Optimal and time-consistent fiscal policy with international mobility of capital: Why does the U.S. tax capital more than Europe?

Paul Klein, Vincenzo Quadrini and José-Víctor Ríos-Rull*

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Abstract

The United States taxes capital more than countries in continental Europe do. In this paper, we ask what can account for this. Our approach is to look at Markov perfect equilibria of a two-country growth model where both governments use labor, capital and corporate taxes to finance exogenously given streams of public expenditure under period-by-period balanced budget constraints. There is no commitment technology and the equilibrium policies are time-consistent. We find that differences in productivity and size, and government spending can account for the heavy American reliance on capital taxation.

*Klein: Institute for International Economic Studies, Stockholm University, 106 91 Stockholm, Sweden, (e-mail: paul.klein@uwc.se); Quadrini: Department of Economics, Stern School of Business, New York University, New York, NY 10012, and CEPR (e-mail: vquadrin@stern.nyu.edu); Ríos-Rull: Department of Economics, 3718 Locust Walk, University of Pennsylvania, Philadelphia, PA 19104, CEPR, NBER, and IAERP. (e-mail: vr0j@econ.upenn.edu). We thank Per Krusell, Enrique Mendoza, Martin Uribe, and, especially, Stephanie Schmitt-Grohé and the attendants to seminars at Duke University, Cemfi, CEPR conference on Optimal Taxation in Tilburg and the 2000 Hydra Macroeconomics Conference on the cost of business cycles. Ríos-Rull thanks the National Science Foundation, the University of Pennsylvania Research Foundation and the Spanish Ministry of Education.
The United States and the United Kingdom tax capital more, and labor less, than their major European competitors. Mendoza, Razin, & Tesar (1994) estimate that in the 1980’s the average capital income tax rates in the United States were about 42 percent and in the United Kingdom 57 percent. Corresponding numbers for France, Germany and Italy—the largest countries in continental Europe—lie between 24 and 27 percent. At the same time, labor taxes are unambiguously higher in continental Europe. The average labor tax rates in France, Germany and Italy are, respectively, 40, 37 and 39 percent while in the United States and in the United Kingdom they are about 25 percent.

The goal of this paper is to explore whether the differences in the structure of taxation can be explained by some of the differences between these two sets of countries. In particular, we explore the following factors: (i) The U.S. and the U.K. have lower government expenditures as a percentage of GDP; (ii) the economic size of the U.S. (but not the U.K.) is larger than each individual country in continental Europe. The difference in the size of the economy derive from both a larger population and a higher total factor productivity.

These differences are studied in a two-country open economy model in which (i) capital is internationally mobile but labor cannot move to the other country; (ii) governments can resort to corporate and capital income taxation, in addition to labor taxes; (iii) they choose these taxes so as to maximize the welfare of their citizens but they are unable to commit to future policies. Within this modeling framework, we show that both factors contribute to generating a higher taxation of capital income (relative to the taxation of labor income) in the U.S. than in continental Europe. Moreover, the first factor—higher government spending—also contribute to generating a higher taxation of capital in the U.K. than in continental Europe. More generally, we show that the government of a country with lower government spending and a larger economy has an incentive to tax capital income proportionally more than labor income. The international mobility of capital is crucial for the second factor (the size of the economy) to have an impact on the taxation structure of a country while the first factor (government size) is important independently of the international mobility of capital.

The fact that in international settings the implications of government policies are affected by other governments actions has attracted the attention of economists for a long
time. A survey can be found in Persson & Tabellini (1995). The consequences of policy competition has always been analyzed in non-quantitative simple theoretical settings with identical countries without explicit attention to the situation in which the interacting countries differ from each other. By contrast, in this paper we explore the quantitative implications for the tax structure when countries are heterogeneous in government spending and size of the economy. The paper is related to previous works that study, in closed economy models, how inequality affects the voting preferences of agents and the endogenous determination of policies (Krusell, Quadrini, & Ríos-Rull (1996, 1997), Krusell & Ríos-Rull (1999)), and the optimal policy of benevolent governments (Klein & Ríos-Rull (1999)).

