Abstract “Does democracy engender equality?”

John E. Roemer
Dept of Political Science
Yale University
PO Box 208301
New Haven CT 06520

Many suppose that democracy is an ethos which includes, inter alia, a degree of economic equality among citizens. In contrast, we conceive of democracy as ruthless political competition between groups of citizens, organized into parties. We inquire whether such competition, which we assume to be concerned with distributive matters, will engender economic equality in the long run.

The society consists of an infinite sequence of generations, each comprised of adults and their children. Adults care about household consumption, and the future wages of their children, which are determined by educational policy. A given generation is characterized by the distribution of wages earned by its adults. Parties form and propose policies to redistribute income among households, and to invest in the education of children; the educational policy that is victorious determines the distribution of wages in the next generation of adults.

A political equilibrium concept is proposed which determines two parties endogenously, and their proposed policies in political competition. One party wins the election (stochastically). This process determines a sequence of wage distributions across the generations.

We show that, whether the limit distribution of wages is one of equality depends upon the nature of intra-party bargaining. If parties are highly ideological, then equality is obtained in the long-run, while if they are opportunists, it is not. The prediction is, in a sense, very different from that of the naïve Downsian model.
“Does democracy engender equality?”

by

John E. Roemer*

§1 Introduction

Among types of political system, the one most identified in contemporary western society with the production of justice is democracy. Even on the political left, democracy has largely replaced socialism as the regime desideratum. Just as those socialists who were dissatisfied with aspects of Soviet society claimed that the Soviet regime was not real socialism, so those who continue to be dissatisfied with, for example, the American system, now argue that it is not an instance of real democracy. Real democracy is thought to be a political system in which genuine representation of all citizens – and even justice – is achieved.

The identification of democracy with justice is not simply a practice of many political theorists: perhaps the most important aspect of political transformation in the world in the last fifty years has been the toppling of authoritarian regimes, and their replacement with democratic ones. Just as socialism was a powerful movement in the first half of the twentieth century – by 1950, fully one-third of the world’s peoples lived under regimes that described themselves as socialist – so democracy has been the massively appealing political doctrine in, let us say, the period since 1960. And as it was

* Departments of Political Science and Economics, Yale University. This paper originated in a series of discussions with Ignacio Ortuno-Ortin. His ideas and critique have been immensely valuable. I am indebted to Roger Howe for many mathematical discussions, and to Herbert Scarf for teaching me how to solve infinite-dimensional optimization problems without optimal control theory. John Geanakoplos and Karine Van der Straeten offered valuable critique at a later stage.
an error of socialists to identify socialism with All Good Things, so now it is an error of
democrats to identify democracy with All Good Things\(^1\). The most common example of
this fallacy is when some say that regime \(X\) cannot be a democracy, because it sustains
Bad Thing \(Y\) (oppression of women, abrogation of civil rights, etc.). If democracy is
defined as a set of political institutions, rather than as an ethos, then the correct approach
is to study what those institutions entail. Perhaps, for example, both the oppression of
women and its absence can co-exist with democracy.

In this article, we undertake a study of this kind: we ask whether democracy,
understood as a system of political competition between parties that represent different
coalitions of citizens, will engender justice, or – as we here interpret justice, \textit{equality}. Of
course, we cannot answer that broad question generally, and so we narrow it down to
something manageable. In particular we focus upon the role of public education as an
instrument for reducing the differentials in human capital that would otherwise obtain,
and we ask whether democracy will entail the long-term equalization of human capital
through political decisions concerning educational investment.

We model the following society, one which reproduces itself over many
generations. At the initial date, there are households led by adults (“mothers”)
characterized by a distribution of human capital, that is, capacities to produce income.
Each mother has one child (“son”). The human capital the son will have, when next
period, he has become an adult, is a monotone increasing function of his mother’s level

\(^1\) There are many people who identify democracy with justice. For instance, Adolfo Perez Esquivel, a
Nobel Peace Prize laureate, recently said, “The vote does not define democracy. Democracy means justice
and equality.” \textit{(The Daily Journal [Caracas], July 12, 2001)}
of human capital and the amount that was invested in his education. (To adhere to the
gender identification of adult mothers and child sons, boys turn into women when they
grow up.) This relationship is deterministic, and describes the educational production
function for all children. Thus, it requires more investment to bring a child from a poor
(low human capital) family up to a given level of human capital than a child from a
richer family. All mothers have the same utility function: a mother cares about her
household’s consumption (that is, her after-tax income), and the earning power her child
will have, as an adult. We will, for simplicity, assume that adults do not value leisure.

Educational finance is, until section 5, purely public. The polity of adults, at each
date, must make four political decisions: how much to tax themselves, how to split the
tax revenues between a redistributive component for households’ consumption and the
educational budget, how to partition the budget for redistribution among adults, and how
to partition the educational budget as investment in particular children, according to their
type (that is, their parental human capital). Once these political decisions are
implemented, a distribution of human capital is determined for the next generation.
When the present children become adults, characterized by that distribution of human
capital, they face the same four political decisions. We wish to study the asymptotic
distribution of human capital of this dynamic process.

In the society we have described, a child is characterized by the family
(household) into which he is born, for his capacity to transform educational investment
into future earning power is determined by his family background, proxied by his
parent’s human capital. We imagine that the transmission of ‘culture’ to the child is
indicated by the parent’s human capital endowment. We view the child’s capacity
successfully to absorb educational investment, and transform it into human capital, as a circumstance beyond his control, and so a society of this kind that wished rapidly to equalize opportunities for all children would compensate children from poorer families with more educational investment. Equality of opportunity will be achieved when all adults have the same human capital, for that means, as children in the previous generation, the compensation for disadvantageous circumstances was complete. (See Roemer (1998) for the theory of equality of opportunity, based upon social compensation for disadvantageous circumstances.) In the real world, equality of opportunity does not require equalizing outcomes in this way, because people may remain responsible for some aspect of their condition, even after the necessary compensation for disadvantage has been made. But in our model there is no such element of personal responsibility, and so, if we take equality of opportunity as our conception of justice, then justice will have been achieved exactly when the wage-earning capacities of all adults are equal.

One might object that it is sufficient to equalize (post-tax) incomes for justice. But it may well be the case that individuals derive welfare not only from consumption, but from their human capital, and so we insist that this more demanding condition of human-capital equality is the one of interest. Indeed, if one’s human capital is an enabler of self-realization, then it is surely the case that justice would require a concern with levels of human capital in a society, not simply income levels.

We will stipulate a democratic process for solving society’s political problems, at each generation. Our question becomes: How close will the asymptotic distribution of human capital engendered by this democratic process be to an equal distribution?
The focus of our model will be on that democratic process. We employ a concept of democratic political equilibrium that takes as data the distribution of preferences of the polity over a given policy space, and produces as its output an endogenous partition of the polity into two political parties, a policy proposal by each party, and a probability that each party will win the election. We suppose that an election occurs, and the policy of the victorious party is implemented. Our procedure will be to begin with a distribution of adult human capital at date 0, which will determine the distribution of adult preferences at date 0. The dynamic process is thus initialized.

Although we have described the political choice as consisting of four independent decisions, we will in fact model the political problem as one on an infinite dimensional policy space. That policy space, denoted \( T \), will consist of pairs of functions \((\psi, r)\) where \( \psi(h) \) is the after-tax household income of an adult with human capital \( h \), and \( r(h) \) is the public educational investment in a child from an \( h \)-family. The only restrictions on these functions are that they be continuous, jointly satisfy a budget constraint, and satisfy two constraints that we call social norms. Thus, the present analysis marks a substantial technical advance over analyses in political economy that must limit their scope to unidimensional policy spaces, or policy spaces of small dimension. But the advance is not merely technical. It is surely artificial to restrict a democratic polity’s choice of policies to ones with simple mathematical properties, such as linearity. Our ability to solve the political problem with no such restrictions means that we are able to model the democratic struggle as ruthless competition: no holds, in the sense of unmotivated restrictions on the nature of policy proposals, are barred, except those precluded by the social norms.
That political equilibrium concept is ‘party unanimity Nash equilibrium with endogenous parties.’ In two recent articles, I introduced the concept of ‘party unanimity Nash equilibrium (PUNE), (Roemer [1999, 1998]). The extension to ‘PUNE with endogenous parties’ is introduced in Roemer (2001, Chapter 13). The endogenous-party aspect is grafted from a model of Baron (1993).

It is probably fair to say that most articles in political economy propose a relatively sophisticated model of the economy, and a trivial model of politics (standardly, political equilibrium consists in both parties’ proposing the median voter’s ideal point, or, more generally, a Condorcet winner in the policy space). Our approach here is just the opposite: the economy is very simple, but the politics are quite complex. Our first justification for the complex politics is that it enables us to solve the problem of political equilibrium with multi- and even infinite dimensional policy spaces, when Condorcet winners do not exist. Our second justification, for the problem at hand, is that our focus is upon the workings of democracy, and therefore, a careful articulation of democratic institutions is appropriate. Of course, a more highly articulated model of the economy would also be desirable, if tractability were not sacrificed.

In section 2, the definition of political equilibrium that we will use, and a companion concept of quasi-equilibrium, are presented. In section 3, we characterize the policies in the political equilibria of the model. Section 4 does the dynamics. Section 5 relaxes the assumption that all educational investment is public. Section 6 concludes.

§2 Party unanimity Nash equilibrium with endogenous parties (PUNEWP)
In this section, I define PUNEPP and a related concept$^2$.

Let $H$ be a set of voter types, where $h \in H$ is distributed according to a probability measure $\mathbf{F}$ in the society in question. Let $T$ be a set of policies. There is a function $v: T \times H \rightarrow \mathbb{R}$ which represents the preferences of types over policies; thus $v(\cdot, h)$ is the utility function of type $h$ on $T$. For each $h$, we assume that $v(\cdot, h)$ is a von Neumann-Morgenstern utility function for lotteries on $T$.

Let $t_i, t_2 \in T$ be two policies; we define $\pi(t_i, t_2)$, the probability that policy $t_i$ defeats policy $t_2$. Our datum is a function $\pi^*:[0,1] \rightarrow [0,1]$, such that $\pi^*(0) = 0, \pi^*(1) = 1$, and $\pi^*$ is strictly increasing on $[0,1]$.

