) f_k(k, s_1) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} - \tag{31} \\ & \max_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} \geq \\ & \left[-\min_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} \right. \\ & \left. - \min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} + \right. \\ & \left. \max_i \left\{ E_{s_0} (MRS^i(c^i(s_1)) \mid s_1 \in S^{nd}) \Pr\{s_1 \in S^{nd}\} + \right. \right. \\ & \left. \left. E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} \right\} \right] f_k(\underline{s}_1^{nd}) \end{aligned}$$

where the term on the lhs is nonnegative because of (29) and the one on the rhs is also nonnegative by construction. Analogously, substituting the expressions for the partial derivatives

into the FOC for $dB = f_k(\underline{s}_1^{nd})dk < 0$ yields:

$$\begin{aligned} & \left[-\max_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} \right. \\ & \left. - \max_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} + p \right] f_k(\underline{s}_1^{nd}) + \\ & -1 + \min_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) f_k(k, s_1) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} + \\ & \min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} \geq 0 \end{aligned}$$

or

$$\begin{aligned} & \left[-\max_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} \right. \\ & \left. - \max_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} + p \right] f_k(\underline{s}_1^{nd}) \\ & \geq 1 - \min_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) f_k(k, s_1) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} - \\ & \min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} \end{aligned} \quad (32)$$

where the term on the lhs is nonnegative because of (28) and the one on the rhs is also nonnegative as it immediately follows from (29). Putting (31) and (32) together,

$$\begin{aligned} & 1 - \max_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) f_k(k, s_1) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} \\ & - \max_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} \geq \\ & \left[-\min_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} \right. \\ & \left. - \min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} + p \right] f_k(\underline{s}_1^{nd}) \geq \\ & \left[-\max_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} - \right. \\ & \left. \max_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} + p \right] f_k(\underline{s}_1^{nd}) \geq \\ & 1 - \min_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) f_k(k, s_1) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} - \\ & \min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} \end{aligned}$$

Since

$$\begin{aligned}
& - \min_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} \\
& - \min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} \geq \\
& - \max_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} \\
& - \max_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \mid s_1 \in S^d) \Pr\{s_1 \in S^d\}
\end{aligned}$$

and

$$\begin{aligned}
& - \min_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) f_k(k, s_1) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} \\
& - \min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} \geq \\
& - \max_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) f_k(k, s_1) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} \\
& - \max_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \mid s_1 \in S^d) \Pr\{s_1 \in S^d\}
\end{aligned}$$

it must be that (30) holds, where recall that

$$\begin{aligned}
p = \max_i \left\{ E_{s_0} (MRS^i(c^i(s_1)) \mid s_1 \in S^{nd}) \Pr\{s_1 \in S^{nd}\} + \right. \\
\left. E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} \right\}
\end{aligned}$$

This implies

$$\begin{aligned}
& \min_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} = \\
& \max_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} \\
& \min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} = \\
& \max_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \mid s_1 \in S^d) \Pr\{s_1 \in S^d\}
\end{aligned}$$

and

$$\begin{aligned}
& \min_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) f_k(k, s_1) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} = \\
& \max_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) f_k(k, s_1) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} \\
& \min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} \\
= & \max_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \mid s_1 \in S^d) \Pr\{s_1 \in S^d\}
\end{aligned}$$

Note that conditions (28), (29) and (30) can be alternatively stated as:

$$\begin{aligned}
& \min_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) \mid s_1 \in S^{nd'} \} \Pr\{s_1 \in S^{nd'}\} + \\
& \min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \mid s_1 \in S^{d'}) \Pr\{s_1 \in S^{d'}\} \geq p \geq \\
\geq & \max_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) \mid s_1 \in S^{nd'} \} \Pr\{s_1 \in S^{nd'}\} + \\
& \max_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \mid s_1 \in S^d) \Pr\{s_1 \in S^d\}.
\end{aligned}$$

$$\begin{aligned}
& \min_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) f_k(k, s_1) \mid s_1 \in S^{nd'} \} \Pr\{s_1 \in S^{nd'}\} + \\
& \min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \mid s_1 \in S^{d'}) \Pr\{s_1 \in S^{d'}\} \geq 1 \geq \\
\geq & \max_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) f_k(k, s_1) \mid s_1 \in S^{nd'} \} \Pr\{s_1 \in S^{nd'}\} + \\
& \max_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \mid s_1 \in S^d) \Pr\{s_1 \in S^d\}
\end{aligned}$$