The organization of the paper is as follows. Section 2 describes the main differences in government spending and taxation between the U.S and the European countries. After the presentation of these stylized facts, section 3 describes the theoretical model and section 4 defines the political equilibrium. Section 5 parameterizes the baseline economy characterized by two countries sharing the main macro aggregates of the U.S and section 6 describes the equilibrium properties of the model. This section also performs a sensitivity analysis with respect to some of the parameters. In section 7 we examine how the three differences between the U.S. and Europe affect the equilibrium taxes, both by looking at each of them in isolation and all them in unison. Finally, section 8 concludes.

2 Some key facts about the U.S. and Europe

2.1 Tax structure

As stressed in the introduction, the United States and the United Kingdom tax capital more, and labor less, than their major European competitors. According to the estimates of Mendoza et al. (1994), in the 1980:s the average capital income tax rates in the United States are about 42 percent and in the United Kingdom 57 percent. Corresponding numbers for France, Germany and Italy—the largest countries in continental Europe—are between 24 and 27 percent.¹

¹King & Fullerton (1984) report more ambiguous numbers for the taxation of capital. According to them, Germany taxes capital at a rate in the range of 44 to 68 percent, the corresponding range for the United States being 32 to 49 percent.
An alternative way to proceed would be to impute differences in the capital output ratio of different countries to differences in the capital income tax rates. When we do this with standard methods, we obtain an implied taxan on anban
the assumption of a Cobb-Douglas production function for all countries, we find that in 1990 total factor productivity in Germany is 98 percent of that of the United States, with Italy and especially France lagging behind at 94 and 80 percent, respectively. The numbers for 1980 are similar, with Italy achieving 99 percent of the total factor productivity of the United States, Germany 91 percent and France only 81 percent. It is safe to conclude that the United States is more productive than its major European competitors.

3 The Economic Model

The model is a two-country, $i \in \{1, 2\}$, neoclassical growth model with competitive markets extended to include a government sector with distortionary taxation, and imperfect within period capital mobility. The government of each country faces the problem of optimally financing an exogenous stream of public spending without being able to commit to policy plans for a long future. We start by describing the model given certain policy rules followed by the governments. Afterwards, we describe how these policy rules are determined. A crucial element of this environment is that governments cannot commit to future policies. We will be looking for time consistent Markov policies, rather than the time inconsistent Ramsey policies.

There are two good reasons for focusing on time–consistent as opposed to Ramsey policies. One is realism in the assumption: it is hard to believe that actual governments are able to commit into the indefinite future. The other is realism in the prediction: the steady state of the Ramsey equilibrium features zero capital taxes, which is not at all what we observe in reality.

Preferences: In each country there is a continuum of consumers of total mass $\mu$ with standard preferences over streams of consumption and leisure. The consumer’s objective is $\sum \beta u(c, h)$, where $c$ and $h$ are consumption and hours worked in country $i$ at time $t$. Households cannot change their country of residence, which makes labor immobile.

Production Possibilities: In each country there is a standard neoclassical technology represented by the production functions $F(Y, L)$. The variable $Y$ is the amount of capital employed in country $i$, and $L$ is the total number of efficiency units of labor employed in country $i$. Note that technologies can differ across the two countries which
justifies the index $i$ in the production function.

Output of each country is perfectly mobile which implies the following world-wide feasibility constraint:

$$\sum_{i=1}^{2} \mu_i (C_i + I_i + G_i) = \sum_{i=1}^{2} \mu_i F_i (Y_i, L_i) \quad (1)$$

where $C_i$ is consumption, $I_i$ investment, and $G_i$ government purchases in country $i$. All variables are expressed in per-capita terms. The law of motion for the capital stock is

$$K_i = (1 - \delta) Y_i + I_i \quad (2)$$

where $K_i$ is the capital that is installed in country $i$ for next period and $\delta$ is the depreciation rate.