Let $\Omega(t_i, t_2)$ be the set of types who prefer $t_i$ to $t_2$ and $I(t_i, t_2)$ be the set of types who are indifferent between $t_i$ and $t_2$. Then we define, pro tem$^3$:  

$$\pi(t_i, t_2) = \pi^*(\mathbf{F}(\Omega(t_i, t_2))) + \frac{1}{2}\mathbf{F}(I(t_i, t_2)). \quad (2.1)$$

In other words, $\mathbf{F}(\Omega(t_i, t_2)) + \frac{1}{2}\mathbf{F}(I(t_i, t_2))$ is the mass of voters who in principle will vote for $t_i$ — but perhaps some voters will make mistakes or perhaps $\mathbf{F}$ is measured imperfectly. Equation (2.1) says that the probability that $t_i$ defeats $t_2$ is an increasing function of the ‘expected’ vote for $t_i$.

A party structure is a partition of $H$ into two elements. We specialize, now, to the case $H = \mathbb{R}_+$, and further specialize by requiring that both elements in a party structure be intervals: thus a party structure is characterized by a pivotal type $h^*$, with $L = [0, h^*)$ and $R = [h^*, \infty)$. We call the two parties Left ($L$) and Right ($R$).

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$^2$ For a more relaxed presentation of PUNE and PUNEPP, see Roemer (2001, Chapters 8 and 13).

$^3$ To be modified below.
Associated with a party is a utility function, which is the average of its members utility functions.

Thus

\[
\begin{align*}
v_{L}(t) &= \int_{0}^{h^*} v(t,h)dF(h) \\
v_{R}(t) &= \int_{h^*}^{\infty} v(t,h)dF(h)
\end{align*}
\]

(We drop a multiplicative constant.) The utility functions \(v(\cdot, h)\) are assumed to be cardinally measurable and unit comparable (CUC), so that averaging them makes sense.

All parties contain three factions: opportunists, reformists, and militants. (These factions are not to be identified with particular citizen types.) Each faction possesses a real-valued payoff function defined on \(T \times T\). The payoff functions of the three factions in Left are defined by:

\[
\begin{align*}
L\Pi_{\text{Opp}}(t',t^2) &= \pi(t',t^2) \\
L\Pi_{\text{Ref}}(t',t^2) &= \pi(t',t^2)v_{L}(t') + \left(1 - \pi(t',t^2)\right)v_{L}(t^2) \\
L\Pi_{\text{Mil}}(t',t^2) &= v_{L}(t')
\end{align*}
\]

with an analogous definition for Right’s three factions. The three factions are interested, respectively, in winning (opportunists), party-member welfare (reformists), and publicity (militants).

**Definition 1.** A party unanimity Nash equilibrium with endogenous parties (PUNEEP) is a party structure \((L,R)\) given by \(L = [0,h^*] \) and \(R = [h^*, \infty)\) with \(h^* > 0\), and a pair of policies \(t', t^r \in T\) such that

(A) there is no policy \(t \in T\) such that
\[ L \Pi^J(t,t^R) \geq L \Pi^J(t^L,t^R), \text{ for } J = O,R,M \]

with at least one of these inequalities strict;

(B) there is no policy \( t \in T \) such that

\[ R \Pi^J(t^L,t) \geq R \Pi^J(t^L,t^R), \text{ for } J = O,R,M \]

with at least one of these inequalities strict;

(C)

\[
\begin{align*}
    h \in L & \Rightarrow v(t^L,h) \geq v(t^R,h) \\
    h \in R & \Rightarrow v(t^R,h) \geq v(t^L,h).
\end{align*}
\]

The three payoff functions of a parties’ factions each represent a complete order on \( T \times T \). Their intersection represents a quasi-order on \( T \times T \). A PUNEEP is a Nash equilibrium of the game played by these two quasi-orders, with the additional requirement (C). Requirement (C) was initially proposed by Baron (1993) as modeling the stability of a party structure.

Remark 1. It is easily shown that the reformists are gratuitous in definition 1. That is, if we eliminate the reformist factions, we do not alter the set of equilibria. But notice, once this is done, we never need mention expected utility, since only the reformists calculate that. It thus suffices that \( \{ v(.,h) | h \in H \} \) be a profile of CUC utility functions (i.e., they need not represent preferences over lotteries).

Remark 2. It is now convenient to alter the convention on how indifferent voters vote, in the presence of parties. When parties are present, we will say that a voter who is indifferent between policies votes for the policy of his party. (Recall that each citizen is a member of one party: this is part of the description of a political environment.) Thus, formally, we now revise the definition of \( \pi \) to:

\[
\pi(t^L,t^R) = \pi^*(F(\Omega(t^L,t^R)) + F(L \cap I(t^L,t^R))). \quad (2.1')
\]
Remark 3. In Roemer (2001, Chapter 8), it is shown that if sufficient convexity is present, then every PUNEEP can be viewed as the outcome of generalized Nash bargaining between the militant and opportunist factions of each party, given the other party’s proposal. There is, in general, a two dimensional manifold of PUNEEP. Each one is characterized by specifying the relative bargaining strengths of the two active factions in each party – thus, two positive numbers. Thus, parties compete with each other à la Nash equilibrium, while internal factions bargain with each other à la Nash bargaining. The PUNEEP concept thus owes its origins doubly to John Nash.

We now further specialize to the case that $F$ has a continuous, strictly increasing distribution function, $F$, on $\mathbb{R}_+$. We next define an auxiliary notion that is useful in the analysis.

Definition 2. A quasi-PUNE is an ordered pair $(h^*, y) \in H \times \mathbb{R}$ and a pair of policies $t^L, t^R \in T$, such that $v(t^L, h^*) = y = v(t^R, h^*)$ and:

2A. $t^L$ solves

$$\max \int_0^{h^*} v(t, h) \, dF(h)$$

subject to

$$t \in T$$

$$h \in [0, h^*) \Rightarrow v(t, h) \geq v(t^R, h) \quad (L0)$$

$$v(t, h^*) \geq y \quad (L1)$$

2B. $t^R$ solves

$$\max \int_{h^*}^{\infty} v(t, h) \, dF(h)$$

subject to
\[ t \in T \]
\[ h \in [h^*, \infty) \Rightarrow v(t, h) \geq v(t^L, h) \quad \text{(R0)} \]
\[ v(t, h^*) \geq y \quad \text{(R1)} \]

2C. Constraints (L1) and (R1) bind at \( t^L \) and \( t^R \) respectively.

We have:

**Proposition 1.** Let \( v \) be continuous in \( h \). If \((t^L, t^R, h^*)\)is a PUNEEP, then \((t^L, t^R, h^*, y)\) is a quasi-PUNE, with \( y = v(t^L, h^*) \).

**Proof:**

Let \((t^L, t^R, h^*)\) be a PUNEEP with \( h^* > 0 \), \( L = [0, h^*] \), and \( R = [h^*, \infty) \). By Remark 2, \( \pi(t^L, t^R) = \pi^r(F(L)) \) and \( 0 < \pi^r(F(L)) < 1 \) by definition of \( \pi^r \) and the fact that \( F \) is strictly increasing on \( R_+ \). By Condition 1A of PUNEEP, there is no policy \( t \) that gives Left’s militants a higher payoff than they receive at \( t^L \) and gives a higher probability of victory against \( t^R \). In particular, there is no policy \( t \) that gives Left’s militants a higher payoff than at \( t^L \) and such that

\[ h \in [0, h^*) \Rightarrow v(t, h) \geq v(t^R, h), \]

and

\[ v(t, h^*) > y, \]

for if there were, than, by continuity of \( v \) in \( h \) there would be an interval \([h^*, h^* + \varepsilon)\) such that

\[ h \in [h^*, h^* + \varepsilon) \Rightarrow v(t, h) > v(t^R, h). \]

It would then follow that at least the set of voters \( L \cup [h^*, h^* + \varepsilon) \) would favor \( t \) and so a higher probability of victory could be achieved for Left at no cost to her militants.
It therefore follows that statement 2A of definition 2 is true, and that \((L1)\) binds.

In like manner, statement 2B of definition 2 is true and \((R1)\) binds, which concludes the proof. ■

The converse of Proposition 1 is not true: there may be quasi–PUNEs that are not PUNEEPs. For if \((t^L, t^R, h^*, y)\) is a quasi-PUNE, it is possible that there exists a policy \(t\) which improves the payoff of both Left’s militants and opportunists, by assembling a set of voters who favor \(t\) over \(t^R\) that is disconnected and does not contain \(h^*\).

We can now give a preview of our strategy. In our politico-economic environment, we can fully characterize the set of quasi-PUNEs: the nice fact is that we can characterize them without recourse to fixed point theorems, only to optimization methods. We will further note that the set of PUNEEPs is a non–empty subset of the set of quasi-PUNEs. We then conduct our dynamic analysis assuming that each generation’s political equilibrium is some quasi-PUNE. Whatever we conclude will hold \textit{a fortiori} for societies whose political equilibria are genuine ones, that is, PUNEEPs. In this manner we avoid ever having to solve the intractable problem of characterizing precisely the set of PUNEEPs.

§3 \textbf{Equilibrium at one date}

Throughout this section, we analyze the society at one date.

A. The politico–economic environment

(i) Preferences
A typical society, in our problem, consists of a continuum of adult types, each characterized by his/her human capital $h$, where $h$ is distributed according to a probability measure $F$, whose mean is denoted $\mu$, and whose support is the positive real line. We denote the distribution function (CDF) of $F$ by $F$. Each adult has one child. Adults care about their own consumption, and their child’s (future) human capital.

We assume:

$$u(x,h') = \log x + \gamma \log h',$$  

where $x$ is the household’s consumption, or after-tax income, and $h'$ is the child’s (future) human capital. Zero consumption is minimal household consumption. Note there is no preference for leisure.

(ii) Technology

If $r$ is invested in the education of a child whose parent is of type $h$ then the child’s future human capital will be

$$h' = \alpha h^br^c$$  

where $\alpha, b, c$ are positive constants.