and

$$\begin{aligned}
& \min_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} = \\
& \max_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} \\
& \min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} = \\
& \max_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \mid s_1 \in S^d) \Pr\{s_1 \in S^d\}.
\end{aligned}$$

$$\begin{aligned}
& \min_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) f_k(k, s_1) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} = \\
& \max_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) f_k(k, s_1) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} \\
& + \min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} = \\
& \max_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \mid s_1 \in S^d) \Pr\{s_1 \in S^d\}
\end{aligned}$$

This completes the proof of the claim. ■

We are now ready to complete the proof of the proposition. The equity price map in the presence of risky debt is given by

$$q(k, B) = \max_i E_{s_0} \{ MRS^i(c^i(s_1)) [f(k; s_1) - B] \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\}$$

The debt price map is

$$\begin{aligned}
p(k, B) = & \max_i \{ E_{s_0} (MRS^i(c^i(s_1)) \mid s_1 \in S^{nd}) \Pr\{s_1 \in S^{nd}\} + \\
& + E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} \}
\end{aligned}$$

The statement only refers to the interior case: $f(k; \underline{s}_1) > B$. Here, the partials of the price maps with respect to B are

$$\begin{aligned}
\frac{\partial q}{\partial B_+} &= - \min_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} \\
\frac{\partial q}{\partial B_-} &= - \max_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial p}{\partial B_+} &= - \min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B^2} \right] \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} \\
\frac{\partial p}{\partial B_-} &= - \max_{i \in I^d} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B^2} \right] \mid s_1 \in S^d) \Pr\{s_1 \in S^d\}
\end{aligned}$$

Analogously, the partials with respect to k are

$$\begin{aligned}
\frac{\partial q}{\partial k_+} &= \max_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) f_k(k, s_1) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\} \\
\frac{\partial q}{\partial k_-} &= \min_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) f_k(k, s_1) \mid s_1 \in S^{nd} \} \Pr\{s_1 \in S^{nd}\}
\end{aligned}$$

and

$$\begin{aligned}\frac{\partial p}{\partial k_+} &= \max_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f_k(k; s_1)}{B} \right] \Big|_{s_1 \in S^d}) \Pr\{s_1 \in S^d\} \\ \frac{\partial p}{\partial k_-} &= \min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f_k(k; s_1)}{B} \right] \Big|_{s_1 \in S^d}) \Pr\{s_1 \in S^d\}\end{aligned}$$

So, if $f(k; \underline{s}_1^{nd}) > B$, the FOCs with respect to B are:

$$\begin{aligned}\frac{\partial V}{\partial B_+} &= \frac{\partial q}{\partial B_+} + \left(\frac{\partial p}{\partial B_+} B + p \right) = \\ &- \min_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) \Big|_{s_1 \in S^{nd}} \} \Pr\{s_1 \in S^{nd}\} \\ &+ \left(- \min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \Big|_{s_1 \in S^d}) \Pr\{s_1 \in S^d\} + p \right) \leq 0 \\ \frac{\partial V}{\partial B_-} &= \frac{\partial q}{\partial B_-} + \left(\frac{\partial p}{\partial B_-} B + p \right) = \\ &- \max_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) \Big|_{s_1 \in S^{nd}} \} \Pr\{s_1 \in S^{nd}\} \\ &+ \left(- \max_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \Big|_{s_1 \in S^d}) \Pr\{s_1 \in S^d\} + p \right) \geq 0\end{aligned}$$

which implies

$$\begin{aligned}p &= \max_i E_{s_0} (MRS^i(c^i(s_1)) \Big|_{s_1 \in S^{nd}}) \Pr\{s_1 \in S^{nd}\} + \\ &E_{s_0} \left(MRS^i(c^i(s_1)) \left[\frac{f(k; s_1)}{B} \right] \Big|_{s_1 \in S^d} \right) \Pr\{s_1 \in S^d\}\end{aligned}$$

and (26). On the other hand, the FOCs with respect to k give:

$$\begin{aligned}\frac{\partial V}{\partial k_+} &= -1 + \frac{\partial q}{\partial k_+} + \left(\frac{\partial p}{\partial k_+} B \right) = \\ &= -1 + \max_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) f_k(k, s_1) \Big|_{s_1 \in S^{nd}} \} \Pr\{s_1 \in S^{nd}\} + \\ &\max_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f_k(k; s_1)}{B} \right] \Big|_{s_1 \in S^d}) B \Pr\{s_1 \in S^d\} \leq 0 \\ \frac{\partial V}{\partial k_-} &= -1 + \frac{\partial q}{\partial k_-} + \left(\frac{\partial p}{\partial k_-} B \right) = \\ &= -1 + \min_{i \in I^e} E_{s_0} \{ MRS^i(c^i(s_1)) f_k(k, s_1) \Big|_{s_1 \in S^{nd}} \} \Pr\{s_1 \in S^{nd}\} + \\ &\min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[\frac{f_k(k; s_1)}{B} \right] \Big|_{s_1 \in S^d}) B \Pr\{s_1 \in S^d\} \geq 0\end{aligned}$$

which implies (27). This completes the proof of Proposition 2. ■

Equilibria with Short sales when intermediation costs are negligible

In Section 4 we established the existence of an equilibrium with intermediated short sales for all levels $\delta > 0$ of the intermediation cost (capturing the default rate on short positions). It is then of interest to investigate the properties of these equilibria as we let δ go to 0. Clearly the spread $\max_i \mathbb{E} [MRS^i(c^i(s))R^e(k, \phi, m, B; s)] - \min_i \mathbb{E} [MRS^i(c^i(s))R^e(k, \phi, m, B; s)]$ must go to zero, since $q(k, \phi, m, B)$ is bounded above for all k, ϕ, m, B and all $\delta > 0$, total resources being finite. We conjecture therefore that the limit of the competitive equilibria with short sales as $\delta \rightarrow 0$ exists, as all variables lie in a compact set.

The previous observation also implies that the marginal valuation for all possible production and financial plans is equalized across all consumers, as in an environment where unlimited short sales are allowed and markets are complete (or a spanning property holds for all admissible production and financial plans of firms). In the limit as $\delta \rightarrow 0$ not only all possible markets, corresponding to all possible choices k, ϕ, m, B , are open, as in the case without short sales, but a larger set of markets are open and active, to ensure the equalization of agents' marginal rates of substitution.

Short sales with endogenous default

We extend here the analysis of Section 4 by examining the case where the consumers' default rate, rather than being exogenous and state and type invariant, is optimally chosen by consumers, and may depend therefore on the state s as well as the type i of the consumer. We show in what follows the required changes in the model. The specification of the intermediation activity and the structure of markets is clearly more complicated, still the main results on unanimity and optimality remain valid.

Since consumers' loans are non-collateralized, we follow Dubey, Geanakoplos and Shubik (2005) in introducing a utility penalty ξ^i for a type i consumer per unit defaulted in any state s , for all i, s . It is convenient to assume here that preferences are additively separable over time, so that they take the following form:

$$u_0^i(c_0^i) + \mathbb{E} [u_1^i(c^i(s)) - \xi^i \delta_s^i [\lambda_-^i (f(k, \phi; s) - B)]] \quad (33)$$

where δ_s^i is the default rate of consumer i in state s . Given this feature of consumers' preferences, the optimal default level in each state s for consumer i is obtained by maximizing (33) with respect to $(\delta_s^i)_s$ subject to the date 1 budget constraint (15), where δ is replaced

by δ_s^i . It is immediate to see that the solution is a well defined map $\delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i)$ for all s and $\theta^i, \lambda_+^i, b^i, \lambda_-^i$, and for any given k, ϕ, B .