The reason for denoting by $K_i$ the beginning of next period’s installed capital, and by $Y_i$ the capital effectively used, is that capital is imperfectly mobile across countries within a period. There is a technology that shuffles capital at a cost. This technology can be written as $T(K_{1,t-1}, K_{2,t-1}, Y_1, Y_2) = 0$. We can interpret the function $T$ as stating which combinations of usable capital $\{Y_1, Y_2\}$ are possible starting with installed capital $\{K_{1,t-1}, K_{2,t-1}\}$. Not changing installed capital is obviously feasible, i.e., $T(K_{1,t-1}, K_{2,t-1}, K_{1,t-1}, K_{2,t-1}) = 0$. We assume that the implicit function $T$ is homogeneous of degree zero (it displays constant returns to scale) and it is strictly convex to reflect increasing adjustment costs. Given these properties, we write the re-installation technology more compactly as $T(x, y_1, y_2) = 0$, where $y_1$ and $y_2$ denotes capital rented in countries 1 and 2 respectively by a firm that owns one unit of capital and that has a fraction $x$ of this unit initially allocated in country 1 and a fraction $1 - x$ allocated in country 2. This technology will give the government some power to tax installed capital, but this power is limited since capital could fly. Figure 3 shows the typical shape of the $T$ function for a situation where half the capital is installed in each country.

We refer to the firms that buy and install capital and later may reallocate it, as investment firms. We refer to the firms that operate the production technologies as production firms. Notice that the former face a dynamic problem (where to install the
Figure 3: Capital Substitutability, $T(x, y_1, y_2) = x - \gamma y_1 + (1 - x) - y_2 - 1$, $x = .5$, $\gamma = .2$

capital) while the problem faced by the latter is static.²

The Government: Public expenditures are composed of government purchases, denoted by $G$, and government transfers, denoted by $T$. These variables are exogenous in the model and they are constant over time. To finance public expenditures, the governments of the two countries have three sources of revenue. In the first place, they tax the capital rents paid by domestic firms at rate $\zeta$. Secondly, they levy taxes on the capital income received by domestic households at rate $\theta$. Finally, households pay taxes on their labor income at rate $\tau$.

The taxation of capital rents includes all taxes that are paid on the non-labor income generated by firms. Corporate taxes are part of this form of taxation and in the rest of the paper we simply refer to $\zeta$ as the corporate tax rate. It should be remembered, however, that this tax includes all forms of capital income taxation, not only corporate taxes, based

²Alternatively, we could assume that households directly decide where to allocate the capital, without the need of the investment firm. The model would be equivalent.
on the location of the production factor (source principle) as opposed to the residency of the recipients (residence principle). The main difference between the corporate tax rate, $\zeta$, and the capital income tax rate, $\theta$, is that the first is paid by the firm, while the second is paid by the households. Without international mobility of capital the two forms of taxation would be economically indistinguishable. With international mobility of capital, however, they have different economic consequences: if a country heavily relies on foreign capital, the corporate tax is a more effective source of tax revenue than the capital income tax as some of the capital income generated by domestic firms is paid to foreign residents.

In each period the government chooses the current labor and corporate income tax rates and the next period capital income tax rate. Therefore, the current capital income tax rate is inherited from the previous period while the current corporate and labor tax rates are decided in the current period. The one-period commitment to the capital income tax rate is necessary to prevent an excessive taxation of capital income received by households. This is because households’ income from capital (and households’ wealth) can be taxed without distortions in the current period. This is in line with the optimal taxation literature (Chari, Christiano, & Kehoe (1994), Judd (1987), Klein & Ríos-Rull (1999), Stockman (1998)). This is not necessary, however, for corporate taxes because the mobility of capital discourages the two governments from excessively taxing the capital income generated by domestic firms (households cannot change their residency but they can reallocate their assets in the foreign country through the investment firms).