$b$ is the elasticity of child’s human capital w.r.t. parental human capital and $c$ is the elasticity of child’s human capital w.r.t. educational investment. Think of the influence of the parent’s human capital on the child’s human capital as operating through family culture, or perhaps neighborhood effects (if neighborhoods are income-segregated). Bénabou (in press) uses a relationship like (3.2), and gives some weak evidence that $b+c<1$; we will, however, concentrate on the case $b+c=1$, for reasons explained below.
If an adult of type $h$ works at her full potential then her (pre-tax) earnings are $h$. Thus human capital is measured in units of income-earning capacity.

(iii) The policy space

Let $C$ be the space of continuous functions on the domain $\mathbb{R}_+$. A policy is a pair of functions $(\psi, r) \in C \times C$ such that

$$
\int_0^\infty (\psi(h) + r(h)) dF(h) \leq \mu \quad (3.3a)
$$

for all $h$, $\psi'(h) + r'(h) \geq 0, \quad (3.3b)$

$$
\psi'(h) + r'(h) \leq 1, \quad (3.3c)
$$

where ‘prime’ indicates derivative, and the inequalities are meant to hold where the derivatives exist. The interpretation is that $\psi(h)$ is the after-tax income of an adult of type $h$, and $r(h)$ is the public educational investment in a child from an $h$-family. We call $\psi(h) + r(h) \equiv X(h)$ the total resource bundle allocated to an $h$ household, so (3.3b,c) restrict the rates at which the total resource bundle changes with $h$. We call (3.3b,c) social norms, as they are not motivated by political competition or incentive compatibility considerations.

In a laissez-faire regime, with no taxation, $X(h) = h$, and so $X'(h) = 1$. This motivates constraint (3.3c): the laissez-faire policy is politically feasible in our democracy.

We assume that all educational investment is public$^4$. Specification of a policy $(\psi, r)$ solves the four political problems described in Section 1.

Thus the indirect utility function $v: T \times H \rightarrow \mathbb{R}$ is given by

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$^4$ This assumption is relaxed below in section 5.
where, in the last line of (3.4), we have dropped a gratuitous constant term.

[There is a natural incentive compatibility condition, that adult utility be non-decreasing in \( h \), so that no adult would have an incentive to work at a lower income-earning capacity than her true capacity. The local version of this condition is:

\[
\frac{\psi'(h)}{\psi(h)} + \gamma c \frac{r'(h)}{r(h)} \geq 0. \tag{3.5}
\]

Some might prefer to substitute (3.5) for (3.3b) in the model, but doing so renders the analysis below much more difficult: it converts what will be a convex optimization problem on an infinite-dimensional space to a difficult, non-convex problem. In the interests of simplicity, and not diffusing attention from our main concern, we use (3.3b) in lieu of (3.5). We conjecture, however, that the results we report would remain the same if (3.3b) were replaced with (3.5).

Thus, our policy space is

\[
T = \{(\psi, r) \in C \times C \mid (3.3a,b,c) \text{ hold}\}.
\]

B. Quasi-PUNEes

For a given point \( (h^*, y) \in \mathbb{R}_+ \times \mathbb{R} \), consider the following two programs:

\[
\begin{align*}
\max_{(\psi, r) \in C^2} & \quad \int_0^{h^*} \log(\psi(h) + \gamma c \log r(h)) dF(h) \\
\text{s.t.} & \quad 0 \leq \psi'(h) + r'(h) \leq 1 \quad \tag{3.61} \\
& \quad \int_0^{\infty} (\psi(h) + r(h)) dF(h) \leq \mu \quad \tag{3.62} \\
& \quad \log \psi(h^*) + \gamma c \log r(h^*) \geq y \quad \tag{3.63}
\end{align*}
\]
Let \((h^*, y)\) be such that solutions \((\psi^L, r^L)\) and \((\psi^R, r^R)\) exist to (3.6) and (3.7), respectively, and such that inequalities (3.63) and (3.73) bind at the solutions. We will show that the following hold:

1. \(0 \leq h \leq h^* \Rightarrow \nu(\psi^L, r^L, h) \geq \nu(\psi^R, r^R, h), \) and
2. \(h^* \leq h \leq \infty \Rightarrow \nu(\psi^R, r^R, h) \geq \nu(\psi^L, r^L, h).\)

It will follow that \((\psi^L, r^L)\) and \((\psi^R, r^R)\) constitute a quasi-PUNE at \((h^*, y)\), and that solutions of programs (3.6) and (3.7) at which (3.63) and (3.73) bind comprise precisely the quasi-PUNEs for our problem.

Our first task is to characterize the set

\[
\Gamma = \{(h^*, y) \in \mathbb{R}_+ \times \mathbb{R} | \text{solutions to (3.6) and (3.7) exist at which (3.63) and (3.73) bind}\}. 
\]

Note that \(T = \{(\psi, r) \in C \times C | \text{(3.61) and (3.62) hold}\}.\) For fixed \(h^*,\) consider the following three programs:

\[
\max \int_0^{h^*} v(\psi, r, h) dF(h) \quad \text{s.t.} \quad 0 \leq \psi(h) + r(h) \leq 1 \quad (3.71) \]

\[
\int_0^{\infty} (\psi(h) + r(h)) dF(h) \leq \mu \quad (3.72) \]

\[
\log \psi(h^*) + \gamma c \log r(h^*) \geq y \quad (3.73) \]

\[
\max \int_0^{h^*} v(\psi, r, h) dF(h) \quad (\psi, r) \in T \quad (3.8) \]

\[
\max \int_0^{\infty} v(\psi, r, h) dF(h) \quad (\psi, r) \in T \quad (3.9) \]
\[
\max_{(\psi, r) \in T} v(\psi, r; h^*)
\] 
(3.10)

Let their solutions be denoted \(\tau^L, \tau^R\), and \(\tau^*\), respectively, where \(\tau = (\psi, r)\) is the generic policy. Let \(y^*(h^*)\) be the value of program (3.10), i.e.

\[
y^*(h^*) = v(\tau^*, h^*),
\]

and define

\[
\begin{align*}
y^L(h^*) &= v(\tau^L, h^*), \\
y^R(h^*) &= v(\tau^R, h^*).
\end{align*}
\]

We have:

**Proposition 2.** For \(h^*\) given, \((h^*, y) \in \Gamma\) iff

\[
\max[y^L(h^*), y^R(h^*)] \leq y \leq y^*(h^*). 
\] 
(3.11)

**Proof:**

1. Suppose \(y > y^*(h^*)\). Then there is no feasible solution to (3.6) or (3.7), for (3.63) will never hold on \(T\). Thus we must have \(y \leq y^*(h^*)\) if \((h^*, y) \in \Gamma\).

2. Suppose \(y < y^L(h^*)\). Then constraint (3.63) is not binding at the solution to (3.6), since the solution to (3.8) is indeed the solution to (3.6). Similarly, if \(y < y^R(h^*)\), then constraint (3.73) is not binding at the solution to (3.7). Thus \((h^*, y) \in \Gamma\) implies \(y \geq \max[y^L(h^*), y^R(h^*)]\).

3. Conversely, if (3.11) holds, then the opportunity sets of (3.6) and (3.7) are non-empty, and at the optimal solutions, (3.63) and (3.73) must bind, because

\[
y \geq \max[y^L(h^*), y^R(h^*)].
\]

Actually, the proof of Proposition 2 has ignored the compactness issue - whether non-emptiness of the opportunity sets for programs (3.6) and (3.7) implies the attainment
of (optimal) solutions. We shall show below that if (3.11) holds, solutions are indeed attained.

Thus we have argued that

$$\Gamma = \{(h^*, y) \mid \max[y^r(h^*), y^r(h^*)] \leq y \leq y^*(h^*)\}.$$  

The virtue of the quasi-PUNE notion is now evident: we can characterize the set $\Gamma$, and thus the set of quasi-PUNEs, merely by solving the three programs (3.8), (3.9), and (3.10). No fixed-point machinery is needed to do this.

We next solve these three programs.

**Proposition 3.** Let $(\psi, r)$ be a solution to (3.6), (3.7), (3.8), (3.9), or (3.10).

Then

$$r(h) = \frac{\gamma_c \psi(h)}{1 + \gamma_c} X(h).$$  

**Lemma.** Let $X$ be the total resource dedicated to household $h$. Then the household’s optimal distribution of $X$ between consumption $\psi$ and educational investment $r$ is

$$\psi = \frac{1}{1 + \gamma_c} X$$

$$r = \frac{\gamma_c}{1 + \gamma_c} X.$$  

**Proof:** The household would choose consumption $\psi$ to maximize its utility, which leads immediately to the claim.  ■
Proof of Proposition 3:

Let \((\hat{\psi}, \hat{r})\) be a solution to program (3.6), and suppose that the claim were false. Let \(\hat{X}(h) = \hat{\psi}(h) + \hat{r}(h)\), and define

\[
\begin{align*}
\psi(h) &= \frac{\hat{X}(h)}{1 + \gamma c}, \\
r(h) &= \frac{\gamma c \hat{X}(h)}{1 + \gamma c}.
\end{align*}
\]

It is straightforward to check that \((\psi, r) \in T\). Furthermore, each household receives the same total resource at \((\hat{\psi}, \hat{r})\) and at \((\psi, r)\). But according to the lemma, for every \(h\), \((\psi(h), r(h))\) is the optimal way for household \(h\) to allocate the total resource assigned to it between consumption and education. Therefore the objective function of (3.6) increases if we substitute \((\psi, r)\) for \((\hat{\psi}, \hat{r})\), a contradiction. (To be precise, the argument shows that \((\hat{\psi}, \hat{r})\) must equal \((\psi, r)\) except possibly on a set of \(F\)-measure zero. Continuity then completes the argument.) □

Remark 4. If we replaced social norm (3.3b) with incentive compatibility (3.5), Proposition 3, although probably true, is much more difficult to prove. It is for this reason that we employ (3.3b).