Thus the default rate in any state s on the loans granted to consumers via the sale of short positions depends not only on the type i of the consumer but also on his overall portfolio holdings. We consider then the case where both the consumer's type and his portfolio holdings are observable by his trading partners. The loan contract offered by intermediaries is so an exclusive contract and the price depends both on the consumer's type and portfolio, $q_{i,\theta^i,\lambda_+^i,b^i,\lambda_-^i}^-$ as well as, obviously, on the return structure of the underlying equity. Hence the budget constraint faced by consumers at date 0 is now

$$c_0^i = w_0^i + [-k + q + p B] \theta_0^i - q \theta^i - p b^i - q^+ \lambda_+^i + q_{i,\theta^i,\lambda_+^i,b^i,\lambda_-^i}^- \lambda_-^i \quad (34)$$

An intermediary who is intermediating m units of the derivative by selling the short positions to consumers of type i , with portfolio $(\theta^i, \lambda_+^i, b^i, \lambda_-^i)$, faces a default rate on its loans equal to $\delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i)$. As a consequence, the shortfall in its revenue at date 1 is:

$$[(f(k, \phi; s) - B) \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i)] m. \quad (35)$$

We consider still the case where only equity, an asset that is 'safe' as it is in positive net supply and backed by real claims, is used to hedge the consumers' default risk. To be able to fully meet the shortfall in (35) due to consumers' default, the intermediary must hold at least

$$\max_s \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i) m$$

units of equity. The total date 0 revenue of the intermediary is then:

$$\max_m \left[q^+ - q_{i,\theta^i,\lambda_+^i,b^i,\lambda_-^i}^- - q \left(\max_s \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i) \right) \right] m \quad (36)$$

The intermediary's choice problem consists in the choice of the amount m to issue of each type $i, \theta^i, \lambda_+^i, b^i, \lambda_-^i$ of derivative so as to maximize its profits, that is its revenue at date 0. Note that the intermediation technology still exhibits constant returns to scale, hence a solution exists provided

$$q \geq \frac{q^+ - q_{i,\theta^i,\lambda_+^i,b^i,\lambda_-^i}^-}{\max_s \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i)};$$

and is characterized by $m(i, \theta^i, \lambda_+^i, b^i, \lambda_-^i) > 0$ only if $q = \frac{q^+ - q_{i,\theta^i,\lambda_+^i,b^i,\lambda_-^i}^-}{\max_s \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i)}$.

The main difference with respect to the reduced form model is then the fact that the market for derivative claims is differentiated according to consumers' types and portfolio choices. This has the following implications for the consumers' optimization problem and the market clearing conditions.

Consumer i chooses his portfolio $\theta^i, \lambda_+^i, b^i, \lambda_-^i$ so as to maximize

$$u_0^i(c_0^i) + \mathbb{E} \left\{ \begin{array}{l} u_1^i [w^i(s) + b^i + (f(k, \phi; s) - B)(\theta^i + \lambda_+^i - \lambda_-^i(1 - \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i)))] \\ - \xi^i \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i) \end{array} \right\}$$

subject to the budget constraint (34), given the asset prices q, q^+, p and q_i^- , and the default map $\delta_s^i(\cdot)$ obtained as above. Let $\bar{\theta}^i, \bar{\lambda}_+^i, \bar{b}^i, \bar{\lambda}_-^i$ denote the consumer's optimal choice in equilibrium. The asset market clearing conditions are then

$$\sum_i m(i, \bar{\theta}^i, \bar{\lambda}_+^i, \bar{b}^i, \bar{\lambda}_-^i) \left[\max_s \delta_s^i(\bar{\theta}^i, \bar{\lambda}_+^i, \bar{b}^i, \bar{\lambda}_-^i) \right] + \sum_{i \in I} \bar{\theta}^i = 1$$

for equity, and

$$\begin{aligned} \bar{\lambda}_-^i &= m(i, \bar{\theta}^i, \bar{\lambda}_+^i, \bar{b}^i, \bar{\lambda}_-^i) \text{ for each } i \\ 0 &= m(i, \theta^i, \lambda_+^i, b^i, \lambda_-^i) \text{ for each } i, (\theta^i, \lambda_+^i, b^i, \lambda_-^i) \neq (\bar{\theta}^i, \bar{\lambda}_+^i, \bar{b}^i, \bar{\lambda}_-^i) \\ \sum_i m(i, \bar{\theta}^i, \bar{\lambda}_+^i, \bar{b}^i, \bar{\lambda}_-^i) &= \sum_i \bar{\lambda}_+^i \end{aligned}$$

for the derivative security.