Governments in both countries are subject to a period by period balance budget constraint, that is,

$$G + T = \zeta (p - \delta)Y + \tau w H + \theta r A, \quad i = 1,2. \tag{3}$$

where $p$ are the capital rents paid by domestic firms per unit of rented capital, $Y$ the total unit of rented capital, $w$ the wage rate, $H$ the aggregate labor supply, $r$ the unit return of households’ wealth and $A$ the aggregate households’ wealth.
4 Recursive representation and policy equilibrium

The analysis is limited to Markov stationary policy rules, where a policy rule is a function that returns the current labor and corporate tax rates and the next period capital income tax rates for both countries, as a function of the current states of the economy. We will denote this function by $\phi(S)$, where $S$ is the vector of aggregate states.

The aggregate state variables include the per-capita wealth held by the households of the two countries, $A_1$ and $A_2$, the aggregation of which is equal to the total amount of capital installed in both countries. Moving capital is costly, so another aggregate variable is needed to describe how much capital is installed in each country. We will use the variable $X$ to denote the fraction of worldwide capital that is installed in country 1. Finally, the economy inherits capital income tax rates as the governments set this rate in advance. Therefore, $S = \{\theta_1, \theta_2, A_1, A_2, X\}$.

The main goal of this section is to define a policy rule $\phi(S)$ that is time-consistent (governments do not have incentive to deviate from this rule) when the two governments choose their policy instruments (tax rates) independently on a competitive basis. This requires three steps. The first step (section 4.1) defines the economic equilibrium for arbitrary policy rules $\phi$. This step is important because allows us to determine the welfare levels of the two countries for arbitrary future policy rules. The second step (section 4.2) defines the optimal equilibrium tax rates in the current period when future tax rates are determined by some arbitrary policy rule. Because the optimal current tax rates depend on the current states, this step defines the optimal current policy rule, given future rules. Finally, the third step (section 4.3) defines the conditions for which the governments will not deviate from the rules assumed for the future (time-consistency). This will be the case if the policy rule assumed for the future is equal to the rule that is optimal in the current period (policy fixed point).

4.1 Step 1: Equilibrium for arbitrary policy rules

Assume that the tax rates in the two countries are determined by an arbitrary function $\phi$, that is, $(\tau_1, \tau_2, \theta_1, \theta_2) = \phi(S)$. Once this function is specified, the agents’ problems are well defined. Following is the description of the problems.
The households’ problem: We define the household’s problem recursively. The state variables for the household are the aggregate states as defined above plus its own wealth that we denote by \( a \) (individual state). The dynamic programming problem is:

\[
V(S, a; \phi) = \max_{c, h} \{u(c, h) + \beta V(S', a'; \phi)\} \tag{4}
\]

s.t.

\[
c = (1 - \tau_i) w_i h + (1 + r(1 - \theta_i)) a + T - a' \tag{5}
\]

\[
r = r(S; \phi) \tag{6}
\]

\[
w = w(S; \phi) \tag{7}
\]

\[
\tau = \phi(S) \tag{8}
\]

\[
\theta' = \phi(S) \text{ for } j = 1, 2 \tag{9}
\]

\[
A' = \Phi(S; \phi) \text{ for } j = 1, 2 \tag{10}
\]

\[
X' = \Phi(S; \phi) \tag{11}
\]

where \( w \) is the wage rate in country \( i \) and \( r \) is the rate of return on its assets. The functions \( \Phi \) and \( \Phi \) define the law of motion for the aggregate states \( A \) and \( X \).

The solution for this problem is given by the working hours, \( h = g(S, a; \phi) \), and the next period claims to the capital of the investment firms, \( a' = g(S, a; \phi) \). Obviously, this problem can only be solved once the functions in equations (6)-(11) are specified. Note that we are indexing these equations with policy functions \( \phi \). We do so to be explicit about the fact that they are equilibrium functions.