Define \(X(h) = \psi(h) + r(h)\). By substituting from (3.12), we can reduce (3.8), (3.9), and (3.10) to the following three programs:
\[
\max_{\mathbf{x} \in \mathbf{C}} \int_{0}^{\mathbf{h}} \log X(h) d\mathbf{F}(h) \\
\text{s.t.} \\
\int_{0}^{\mathbf{h}} X(h) d\mathbf{F}(h) \leq \mu \\
0 \leq X'(h) \leq 1.
\] (3.8a)

\[
\max_{\mathbf{x} \in \mathbf{C}} \int_{\mathbf{h}}^{\infty} \log X(h) d\mathbf{F}(h) \\
\text{s.t.} \\
\int_{0}^{\infty} X(h) d\mathbf{F}(h) \leq \mu \\
0 \leq X'(h) \leq 1,
\] (3.9a)

and

\[
\max_{\mathbf{x} \in \mathbf{C}} \log X(h^*) \\
\text{s.t.} \\
\int_{0}^{\infty} X(h) d\mathbf{F}(h) \leq \mu \\
0 \leq X'(h) \leq 1.
\] (3.10a)

In other words, Proposition 3 enables us replace optimization problems on \( \mathbf{C} \times \mathbf{C} \) with optimization problems on \( \mathbf{C} \).

We have:

**Proposition 4.**

a. The solution to (3.8a) is

\[ X^L_{\psi}(h) \equiv \mu; \]

b. The solution to (3.9a) is
where \((x,y)\) solves the following two simultaneous equations:

\[
\begin{align*}
    x + \int_0^y hdF(h) + y(1 - F(y)) &= \mu \\
    \int_{h^*}^y \frac{dF(h)}{h+x} &= \frac{F(y)}{x+y}.
\end{align*}
\]  

(3.13a) \hspace{1cm} (3.13b)

We have \(x > 0\). The solution is illustrated in Figure 1.

c. The solution to (3.10a) is illustrated in Figure 1. It is given by

\[
X_{h^*}(h) = \begin{cases} 
    X_0^* + h, & \text{if } 0 \leq h \leq h^*, \\
    X_0^* + h^*, & \text{if } h > h^* 
\end{cases}
\]

where \(X_0^*\) is the solution of the equation

\[
X_0^* + \int_0^\infty h dF(h) + h^*(1 - F(h^*)) = \mu.
\]

(3.14)

The solution is illustrated in Figure 1.

Proof:

In the appendix, we prove that the various policies are as described. Here, we prove that

\(x > 0\). Using integration by parts:

\[
\int_0^y h dF(h) = hF(h)\big|_0^y - \int_0^y F(h)dh = yF(h) - \int_0^y F(h)dh.
\]
Hence, (3.13a) reduces to

\[ x = \mu + \int_0^y F(h)dh - y = \mu + \int_0^y (1 - F(h))dh = \int_y^\infty (1 - F(h))dh, \]

where the last step uses the fact that \( \mu = \int_0^\infty (1 - F(h))dh \). Since \( F \)'s support is the positive real line, we have that \( x > 0 \) if \( y < \infty \). Suppose \( y = \infty \). Then (3.13b) becomes

\[ \int_{h^*}^\infty \frac{dF(h)}{h} = 0, \]

a contradiction. Therefore \( x > 0 \).

Here is an intuitive argument for part of Proposition 4.

Consider program (3.8a). The benefit to household \( h \) is \( \log X(h) \); the cost (to the optimizer) of supplying household \( h \) is \( X(h) \); hence the benefit-cost ratio, \( \frac{\log X(h)}{X(h)} \), is non-increasing in \( h \), because \( X'(h) \geq 0 \) is required. So the optimizer should give as much of the resource as possible to low \( h \): front-loading, so to speak. The binding constraint is \( X'(h) \geq 0 \): so the planner allocates \( X(h) = \mu \).

Second, consider (3.10a). Clearly it is a waste to give any resource to \( h > h^* \), so we must have

\[ X(h) = X(h^*) \text{ for } h > h^*. \]
Now the optimizer wants to minimize what goes to [0, \( h^* \)), conditional upon reaching a high value at \( h^* \), so \( X \) should descend rapidly (at rate 1) to the left of \( h^* \). The stated function \( X_{h^*} \) makes the value at \( h^* \) as large as possible.

Claim (3.9a) is harder to motivate, and so we do not do so.

In like manner using Proposition 3, we can reduce programs (3.6) and (3.7) to:

\[
\begin{align*}
\max_{X_{\infty}} \int_0^{h^*} \log X(h) dF(h) \\
\text{s.t.} \\
0 \leq X'(h) \leq 1 \\
\int_0^\infty X(h) dF(h) \leq \mu \\
\log X(h^*) \geq \hat{y}, \quad (3.6a)
\end{align*}
\]

and

\[
\begin{align*}
\max_{X_{\infty}} \int_h^{\infty} \log X(h) dF(h) \\
\text{s.t.} \\
0 \leq X(h) \leq 1 \\
\int_0^\infty X(h) dF(h) \leq \mu \\
\log X(h^*) \geq \hat{y}, \quad (3.7a)
\end{align*}
\]

where \( \hat{y} = \log(1 + \gamma^c) + \frac{y - \gamma^c \log \gamma^c}{1 + \gamma^c} \).

Of course, the analogous result to Proposition 2 holds, that is:

**Proposition 2a.** Let \( \hat{\Gamma} = \{ (h^*, \hat{y}) \in \mathbb{R}_+ \times \mathbb{R} \} \) (3.6a) and (3.7a) have solutions at which (3.63a) and (3.73a) bind. Define
\[ \hat{y}^L(h^*) = \log X^L_{h^*}(h^*) \]
\[ \hat{y}^R(h^*) = \log X^R_{h^*}(h^*) \]
\[ \hat{y}^*(h^*) = \log X^*_{h^*}(h^*) \]

*Then*

\[ \max \{ \hat{y}^L(h^*), \hat{y}^R(h^*) \} \leq \hat{y} \leq \hat{y}^*(h^*) . \]

(3.11a)

*Conversely, if* (3.11a) *holds, then* \((h^*, \hat{y}) \in \hat{\Gamma} \).

*Proof:* As in Proposition 2.

\( \hat{\Gamma} \) is our parameterization of the set of quasi-PUNEs associated with the ‘reduced’
problem, where we work with the total-resource bundle function, \( X \). From consideration
of the three programs (3.8), (3.9), and (3.10), it is clear that the interval of admissible
values \( \hat{y} \) is non-empty for every \( h^* \).

In figure 2, we illustrate the manifold \( \hat{\Gamma} \). Although the picture is not accurate, it
is the case that quasi-PUNEs exist for every \( h^* > 0 \).

We next derive what the quasi-PUNE looks like at \((h^*, \hat{y}) \in \hat{\Gamma} \).

*Proposition 5.* Let \((h^*, \hat{y}) \in \hat{\Gamma} \). Then:

a. *The solution to* (3.6a) *is illustrated in Figure 3. It is defined by:

\[ X^L(h) = \begin{cases} 
\hat{X}^L_0, & 0 \leq h \leq h_L \\
\hat{X}^L_0 + (h - h_L), & h_L \leq h \leq h^* \\
\hat{X}^L_0 + e^\delta, & h > h^* 
\end{cases} \]
where $(\hat{X}_0^L, h_L)$ is the simultaneous solution of the two equations:

\[
\log(\hat{X}_0^L + (h^* - h_L)) = \hat{y},
\]

\[
\hat{X}_0^L + \int_{h_L}^{h^*} (h - h_L) dF(h) + (1 - F(h^*)) (h^* - h_L) = \mu.
\]  (3.15b)

We have $\hat{X}_0^L > 0$.

b. The solution of (3.7a) is illustrated in Figure 3. It is defined by:

\[
X^R(h) = \begin{cases} 
\hat{X}_0^R + h, & 0 \leq h \leq h_R \\
\hat{X}_0^R + h_R, & h > h_R 
\end{cases}
\]

where $(\hat{X}_0^R, h_R)$ is the simultaneous solution of:

\[
\log(\hat{X}_0^R + h^*) = \hat{y},
\]  (3.15c)

\[
\hat{X}_0^R + \int_0^{h_R} h dF(h) + (1 - F(h_R)) h_R = \mu.
\]  (3.15d)

We have $\hat{X}_0^R > 0$.

Proof:

The proof that the optimal policies are as stated follows the template of the proof of Proposition 4(b), presented in the appendix. We will not go through those analogous constructions. We do prove here that $\hat{X}_0^R > 0$, and hence, that $\hat{X}_0^L > 0$. From (3.11a), we have that

\[
\text{Exp}(\hat{y}) \geq \text{Exp}(\hat{y}^R(h^*)) = X^R_{h^*}(h^*) = x + h^*.
\]
where the last part follows from Proposition 4(b). Therefore, since $X^R(h)$ dominates $X_{h^*}^R(h)$ on $[0, h^*]$, we have $X^R(0) = \hat{X}_0^R \geq x$. Now use the fact that $x > 0$ (Prop.4).

Proposition 4 and Proposition 5 completely characterize the manifold of quasi-PUNEes.

We have one item left to check: that every member of each party weakly prefers her party’s policy to the other party’s policy. This claim is easy to verify. Indeed, from figure 3 we see that the total resource bundle functions of the two parties coincide on the interval $[h_L, h^*]$ of types, and indeed, each member of a party weakly favors her party’s policy to the other’s.

The two educational investment functions are just multiples of the functions graphed in Figure 3, for according to Proposition 3, if $X(\cdot)$ is proposed by either party in a quasi-PUNE, then

$$r(h) = \frac{\gamma c}{1 + \gamma c} X(h), \text{ and}$$
$$\psi(h) = \frac{1}{1 + \gamma c} X(h).$$

From Figure 3, we see that, any quasi-PUNE, the Left policy is more egalitarian than the Right policy. Furthermore, each is more egalitarian than the laissez-faire policy, a fact we state as:

**Proposition 6.** Let $b+c=1$. Let $h^2 < h^1$ be two levels of human capital, and let $h^{id}$ be the human capital level of the son of $h^i$, for $i=1,2$ in regime $J$, where $J$ can be $L,R$, or $lf$. 


standing for Left victory, Right victory, or laissez-faire. (The L and R policies are
associated with a given quasi-PUNE.) Then:

\[
\frac{h^{1L}}{h^{2L}} \leq \frac{h^{1R}}{h^{2R}} < \frac{h^{1f}}{h^{2f}} = \frac{h^1}{h^2}.
\]

Proof:

1. We may view figure 3 as a graph of the two educational investment functions,
   \( r^L(\cdot) \) and \( r^R(\cdot) \). Observe this geometric fact: Any chord on the graph of either two
   function cuts the vertical axis above the origin, when extended. (It is crucially
   important that the graphs of these two functions cut the vertical axis above the origin.)