The consistency condition M) on the firms' equity conjectures must also be properly modified to reflect the different specification of the value of intermediation in the present context:

$$M') \quad q(k, \phi, B) = \max \left\{ \begin{array}{l} \max_i \mathbb{E} [MRS^i(c^i(s)) (f(k, \phi; s) - B)], \\ \max_{i, \theta^i, \lambda_+^i, b^i, \lambda_-^i} \frac{\max_i \mathbb{E} [MRS^i(c^i(s)) (f(k, \phi; s) - B)] - q^-(i, \theta^i, \lambda_+^i, b^i, \lambda_-^i; k, \phi, \bar{U}^i)}{\max_s \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i; k, \phi)} \end{array} \right\}, \forall k, \phi, B$$

where $q^-(i, \theta^i, \lambda_+^i, b^i, \lambda_-^i; k, \phi, B, \bar{U}^i)$ is constructed as follows. For any k, ϕ, B and $i, \theta^i, \lambda_+^i, b^i, \lambda_-^i$, set $q^-(i, \theta^i, \lambda_+^i, b^i, \lambda_-^i; k, \phi, B, \bar{U}^i)$ as the value of q^- that satisfies the following equation:

$$\begin{aligned} \bar{U}^i &= u_0^i(w_0^i + [-\bar{k} + \bar{q} + \bar{p} \bar{B}] \theta_0^i - \bar{q} \theta^i - \bar{p} b^i - \bar{q}^+ \lambda_+^i + \bar{q}^- \lambda_-^i) + \\ &\quad \mathbb{E} \left[\begin{array}{l} u_1^i [w^i(s) + b^i + (f(\bar{k}, \bar{\phi}; s) - \bar{B})(\theta^i + \lambda_+^i) - \lambda_-^i [f(k, \phi; s) - B] (1 - \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i; k, \phi, B))] \\ - \xi^i \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i; k, \phi, B) [\lambda_-^i (f(k, \phi; s) - B)] \end{array} \right] \\ &= \bar{U}^i \end{aligned}$$

where \bar{U}^i denotes the utility level of type i consumers at the equilibrium choices $\bar{\theta}^i, \bar{\lambda}_+^i, \bar{b}^i, \bar{\lambda}_-^i$ and the map $\delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i; k, \phi, B)$ is similarly obtained by maximizing the expected utility term on the right hand side of the above expression with respect to δ_s^i . That is, $q^-(i, \theta^i, \lambda_+^i, b^i, \lambda_-^i; k, \phi, B, \bar{U}^i)$ identifies the maximal willingness to pay in equilibrium of consumer i for a short position equal to λ_-^i in the firm with plan k, ϕ, B when the rest of his portfolio is given by $\theta^i, \lambda_+^i, b^i$.⁴¹ At these prices consumers are indifferent between choosing the equilibrium portfolio $\bar{\theta}^i, \bar{\lambda}_+^i, \bar{b}^i, \bar{\lambda}_-^i$ and any other portfolio with a short position λ_-^i in the equity of a firm with plan k, ϕ, B .

An important difference with respect to the previous analysis is the fact that here the price of short positions is no longer defined at the margin. This is due to the exclusive nature of loan contracts corresponding to short positions. Also, at the same prices intermediaries are indifferent between issuing the derivatives traded in equilibrium and any other derivative on equity of firms with plan k, ϕ, B such that $q = \frac{\max_i \mathbb{E}[MRS^i(c^i(s))(f(k, \phi; s) - B)] - q^-(i, \theta^i, \lambda_+^i, b^i, \lambda_-^i; k, \phi, B, \bar{U}^i)}{\max_s \delta_s^i(\theta^i, \lambda_+^i, b^i, \lambda_-^i; k, \phi, B)}$.

The unanimity and constrained optimality properties still hold in this environment. The argument again is very similar and relies on the the fact that, given the above specification of the intermediation technology and the price conjectures, the model is equivalent to a setup where the markets for all types of equity and all types of corresponding derivatives are available for trade. The notion of completeness here also requires the exclusivity of the loan contracts associated to short positions, so that the market for all types of derivatives can also be differentiated according to the type of a consumer and the level of his trades.

⁴¹In the specification of $q^-(i, \theta^i, \lambda_+^i, b^i, \lambda_-^i; k, \phi, B, \bar{U}^i)$ we have implicitly assumed that all the long positions of a consumer are in the assets corresponding to the firms' equilibrium choices. This is with no loss of generality and to avoid excessive notational complexities.