The investment firms’ problem: For an investment firm, the individual state variables are given by the total stock of capital they invested in the previous periods and the location (in country 1 and 2) of this capital. We use the pair \((k, x)\) to denote the individual states of the investment firm, where \( k \) is the capital owned by the firm and \( x \) the fraction of this capital initially installed in country 1.

Given the initial location of the owned capital, \( x \), the firm decides how to change this location. At the end of the period, the firm returns all the proceeds to the shareholders. Therefore, there are two stages in which the firm makes decisions. At the end of each period it decides where to allocate the new capital between the two countries. At this stage, the firm has full flexibility in choosing the location of its investment (installation).
At the beginning of the next period, starting with this allocation of capital, the firm decides how to reallocate it. At this stage, the reallocation of capital is not completely flexible but it depends on the technology $T$.

Given the constant return-to-scale property of the investment technology, the choices of the investment firm are independent of its scale. Therefore, without loss of generality, we consider the problem of a firm that owns (has purchased) one unit of capital. Let’s start with the problem solved at the beginning of the period, when the firm starts with the allocation $x$. Let $\Omega(S, x; \phi)$ be the value of such a firm. The value of a firm with $k$ units of capital is simply given by $k \cdot \Omega(S, x; \phi)$. The problem that this firm solves is:

$$\Omega(S, x; \phi) = \max_{\{1, 2\}} \left\{ \sum_{i=1}^{2} \left[ 1 + (p(S; \phi) - \delta)(1 - \zeta) \right] y \right\}$$

s.t. $T(x, y_1, y_2) = 0$

where $p$ is the rental rate of capital in country $i$. These rents are considered profits in country $i$ and they are taxed at rate $\zeta$. The first order conditions of this problem give:

$$\frac{1 + [p_1(S; \phi) - \delta](1 - \zeta_1)}{1 + [p_2(S; \phi) - \delta](1 - \zeta_2)} = \frac{T_2(x, y_1, y_2)}{T_3(x, y_1, y_2)}$$

where $T_2$ and $T_3$ are the derivatives of function $T$ with respect to their second and third arguments respectively. Given the constraint $T(x, y_1, y_2) = 0$, the first order conditions give the pair of solution functions for the allocation of capital $\{g(S, x; \phi)\}_{i=1}^{2}$.

Taking into consideration this reallocation policy that will be implemented in the next period, the investment firm decides today where to install the new capital. Formally

$$\max_{S'} \Omega(S', x'; \phi)$$

s.t. $S' = (\phi(S), \Phi(S; \phi), \Phi(S; \phi))$

with solution $x' = g(S; \phi)$.

Notice that in this non-stochastic environment, the initial allocation of capital is always equal to the next period allocation because the investment firms perfectly forecast the future. This would not be case in the presence of shocks.
The ex-post household’s return from owning one share (one unit of capital) of the investment firm is simply given by \(1 + r' = \Omega(S', g(S; \phi); \phi)\).

**Production firms’ problem:** Production firms of each country solve a static problem. They rent capital and hire labor to maximize the following profit function:

\[
\max \ F(y, h) - p(S; \phi) y - w(S; \phi) h
\]  
(17)

The first order conditions for these firms are:

\[
p(S; \phi) = \frac{\partial}{\partial y} F(y, h) \tag{18}
\]

\[
w(S; \phi) = \frac{\partial}{\partial h} F(y, h) \tag{19}
\]

Notice that free entry guarantees that profits are zero and these two expressions define the equilibrium prices \(p\) and \(w\).