2. Consider the R policy and let the equation of the chord connecting
   \((h^1, r^R(h^1))\) and \((h^2, r^R(h^2))\) be denoted \( r = mh + d \): we know that \( m \geq 0 \) and \( d > 0 \).

Therefore:

\[
\frac{h^{1R}}{h^{2R}} = \frac{\alpha h^i r^R(h^1)c}{\alpha h^i r^R(h^2)c} = \frac{h^i (mh^1 + d)c}{h^2c (mh^2 + d)c} < \frac{h^i}{h^2c} = \frac{h^{1f}}{h^{2f}}
\]

because \( \frac{mh^1 + d}{mh^2 + d} < \frac{h^1}{h^2} \). Therefore

\[
\frac{h^{1R}}{h^{2R}} < \frac{h^1}{h^2},
\]

because \( b+c = 1 \). The rest of the claim is straight-forward. 

If \( b+c < 1 \), the equalizing effects of Left and Right policies are only magnified.
C. Existence of PUNEEP

We know every PUNEEP is a quasi-PUNE. We now show that the set of PUNEEPs is non-empty. To do so, we compute the PUNEEP where each party plays the ideal policy of its militants.

Let $h^*$ be any type, and let $L = [0, h^*)$ and $R = [h^*, \infty)$. Let each party play the ideal policy of its militants. We have denoted these policies $X_{h^*}^L$ and $X_{h^*}^R$. This is clearly a PUNE because the militant factions will not deviate to any other policy. It will, however, generally not satisfy the endogenous party constraint [Definition 1, part (C)]. It will satisfy that constraint exactly when $X_{h^*}^L(h^*) = \mu = X_{h^*}^R(h^*) = x(h^*) + h^*$, where $x(h^*)$ is the number $x$ that solves equation (3.13b). Thus, there is a PUNEEP where both parties play the ideal policies of their militant factions at $h^*$ when the triple $(x, y, h^*)$ solves equations (3.13a), (3.13b), and

$$x + h^* = \mu,$$

simultaneously.

It would be distracting to show that such a solution exists for any $F$. Without proving this, we simply display the solution for $F$ taken as the lognormal distribution with mean 40 and median 30: it is

$$h^* = 39.3739, \quad x(h^*) = .626149, \quad y(h^*) = 171.38.$$

Unfortunately, this is the only PUNEEP whose existence is easy to prove, because it is trivial to observe that neither parties’ militants will accept any deviation.
Hence the set of PUNEEP is non-empty. I conjecture that the set of PUNEEPs is indeed a 2-manifold in the set of quasi-PUNEES, but studying that question is beyond this paper’s scope.

§4 Democratic dynamics
A. Introduction

We now imagine a sequence of overlapping generations, at dates $t = 0, 1, \ldots$. The probability distribution of adult wages at date 0 is $F^0$. Political competition is organized over the questions of taxation and educational investment, and a PUNEEP is realized, inducing a policy lottery. One party wins the election, and its educational investment policy is implemented, giving rise to a distribution of wages at date 1, $F^1$. This process continues forever, inducing a sequence $\{F^t\}$ of wage distributions. We are interested in the asymptotic distribution of human capital.

Over time, it is not reasonable to suppose that $\alpha$ remains constant. We therefore denote its time-dated value by $\alpha^t$. Let $\mu^t$ be the average human capital at date $t$.

The coefficient of variation (CV) of $F^t$ is

$$\eta^t = \int \left( \frac{h}{\mu^t} - 1 \right)^2 dF^t(h). \quad (4.1)$$

We are, in particular, interested in the limit of $\{\eta^t\}$. Does it exist, and if so, is it positive or zero? If it is zero, we say that the distribution of human capital converges to equality. (In that case, given any pair of dynasties, the ratio of levels of human capital of their representatives tends to one with time.)
B. Laissez-faire

Under laissez-faire, $X(h) = h$. The optimizing mother divides her income between household consumption and investment as follows, from the lemma:

$$r(h) = \frac{\gamma c}{1 + \gamma c} X(h), \quad \psi(h) = \frac{1}{1 + \gamma c} X(h).$$

Consequently, her child has human capital

$$h' = \alpha \left(\frac{\gamma c}{1 + \gamma c}\right)^t h^{b+c}.$$

Suppose that $b + c = 1$. Then it follows that the distribution of human capital tomorrow is identical to the distribution today: all human capitals are multiplied by a constant.

Consequently, the coefficient of variation of human capital is constant across time.

If $b + c < 1$, then it follows immediately from the above formula that the ratio of human capitals in any two dynasties approaches one over time, and hence the coefficient of variation of human capital approaches zero.

C. Democracy

We will first work with an altered sequence of distributions, normalized to maintain the mean constantly at $\mu^0$. Define the distribution function

$$\hat{F}'(h) = F'(\frac{\mu^0}{\mu'} h),$$

and let $\hat{F}'$ be the associated probability measure. Then the mean of $\hat{F}'$ is $\mu^0$. Since $\hat{F}'$ has the same coefficient of variation as $F'$, it will suffice to study the coefficients of variation of the sequence $\{\hat{F}'\}$. 
Proposition 6. (a) If \( b + c \leq 1 \), then the distribution function \( \hat{F}^{t+1} \) cuts the distribution function \( F^t \) once from below. That is,

\[
(\exists h')(0 < h < h' \Rightarrow \hat{F}^{t+1}(h) < \hat{F}^t(h) \text{ and } h > h' \Rightarrow \hat{F}^{t+1}(h) > \hat{F}^t(h)).
\]

(b) The sequence \( \{ \eta^t \} \) is monotone decreasing, and hence converges.

Proof.

Part (a). Let \((\psi, r)\) be the PUNEEP at date \( t \). Since the mapping \( h \mapsto \alpha'h^b r(h)^c \) is strictly monotone increasing, mothers and sons occupy the same ranks in their respective wage distributions, that is:

\[
\forall h \quad F^{t+1}(\alpha'h^b r(h)^c) = F^t(h) \tag{4.3}
\]

Hence, from (4.2):

\[
\forall h \quad \hat{F}^{t+1}(\frac{\mu^{t+1}}{\mu^t} \alpha'h^b r(h)^c) = \hat{F}^t(\frac{\mu^t}{\mu^t} \alpha'h^b r(h)^c) \tag{4.4}
\]

Let \( \theta : \mathbb{R}_+ \to \mathbb{R}_+ \) be defined by:

\[
\forall h \in \mathbb{R}_+ \quad \frac{\mu^t}{\mu^0} h \to \frac{\mu^{t+1}}{\mu^0} \alpha'h^b r(h)^c.
\]

Then we may rewrite (4.4) as \( \hat{F}^{t+1}(h) = \hat{F}^t(\theta^{-1}(h)) \), and so \( \hat{F}^{t+1}(h) \geq \hat{F}^t(h) \) as

\[
\hat{F}^t(\theta^{-1}(h)) \geq \hat{F}^t(h) \text{ as } \theta^{-1}(h) \geq h \text{ as } h \geq \theta(h) \text{ as } \frac{h^{1-b}}{r(h)^c} \geq \alpha^* \text{ as } \frac{h^{1-b}}{r(h)^c} \geq \alpha^* \Rightarrow \frac{\mu^{t+1}}{\mu^t} \alpha'h^b r(h)^c.
\]

We next argue that the function \( \zeta(h) = \frac{h^{1-b}}{r(h)^c} \) is strictly increasing on \( \mathbb{R}_+ \), taking on values from zero to “infinity,” which means that

\[
(\exists h')(0 \leq h < h' \Rightarrow \zeta(h) < \alpha^* \text{ and } h > h' \Rightarrow \zeta(h) > \alpha^*).
\]

This will prove part (a).
Suppose Left won the election at date $t$. The graph of $r(h)$ is a multiplicative constant of the graph of $X^L$ pictured in Figure 3. Obviously $\zeta(h)$ is strictly increasing on the intervals $[0,h_L)$ and $[h^*,\infty)$, where $r$ is constant. On the interval $[h_L,h^*]$, we have

$$r(h) = \beta_0 + \frac{\gamma c}{1 + \gamma c} h,$$

where $\beta_0 > 0$. Therefore on this interval

$$\zeta(h) = \frac{h^{1-b}}{(\beta_0 + \frac{\gamma c}{1 + \gamma c} h)^c}.$$

Therefore we have

$$\frac{d}{dh} \log \zeta(h) = \frac{1-b}{h} - \frac{\gamma c^2}{(1 + \gamma c)\beta_0 + \gamma ch}$$

and so

$$\frac{d}{dh} \log \zeta(h) > 0 \iff \frac{1-b}{c} > \frac{\gamma ch}{(1 + \gamma c)\beta_0 + \gamma ch}.$$  (4.5)

Since $\beta_0 > 0$, the r.h.s. of the last inequality is smaller than unity, and hence $\zeta(h)$ is strictly increasing on $[h_L,h^*]$ if $\frac{1-b}{c} \geq 1$. But this means $b + c \leq 1$, which is our premise.

Now suppose that Right won the election at date $t$. Again consult figure 3. Exactly, the same kind of argument shows that $\zeta$ is strictly increasing.

Part (b). Since the sequence $\{\hat{F}^i\}$ is mean-preserving and $\hat{F}^{i+1}$ cuts $\hat{F}^i$ once from below, we have that $\hat{F}^{i+1}$ second-order stochastic dominates $\hat{F}^i$. It therefore follows that the sequence of CVs is monotone decreasing, and therefore converges$^5$.