**Equilibrium:** We are now in a condition to define a competitive equilibrium for given policy rules.

**Definition 1** A Recursive competitive equilibrium for given policies \(\phi\) is a list of aggregate functions \(\{\Phi_Y, i, \Phi_H, i, p, w, \phi, \Phi_A, i, \Phi_X, r, \Omega\}\), household values and decision rules, firms values and decisions so that:

(i) Factor prices are marginal productivities in each country,

\[
p(S; \phi) = \frac{\partial}{\partial Y} F[\Phi_Y(S; \phi), \Phi_H(S; \phi)] \quad i = 1, 2, \tag{20}
\]

\[
w(S; \phi) = \frac{\partial}{\partial H} F[\Phi_Y(S; \phi), \Phi_H(S; \phi)] \quad i = 1, 2. \tag{21}
\]

(ii) Households are representative,

\[
g(S, A; \phi) = \Phi(S; \phi) \quad i = 1, 2, \tag{22}
\]

\[
g(S, A; \phi) = \Phi(S; \phi) \quad i = 1, 2. \tag{23}
\]
(iii) Investment firms solve their problem and are representative,
\[
\frac{T_2(\Phi (S; \phi), \Phi_1(S; \phi), \Phi_2(S; \phi))}{T_3(\Phi (S; \phi), \Phi_1(S; \phi), \Phi_2(S; \phi))} = \frac{1 + (p_1(S; \phi) - \delta)(1 - \phi_1(S))}{1 + (p_2(S; \phi) - \delta)(1 - \phi_2(S))}
\]
(24)

\[
T(\Phi (S; \phi), \Phi_1(S; \phi), \Phi_2(S; \phi)) = 0,
\]
(25)

\[
g(S; \phi) = \Phi (S; \phi).
\]
(26)

(iv) The return on investment is determined as the maximized profits of the investment firms.

\[
1 + r' = \Omega (S', g(S; \phi); \phi).
\]
(27)

(v) Both governments balance their budget every period. For \(i \in \{1, 2\}\),

\[
G + T = \Phi (S)(p(S; \phi) - \delta)\Phi (S; \phi) + \Phi (S)w(S; \phi)\Phi (S; \phi) + \theta r(S; \phi) A
\]
(28)

Conditions (i) and (ii) are standard. Condition (iii) requires some further explanation. It has three parts and all of them are implications of the fact that under constant returns to scale in the investment sector what firms consider optimal is what the economy as a whole does. The first and second parts, equations (24) and (25), state that what the economy as a whole allocate as capital in each country has to satisfy the first order conditions of the firms that make that decision. The third part, equation (26) states that what an investment firm considers to be an optimal allocation rule of new investment, is also what the whole economy chooses as an allocation rule. Condition (iv) describes how the return on assets \(r\) is determined. Note that absence of arbitrage opportunities requires that it be the same in both countries in each period. Condition (v) is obvious.


Governments take as given the function that determine future policies—the function \(\Phi\)—and the other country’s current policies \(\{\zeta_*, \tau_*, \theta_*\}\). The star denotes the other country
variable. In order to define equilibrium current policies, we need to know how the private sectors will react to these current policies, which are known to households before they choose their current actions. Therefore, we start defining a competitive equilibrium for arbitrary feasible policies today while assuming that in the next period, policies revert to those implied by $\phi$ (we call this a “hat” equilibrium).

We use $\pi$ as a compact notation for an arbitrary current policy, that is, $\pi = \{\pi_1, \pi_2\} = \{\zeta, \tau, \theta_i, \zeta, \tau, \theta_j\}$. These policies are determined through a strategic game that the two governments play with each other. We confine the analysis to perfect Markov equilibria.

In order to determine the equilibrium policies, we start looking at the problem solved by individual households when the two governments choose arbitrary policies $\pi$ in the current period and future policies will be determined by some policy function $\phi$. The next period value functions are the value functions for the problem with given policy functions $\phi$ as defined in the previous section. For the current period, instead, the value functions are indexed by the arbitrary current policies $\pi$. We denote all functions that are affected by the arbitrary current policies $\pi$ with hats. We then have:

$$V(S, \pi, a; \phi) = \max_c u(c, h) + \beta V(S', a'; \phi)$$

(29)

s.t.

$$c = (1 - \tau) w h + [1 + r (1 - \theta)] a + T - a'$$

(30)

$$r = r(S, \pi; \phi)$$

(31)

$$w = w(S, \pi; \phi)$$

(32)

$$\tau = \pi_i$$

(33)

$$\theta_j = \pi_j$$ for $j = 1, 2$ (34)

$$A' = A(S, \pi; \phi)$$ for $i = 1, 2$ (35)

$$X' = X(S, \pi; \phi)$$ (36)

with solution $a' = g(S, \pi, a; \phi)$ and $h = g(S, a, \pi; \phi)$.

The problem solved by the investment firm is similar to the problem solved in the
previous section. The solutions are denoted by \( \{ \hat{y}(S, \pi; \phi) \}_{1,2} \) and \( \hat{x}(S, \pi; \phi) \).

We now describe the “hat” equilibrium.

**Definition 2** A recursive equilibria for arbitrary policy actions \( \pi \) this period followed by policy functions \( \hat{\phi} \) is a list of aggregate functions \( \{ \{ \hat{\Phi}^Y, \hat{\Phi}^H, \hat{p}, \hat{w}, \hat{\Phi}^A \} =_{1,2}, \hat{\Phi}^r \} \), household values and decision rules, firms values and decisions so that:

(i) Factor prices are marginal productivities in each country,

\[
p(S, \pi; \phi) = \frac{\partial}{\partial Y} F(\hat{\Phi}^Y(S, \pi; \phi), \hat{\Phi}^Y(S, \pi; \phi)) \quad i = 1, 2, \quad (37)
\]

\[
w(S, \pi; \phi) = \frac{\partial}{\partial H} F(\hat{\Phi}^H(S, \pi; \phi), \hat{\Phi}^H(S, \pi; \phi)) \quad i = 1, 2. \quad (38)
\]

(ii) Households are representative,

\[
g(S, \pi, A; \phi) = \hat{\Phi}^A(S, \pi; \phi) \quad i = 1, 2, \quad (39)
\]

\[
g(S, \pi, A; \phi) = \hat{\Phi}^A(S, \pi; \phi) \quad i = 1, 2. \quad (40)
\]

(iii) Investment firms solve their problem and are representative,

\[
\frac{T_2(\hat{\Phi}^2(S, \pi; \phi), \hat{\Phi}^1(S, \pi; \phi), \hat{\Phi}^2(S, \pi; \phi))}{T_3(\hat{\Phi}^1(S, \pi; \phi), \hat{\Phi}^1(S, \pi; \phi), \hat{\Phi}^2(S, \pi; \phi))} = \frac{1 + (\hat{p}_1(S, \pi; \phi) - \delta)(1 - \pi_1)}{1 + (\hat{p}_2(S, \pi; \phi) - \delta)(1 - \pi_2)} \quad (41)
\]

\[
T(\hat{\Phi}^r(S, \pi; \phi), \hat{\Phi}^1(S, \pi; \phi), \hat{\Phi}^2(S, \pi; \phi)) = 0, \quad (42)
\]

\[
g(S, \pi; \phi) = \hat{\Phi}^r(S, \pi; \phi). \quad (43)
\]

(iv)

\[
1 + r' = \Omega(S', g(S, \pi; \phi); \phi). \quad (44)
\]

(v) The governments balance their budget every period, that is, for \( i \in \{1,2\}, \)

\[
G + T = \zeta(\pi)(\hat{p}(S, \pi; \phi) - \delta) \hat{\Phi}^Y(S, \pi; \phi) + \tau(\pi) \hat{w}(S, \pi; \phi) \hat{\Phi}^H(S, \pi; \phi) + \theta r(S, \pi; \phi) A \quad (45)
\]
The derivation of the functions $\hat{V}$ allows us to define the current return for the governments from a policy $\pi$. The maximization of this return will then define the optimal current policy. Let

$$V(S, \pi, \pi^*; \phi) = \hat{V}(S, \pi, \pi^*; A; \phi)$$

Both governments choose $\pi$ simultaneously to maximize $V$. We then have the following definition.