---

$^5$ We can prove that the sequence of distribution functions $\hat{F}^i$ converges weakly to a limit. But all we need in what follows is the convergence of the sequence of CVs.
We have noted that if $b+c<1$, then, under laissez-faire, the distribution of human capital ‘converges to equality.’ It is easy to show that the same thing happens under democracy, regardless which quasi-PUNE is realized at each date: this follows from the argument in Proposition 6, which shows that the coefficient of variation under democracy decreases at least as fast as under laissez-faire. Thus to compare the performance of democracy and laissez-faire, with regard to equality, for the case $b+c<1$, would require comparing speeds of convergence, a delicate undertaking. Rather than attempting this, we will study, instead, the asymptotic properties of the democratic regime under the assumption that $b+c=1$, for we know that under laissez-faire, there is no change in the CV over time in this case. Thus, laissez-faire provides a clean bench-mark.

Indeed, for simplicity of exposition, we now further assume that $b=c=0.5$, although the argument below holds for any $b \in (0,1), c = 1 - b$.

We will study the asymptotic distribution of human capital for certain special sequences of quasi-PUNEs. Fix a type $h^*>0$, and define the sequence $A(h^*)$ as those quasi-PUNEs which lie on the lower boundary of the manifold $\hat{\Gamma}^t$ at each date $t$, and the pivotal type $h^*_t$ which demarcates the partition of the type space into the two parties is the $t^{th}$ descendent of $h^*$. Let $B(h^*)$ be that sequence of quasi-PUNEs which, at each date $t$, lie on upper boundary of the manifold $\hat{\Gamma}^t$, and where $h^*_t$ is the $t^{th}$ descendent of $h^*$. See figure 2.

Theorem. Let $b=c=0.5$. For any $h^*>0$, the limit CV of the distribution of human capital for the sequence $A(h^*)$ is zero, and the limit CV of distribution of human capital for the sequence $B(h^*)$ is positive.
The sequence $B(h^*)$ is associated with quasi-PUNEs at which both parties play the ideal policy of type $h^*_t$: these are quasi-PUNEs where the opportunists in the two parties are all-powerful. In the sequence $A(h^*)$, in contrast, the militants in the two parties are powerful: in at least one party, at those quasi-PUNEs, the party plays the ideal policy of its militants.

Note that, if $F_t$ is the CDF of the distribution of types at date $t$, for either the sequence $A(h^*)$ or $B(h^*)$, then $F'(h^*_t)$ is constant for all $t$, because the members of a dynasty occupy the same rank in their respective type-distributions forever. Since the probability of Left victory at a quasi-PUNE in either of these sequences is $\pi'(F'(h^*_t))$, this probability is a positive constant over time, in the open interval $(0,1)$. Therefore, in both sequences of quasi-PUNEs, each of Left and Right win elections an infinite number of times. We use this fact below.

Proof of Theorem:

1. We prove the second claim first. Fix $h^* > 0$; without loss of generality, normalize by setting $h^*=1$. At date 0, both parties play the policy defined in Proposition 4, part (c).

We shall, at each date, renormalize so that the descendents of $h^*$ always have one unit of human capital -- that is, we divide all human capitals by the level of human capital of the contemporaneous member of the $h^*$ dynasty. This does not affect coefficients of variation.

Therefore, at date 1, denoting the human capital of the son of $h$ by $S_1(h)$, we have:
where \( X_0^* \) is defined by (3.14). Eqn. (4.9) follows directly from Proposition 4(c), and our normalization procedure.

2. Denote the distribution of human capital at date \( t \) in the sequence \( B(h^*) \) by \( F^t \). Then (Prop.4(c)) we have that the total resource bundle function at date \( t \) is

\[
X^t(h) = \begin{cases} 
X^*_t + h, & 0 \leq h \leq 1 \\
X^*_t + 1, & h > 1,
\end{cases}
\]

where \( X^*_t \) is defined by

\[
X^*_t + \int_0^1 h \, dF^t(h) + (1 - F^t(1)) = \mu^t, \tag{4.10}
\]

where \( \mu^t \) is the mean of distribution \( F^t \). Thus:

\[
S_2(h)^2 = \frac{S_1(h)(S_1(h) + X^*_t)}{1 + X^*_t}, \quad \text{for } h \leq 1
\]

\[
= \frac{h^2 + X^*_0 h}{(1 + X^*_0)(1 + X^*_t)} + \frac{X^*_t S_1(h)}{1 + X^*_t},
\]

using (4.9), where \( S_2(h) \) is the (normalized) human capital of the grandson of \( h \). By induction, for the \( T \)th descendent we have:

\[
\text{for } h \leq 1, \quad S_t(h)^2 = \frac{h^2 + X^*_0 h}{\prod_{j=1}^{T-1} (1 + X^*_j)} + \sum_{t=1}^{T-1} \lambda_t S_t(h), \tag{4.11}
\]

where \( \lambda_t = \frac{X^*_t}{\prod_{j=t}^{T-1} (1 + X^*_j)} \), \( t = 1, \ldots, T - 1 \).
3. Let $0 < h^2 < h^3 < 1$ be two levels of human capital at date 0. If the product

$$\Delta = \prod_{j=1}^{\infty} (1 + X^*_j)$$

converges, then from (4.11), it follows that $S^\omega(h^2)^2 < S^\omega(h^3)^2$, and so the

CV of $F^\omega$ does not converge to zero: that is, the ratio of human capitals of pairs of
dynasties does not converge to unity.

4. (the key step) Thus, to prove the claim, we need only show convergence of the infinite
product $\Delta$. Integrating by parts, note that:

$$\int_0^1 h\,dF'(h) = F'(1) - \int_0^1 F'(h)dh,$$

and so from (4.10) we deduce:

$$X^*_i = \int_1^\infty (1 - F'(h))dh; \quad (4.12)$$

that is, $X^*_i$ is the area ‘above’ the CDF on the interval $[1, \infty)$. (Use the fact that

$$\mu' = \int_0^\infty (1 - F'(h))dh.)$$

By definition of $F^i$ we have:

$$X^*_i = \int_1^\infty (1 - F^i(h))dh = \int_1^\infty (1 - F^i(S_i(h)))dS_i(h) =
\int_1^\infty (1 - F^0(h))dS_i(h),$$

because $F^0(h) = F^i(S_i(h))$ (i.e., members of a dynasty occupy the same rank in their
respective distributions). For $h>1$ we have:

$$S_i(h)^2 = \frac{h(1 + X^*_0)}{1 + X^*_0} = h \quad \text{(from Prop.4(c))},$$

and so $\frac{dS_i(h)}{dh} = \frac{1}{2\sqrt{h}}$. Therefore, continuing the above expansion:
\[ X_t^* = \int_1^\infty (1 - F^0(h)) \frac{dh}{2\sqrt{h}} \leq \frac{1}{2} \int_1^\infty (1 - F^0(h))dh. \]

By induction, it follows that:
\[ X_t^* \leq \frac{1}{2} \int_1^\infty (1 - F^0(h))dh. \]

Therefore \( \sum X_t^* \) converges, and, in particular, \( X_t^* \to 0 \). But note that
\[ \log \Delta = \sum \log(1 + X_t^*), \]
which converges iff \( \sum X_t^* \) converges, because for \( X_t^* \) near zero, \( \log(1 + X_t^*) \equiv X_t^* \).

Therefore \( \Delta < \infty \), as we set out to prove.

5. We next consider the sequence \( A(h^*) \). The lower boundary of the manifold \( \hat{t} \) consists of two segments. On the first segment, the Left party plays the ideal policy of its militants which is the constant function \( X'(h) = \mu' \). On the second segment, the Right party plays the ideal policy of its militants, given by Prop 4(b). If the sequence \( A(h^*) \) spends an infinite number of periods on the first segment, then, since the Left wins an infinite number of times in that subsequence, the CV converges to zero, because the ratio of human capitals in any two dynasties approaches unity. We therefore assume, w.l.o.g., that the quasi-PUNEs in the sequence \( A(h^*) \) always lie on the second segment.

Thus, at date \( t \), the Right plays the policy
\[ X_t^R(h) = \begin{cases} 
  x_t + h, & h \leq y_t, \\
  x_t + y_t, & h > y_t,
\end{cases} \]
where \( (x_t, y_t) \) are defined by the time-dated versions of equations (3.13a, 3.13b), and the Left plays the policy
\[ X^L_i(h) = \begin{cases} 
  x_i + h^L_i, & h \leq h^L_i \\
  x_i + h, & h^L_i < h \leq h^*_i \\
  x_i + h^*_i, & h > h^*_i, 
\end{cases} \]

where \( h^L_i \) is defined by

\[ x_i + h^L_i F(h^L_i) + \int_{h^L_i}^{h^*} h dF(h) + (1 - F(h^*))h^* = \mu^*. \]

6. We next observe that, to show the CVs of \( \{F^i\} \) converges to zero, it suffices to show that the CVs would converge to zero if Left won every election in the sequence \( A(h^*) \).

To see this, suppose that the actual sequence of L/R victories and associated policies is

\[ X_0^L, X_1^R, X_2^R, X_3^L ... \]  

(i)

Now replace the Right policies in this sequence with the laissez-faire policy, which is

\[ X^L (h) \equiv h; \]

thus:

\[ X_0^L, X^L, X^L, X_3^L ... \]  

(ii)

The laissez-faire policy leaves the CV unchanged. Thus, the limit CV of (ii) is the limit CV of:

\[ X_0^L, X_3^L ... \]  

(iii)

that is, of the sequence of Left policies. But the limit CV of (i) is surely no larger than the limit CV of (ii), because the right policy at every date reduces the CV (see Prop. 6).

So if the limit CV of (iii) is zero, so is the limit CV of (i), a fortiori.

7. Therefore it suffices to show that the limit CV of a sequence of Left victories is zero.

8. Consider the graph of the CDF of a distribution function at some date, illustrated in Figure 4, with various areas labeled. Integrating (3.13a) by parts shows that
\[ x = \int_{y}^{\infty} (1 - F(h))dh; \]

that is, area \( D = x \). The mean of the distribution, \( \mu \), is the area above the CDF; that is:

\[
\mu = D + A + H + G + J + B + I, \quad \text{and} \\
h^{*} = H + G + E + J + B + I + C.
\]

Therefore \( \mu - (x + h^{*}) = A - (C + E) \).

But (4.14) says \( \mu - (x + h^{*}) = B - E \), and so, reverting to the time-dated notation:

\[ A' = B' + C'. \quad (4.15) \]

9. We next note that \( A' + D' \to 0 \) with \( t \); this employs the same argument as in step 4 above, because \( X_t^L \) is always a constant function for \( h > h^{*}_t \). Therefore, \( A' \to 0 \), and so from (4.15), \( \lim B' = \lim C' = 0 \).

10. Denote \( r \equiv F'(h^{*}_t) \), so by definition, area \( B' = h^{L}_t(r - F'(h^{L}_t)) \). By step 9, since \( B' \to 0 \), we have that either \( h^{L}_t \to 0 \) or \( F'(h^{L}_t) \to r \). We claim that \( h^{L}_t \) does not approach zero as a limit. Recall that \( x_t \to 0 \) (since \( x_t = \int_{y_t}^{\infty} (1 - F'(h))dh < \int_{1}^{\infty} (1 - F'(h))dh \to 0 \)).

Integrating (3.13b) by parts gives:

\[
\int_{1}^{y_t} \frac{F'(h)}{x_t + h^{*}} dh = \frac{F'(h^{*}_t)}{x_t + h^{*}},
\]

and therefore \( \lim_{t \to \infty} \int_{1}^{y_t} \frac{F'(h)}{h^{2}} dh = r \).

But \( \int_{1}^{y_t} \frac{F'(h)}{h^{2}} dh < \int_{1}^{y_t} \frac{1}{h^{2}} dh = 1 - \frac{1}{y_t} \), and so \( \lim y_t \geq \frac{1}{1 - r} > 1 \). Letting \( y^{*} = \lim y_t \), we therefore have that in the limit the total resource bundle function proposal by Right, in the sequence \( A(h^{*}) \) is:
If $h_t^r \to 0$, then in the limit the total resource bundle function proposal of Left is:

$$X^R_\infty(h) = \begin{cases} h, & \text{for } h \leq y^* \\ y^*, & \text{for } h > y^*. \end{cases}$$

Now both $X_t^L$ and $X_t^R$ must integrate to $\mu'$, which we next show gives a contradiction:

$$0 = \lim_{0}^{\infty} (X_t^R(h) - X_t^L(h)) dF'(h) = \lim_{y^*}^{\infty} \int_1^h dF'(h) + (1 - F'(y^*)) (y^* - 1) = \lim_{y^*}^{\infty} \left[ h dF'(h) - \int_1^h F'(h) dh + (1 - F'(y^*)) (y^* - 1) \right]$$

$$= (y^* - 1) + \lim_{y^*}^{\infty} (F'(y^*)) - r - \int_1^{y^*} F'(h) dh. \quad (4.16)$$

Since $y^* > 1$, $\lim F'(y^*) = 1$. (Recall, Left policies squeeze all $h > 1$ eventually to 1.) And

$$\lim_{y^*}^{\infty} \int_1^{y^*} F'(h) dh \leq \int_1^{y^*} dh = y^* - 1.$$ 

Therefore, (4.16) says that

$$0 = (y^* - 1) + (1 - r) - (y^* - 1),$$

where the last term on the r.h.s. is a number no larger than $y^* - 1$. But this is impossible.

The contradiction demonstrates that $\lim h_t^L \neq 0$, and so it follows that $\lim F'(h_t^L) = r$.

11. There are now two cases to consider: either

(a) 1 is not a limit point of $\{h_t^L\}$, or

(b) 1 is a limit point of $\{h_t^L\}$.

In case (b), for large $t$, the policy $X_t^L$ is arbitrarily close to giving all types the same total resources, and so almost the same is spent on the education of all children, and so the
limit CV of the human-capital distribution is zero. We will therefore complete the proof of the theorem by showing that case (a) cannot occur.

Suppose, then, case (a). Then for large $t$, $h_t^L$ is bounded away from 1 from below. In particular, there exist types $h^2 < h^1 < 1$ such that, for all large $t$, $h_t^L < h^2$. For such $t$, we have:

$$S_i(h^i) = \frac{h^i(h^i + x_t)}{1 + x_t}, \quad i = 1, 2.$$

It follows, again by application of the argumentation of step 4, that in the limit, the human capital possessed by the descendents of $h^2$ and $h^1$ are different, that is:

$$S_\infty(h^2) < S_\infty(h^1).$$

Thus, there is an interval of measure $F^0(h^1) - F^0(h^2)$ that lies between the largest limit point of the sequence $\{h_t^L\}$ and 1. This is impossible, since $\lim F^i(h_t^L) = \lim F^i(1) = r$. The contradiction establishes that, indeed, case (b) holds, which proves the theorem.

§5 Topping Off

We have assumed until now that educational funding is purely public. But winning publicly financed education has been itself a significant victory of democracy. So it would have been more convincing to begin with the supposition that education could be privately or publicly financed.

First, note that at any quasi-PUNE, under our assumptions, no household will desire to top off public education with additional private education, because every quasi-PUNE partitions the household’s total resource bundle just as the optimizing household would. So there will be no demand for further private education at these equilibria.
Now suppose that it is not assumed, initially, that education will be publicly financed. Thus, a party may propose a policy \((\psi, r)\) assuming that citizens will top off privately, if \(r(h) < \frac{\gamma c}{1 + \gamma c} X(h)\). Thus, the \(h\)-household facing the policy \((\psi(h), r(h))\) solves for its private educational investment, which we denote \(r^P(h)\):

\[
\begin{align*}
\max_{r^P(h)} \log(\psi(h) - r^P(h)) + \gamma c \log(r(h) + r^P(h)).
\end{align*}
\]

The solution is

\[
r^P(h) = \frac{\gamma c \psi(h) - r(h)}{1 + \gamma c}.
\]

Then

\[
\begin{align*}
\psi(h) - r^P(h) &= \frac{X(h)}{1 + \gamma c}, \text{and} \\
r^P(h) + r(h) &= \frac{\gamma c X(h)}{1 + \gamma c}.
\end{align*}
\]

Without loss of generality, we may therefore write the household’s indirect utility function as

\[
v(\psi, r; h) = \log \frac{X(h)}{1 + \gamma c} + \gamma c \log \frac{\gamma c X(h)}{1 + \gamma c} \equiv \log X(h).
\]

Now each party takes account of the fact that its members will top off, if need be, and so we may write the program of the Left party (for instance) at \(h^*\) as:

\[
\begin{align*}
\max_{\psi, r} & \int_0^{h^*} \log X(h) \, dF(h) \\
\text{s.t.} & \ 0 \leq X'(h) \leq 1 \\
& \int_0^{h^*} X(h) \, dF(h) \leq \mu \\
& \log X(h^*) \geq y^*.
\end{align*}
\]
It is thus clear that the set of quasi-PUNEs where private financing of education is not precluded is isomorphic to the set of quasi-PUNEs where only public financing is possible. It is a matter of indifference whether education is publicly funded or whether households finance some or all education privately: the children receive identical educational investments in both cases.

In other words, our model is not constructed to elucidate why publicly financed education is an almost ubiquitous institution of advanced democracies.

§6 Discussion

In our hypothetical laissez-faire benchmark society, the coefficient of variation (CV) of the distribution of human capital stays constant over time, when the educational production function exhibits ‘constant returns to scale.’ Under democracy, the CV of that distribution decreases monotonically; whether it decreases to zero depends, in our model, on the nature of intra-party struggle. If militants are relatively powerful, then the limit distribution is one of perfect equality. If the opportunists are relatively powerful, it is not. One might paraphrase by saying that, to the extent that democratic politics is ideological (that is, where militants dominate), then democracy engenders equality, but if democratic politics become dominated by political entrepreneurs (whose goal is to remain in office--hence, maximize the probability of victory), then equality is not achieved in the limit. If there are decreasing returns to scale \((b+c < 1)\), then democracy and laissez-faire both produce equality in the long-run, and we suggest that our results translate into statements about relative speeds of convergence to equality.
It is worth noting that our conclusion is very different from what Downsian intuitions would suggest. In a Downsian model, both parties propose the median voter’s ideal policy (of course, on a unidimensional policy space). If median human capital is less than average human capital, then the median voter advocates a policy of complete leveling; the intuition would therefore be that Downsian politics produces equality. In our model, however, if Downsian political actors (the opportunists) dominate, then equality is not achieved. We propose that this difference shows the pitfalls of the unidimensional Downsian analysis, and underscores the point that large-dimensional policy spaces are not simply a mathematical nicety.

Why have we not observed more rapid convergence to equality of wages in advanced democracies? Besides the fact (if it is one) that politics are dominated by opportunists, a number of reasons can be suggested -- reasons which take the form of divergence from the premises of our model. These include:

- random talent or effort
- technological shocks
- imperfectly representative democracy
- elastic labor supply
- non-economic issues

We take these up in turn.

In the model, the human capital of the son is a deterministic function of the human capital of the mother and educational investment. In reality, sons from families with similar mothers differ according to their talent and their effort, which we could model by inserting a stochastic multiplicative term in the educational production
function. Doing this does not complicate the analysis very much. Statements about convergence to equality in the deterministic model become translated into analogous statements about the non-persistence of the effect of initial conditions on the wages of distant descendents. That is, in the model with stochastic talent, in the sequence of PUNEs that lie on the lower boundary of the manifold, it is the case that there is eventually no influence of the wage of the initial mother (Eve) on the wages of members of her dynasty, whereas, for the sequence of PUNEs on the upper boundary of the manifold, there is persistence. Thus, ‘convergence to equality’ becomes translated into ‘equality of opportunity,’ in the sense that the socio-economic status of an individual’s family origin eventually has no influence on his own level of human capital.

In our model, the only kind of technological change allowed was neutral, in the sense of time-dating the constant $\alpha$. In reality, technological change is often non-neutral. This is the case with the shock to the educational production function in the US and UK in the last twenty years. Clearly, if non-neutral technological change is historically important, it can upset the process of convergence to equality.

Our model assumes that every citizen is a member of a party, and that parties aggregate the interests of their members in an unbiased way: these two premises constitute the assumption of ‘perfectly representative democracy.’ There are very few countries today where an approximation of these premises holds, and, in every case, this ideal has only existed for at most two generations.

Because we have assumed an inelastic labor supply, parties put forth policies which involve 100% marginal tax rates on certain intervals of the income distribution.
With elastic labor supply, this will not happen, and convergence to equality will be retarded, if not eliminated.

Non-economic issues, especially concerning racial and ethnic questions, are politically salient in many countries, and these issues can retard redistribution. Thus, poor natives (or whites) of a country may vote for the party of the Right because that party opposes immigration (or income redistribution to minorities). There is reason to believe that American racism can explain a large part of the difference in the degrees of redistribution between the US and Europe. (See Roemer [1998] and Lee and Roemer [2002].) But, in our model, citizens are presumed to have only economic interests.

On the other hand, there are features of reality, not present in the model, that render reality more prone to equalization than the model: principally, citizens do have some degree of ‘altruism,’ which is to say, concern for the children of others. Even without altruism, because education is to some degree a public good — parents want the children of others to be educated because that will enhance the welfare of their children — convergence to equality will be at least as rapid as in our model.

Thus, our model should be viewed as asking the question: In an ideal type of democracy, where citizens are interested only in their own dynasty, and education is a private good, will competitive politics induce equalization of human capital? Clearly this is only the beginning of a thorough analysis.

In the United States, funding for public education of $h$-households does increase with $h$: this is accomplished through the linking of educational finance with the local property tax base. In the political equilibria of our model, this is the case — that is, both parties propose policies $r(h)$ that are everywhere non-decreasing in $h$, and increasing in $h$
in some intervals. In many European countries, equal public educational investment in children of all backgrounds is closer to the truth. Section 5 tells us that, in these equilibria, rich parents will top off the public investment in their children. I conjecture that, at least in the Nordic countries, this topping-off does not occur. We may understand this as the consequence of the operation of another social norm – not one we have modeled here. There is, however, an alternative explanation, that the education of other people’s children is a public good.

Our theorem has an implication for a debate in democratic theory. Democratic theorists are divided into two groups, according to whether they define democracy in a minimalist or maximalist fashion. The minimalist view (see, for example, Przeworski et al (2000)) is that democracy is best conceived as a system with political competition between parties, *tout court*. The maximalist version frequently goes by the name of deliberative democracy (see, for instance, Elster (1998)); here democracy requires as well as political competition, a thorough-going discussion among citizens – a forum – at which citizens convince each other to take account of their mutual needs. Maximalists tend to think that political competition alone will not suffice to bring about a decent society (read: equality or justice).

Our analysis tends to support this conclusion: something besides democracy is needed to guarantee convergence to equality.

As a final exercise, we report the results of a simulation. In the economy simulated, \( b=c=0.5=\gamma \), and we begin with the lognormal distribution whose mean and median are 40 and 30, respectively -- this looks like the distribution of income in the US, in thousands of dollars, in the 1990s. We take as the quasi-PUNE realized the one with
at the median of the distribution at each date, which means each party represents exactly one-half the population, and which is located half-way between the upper and lower boundaries in the manifold of quasi-PUNEs – thus, we attempt to capture a political system where the opportunists and militants each have some bargaining power inside the parties.

We present some results in Table 1 and Figure 5. At each date, each party wins with probability one-half; there is a different sequence of L-R victories in the nine simulations displayed in the table, depending on the realization of this random variable. We see that the CV of the distribution of human capital appears to converge very rapidly to zero in the six generations of our simulations\(^6\). (Our theorem, however, does not tell us whether, in fact, convergence to zero occurs. It may not.)

Thus, we see that the convergence to equality – or at least to very low levels of inequality -- which occurs at certain quasi-PUNEs, even when \(b+c=1\), is very dramatic in the model. Ruthlessly competitive politics are radically different from laissez-faire.

\(^{6}\) I chose a small number for \(\alpha\), and so there is economic contraction over time in the table. We know that the size of \(\alpha\) is irrelevant as far as the sequence of CVs is concerned.
Appendix

Proof of Proposition 4

We prove part (b). Proofs of the other parts are somewhat simpler, and of the same character. Our task is to solve for the ideal policy of the Right militants:

\[
\max_{\psi} \int_{h^*}^\infty \log X(h)dF(h)
\]

s.t. \(0 \leq X'(h) \leq 1\)

\[
\int_0^\infty X(h)dF(h) \leq \mu.
\]

The solution is shown in the figure 1, where \((x, y)\) is the simultaneous solution of the following two equations:

\[
x + \int_0^y h dF(h) + y(1 - F(y)) = \mu \quad (A1)
\]

\[
\int_{h^*}^y \frac{dF(h)}{h + x} = \frac{F(y)}{x + y}. \quad (A2)
\]

(A1) says that the function \(X\) integrates to \(\mu\), as required; (A2) fixes a particular pair \((x, y)\).

Denote this policy by \(X^*\).

Define the function

\[\text{Proof of Proposition 4}^7\]

We construct a proof that is ‘elementary,’ in the sense of not requiring any knowledge of optimal control theory or the calculus of variations.
\[ \Delta(\varepsilon) = \int_{h^*}^{\infty} \log(X^*(h) + \varepsilon g(h))dF(h) + \int_0^{h^*} \lambda(h)(1 - (X^*(h) + \varepsilon g(h))')dh + \delta(\mu - \int_0^{\infty} (X^*(h) + \varepsilon g(h))dF(h)). \]

I will produce a non-negative function \( \lambda \) and a positive number \( \delta \) such that, for any function \( g \), \( \Delta \) is maximized at \( \varepsilon = 0 \). In particular, it follow that \( \Delta(0) \geq \Delta(1) \).

The second and third terms on the r.h.s of the definition of \( \Delta \) vanish at \( \varepsilon = 0 \). This will thus imply that

\[ \int_{h^*}^{\infty} \log X^*(h)dF(h) \geq \int_{h^*}^{\infty} \log (X^*(h) + g(h))dF(h), \]

for any variation \( g \), proving the claim.

Note that \( \Delta \) is a concave function\(^8\). It therefore suffices to show that \( \Delta'(0) = 0 \).

Define \( \lambda \) and \( \delta \) as follows, where \( f \) is the density of \( F \) and \((x,y)\) are defined above:

(i) \( \lambda(0) = 0 \),

(ii) \( \lambda'(h) = \delta f(h) \) on \([0,h^*]\),

(iii) \( \lambda'(h) = \delta f(h) - \frac{f(h)}{h + x} \) on \([h^*,y]\),

(iv) \( \lambda(y) = 0 \), and

(v) \( \delta = \frac{1}{x + y} \).

We must show that (i)-(iv) are consistent, given (v). Note that \( \lambda' \geq 0 \) on \([0,h^*]\) from (ii) and \( \lambda' \leq 0 \) on \([h^*,y]\) from (iii) and the definition of \( y \). From (i), (ii) and (v):

\[ \lambda(h^*) = \int_0^{h^*} \delta f(h)dh = \frac{F(h^*)}{x + y}. \]

From (iii):

\(^8\) Here is where we exploit the fact that the optimization program is convex.
\[ \lambda(y) - \lambda(h^*) = \int_{h^*}^{y} f(h) \left( \frac{1}{x + y} - \frac{1}{h + x} \right) dh. \]

Therefore (iv) is true if

\[ 0 - \frac{F(h^*)}{x + y} = \int_{h^*}^{y} \left( \frac{1}{x + y} - \frac{1}{h + x} \right) f(h) dh = \frac{F(y) - F(h^*)}{x + y} - \int_{h^*}^{y} \frac{dF(h)}{h + x}, \]

which is true if and only if:

\[ \int_{h^*}^{y} \frac{dF(h)}{h + x} = \frac{F(y)}{x + y}. \]

But the last equation is true by definition of \((x,y)\).

Thus the function \(\lambda\) is well-defined and non-negative on its domain, as required.

We now differentiate \(\Delta\), where \(g\) is an arbitrary, differentiable function:

\[
\Delta'(0) = \int_{h^*}^{y} g(h) dF(h) - \int_{0}^{y} \lambda(h) g'(h) - \delta \int_{0}^{y} g(h) dF(h) \\
\]

\[
= \int_{h^*}^{y} \frac{g(h) dF(h)}{h + x} + \int_{0}^{h^*} \frac{g(h) dF(h)}{x + y} + \int_{0}^{y} \lambda'(h) g(h) dh + \int_{h^*}^{y} \lambda'(h) g(h) dh - \lambda(h) g(h) \bigg|_{0}^{y} - \\
\delta \int_{0}^{h^*} g(h) dF(h) - \delta \int_{0}^{h^*} g(h) dF(h) \\
\]

where we used integration by parts,

\[
\]

\[
= \int_{h^*}^{y} \left( \frac{f(h)}{h + x} + \lambda'(h) \right) g(h) dh + \left( \frac{1}{x + y} - \delta \right) \int_{y}^{\infty} g(h) dF(h) - \delta \int_{h^*}^{y} g(h) dF(h) + \int_{h^*}^{y} \lambda'(h) - \delta f(h) \right) g(h) dh - \\
- \lambda(y) g(y) + \lambda(0) g(0) \\
\]

\[
= \int_{h^*}^{y} \left( \frac{f(h)}{h + x} + \lambda'(h) - \delta f(h) \right) g(h) dh + \left( \frac{1}{x + y} - \delta \right) \int_{y}^{\infty} g(h) dF(h) + \int_{0}^{h^*} (\lambda'(h) - \delta f(h)) g(h) dh + 0 \\
\]

= [by definition of \( \lambda' \) and \( \delta \)] 0,

as was to shown. \( \blacksquare \)
References


Figure 1 The policies of Proposition 4(b) and 4(c)
Figure 2 The manifold of quasi-PUNEs
Figure 3  Left (bold) and Right policies in a quasi-PUNE (Prop. 5)
Figure 4  Aid in the proof of main theorem
Figure 5a  CDF of human capital after five Right victories: simulation

This is generation 5 after five Right victories

CDF
Figure 5b CDF of human capital after five Left victories: simulation

This is generation 5 after five Left victories
Table 1  Coefficients of variation in six-generation simulations
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<th>median</th>
<th>cvar</th>
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<td>29.4005</td>
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