**Definition 3** Given the function $\phi$, a Nash equilibrium of the policy game is a pair of functions $\{\pi(S; \phi)\}_{i=1}^2$ such that, for $i = 1, 2$, $\pi(S; \phi)$ maximizes $V(S, \pi, \pi^*; \phi)$ given $\pi^*(S; \phi)$.

### 4.3 Step 3: The Policy Fixed Point

We now have all the elements required to define the equilibrium time-consistent policies.

**Definition 4** We say that the policy function $\phi$ defines the time-consistent policies if it is the Nash solution of the policy game when the governments expect $\phi$ to determine future policies. Formally

$$\phi(S) = \pi(S; \phi) \quad i = 1, 2, \quad S \in \mathbb{R}^5.$$  

### 4.4 Steady state

In the numerical experiments that follow, we will confine our attention to steady state equilibria. It is therefore worth noting a technical feature of the model, namely that its steady state is determinate. In particular, there is a unique steady state distribution of wealth across the two countries.

This is because a steady state equilibrium requires that the capital income tax rate is the same in the two countries. Otherwise, the citizens of the two countries would save at different rates. If they save at different rates, the world distribution of assets will change, meaning that the equilibrium cannot be a steady state.

Given that policies are endogenous, a steady state equilibrium requires that the two governments desire the same capital income tax rate. When countries are identical in
their fundamentals, this will happen when the world distribution of wealth is equal. More generally, the distribution of wealth must be such as to make governments desire the same capital tax rate.

5 The baseline model economy

In this section we describe the equilibrium properties of the tax rates for what we label the baseline model. This is a case of two identical countries (symmetry) that are similar to the U.S. These countries inherit capital income tax rates from the previous period and set simultaneously labor and corporate income taxes. We start describing the functional forms in section 5.1 and we go on with the calibration details in section 5.2.

5.1 Functional forms

We start choosing functional forms. We choose a utility function of the form

$$u(c, h) = \frac{c^{1-\sigma}}{1-\sigma} + \alpha \frac{(1-h)^{1-h}}{1-\sigma}.$$ (48)

This utility function is not standard in the study of business cycles. However, it is increasingly being used when authors care about the response of hours to wages. For example Castañeda, Díaz-Giménez, & Ríos-Rull (2000), in their study of the distribution of income and wealth, chose this utility function to ensure that the cross-sectional distribution of hours generate a reasonable profile, given the wage distribution and the tax system. In addition, this functional form is convenient because allows us to control, separately, the elasticity of labor from the steady state hours worked. From the first order conditions it can be verified that, in order to keep the steady state labor supply constant, the expression $\alpha/(1-h)^h$ must remain constant. Because $1-h < 1$, an increase in $\sigma$ must be associated with a decrease in $\alpha$.

Production takes place according to a Cobb-Douglas technology, so that output in country $i$, at time $t$, is $\lambda Y L^{1-\eta}$, with geometric depreciation at rate $\delta$. 

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The shifting technology is not standard; we assume that it takes the normalized form:

\[ T(x, y_1, y_2) = x^+ y_1^+ + (1 - x)^- y_2^- - 1 \]  \hspace{1cm} (49)

Notice that if \( \gamma = 0 \), then capital can be shifted across countries without cost, which implies perfect mobility of capital. The higher the value of \( \gamma > 0 \), the more costly it is to move capital internationally. Therefore, we will refer to \( \gamma \) as the parameter of capital mobility. This functional form is very convenient because it is a one parameter function. Figure 3 depicted this function.

5.2 Calibration targets and parameterization

We assume that policies are reset every two years which determines the length of a period in the model. We do this as a compromise between the time between elections and the legal definition of the length of time that a budget applies. It also agrees with the political cycle of several countries. The targets that we choose to calibrate the baseline model economy are listed in Table 1. They are chosen to match the target for the U.S., with some qualifications. We want to get a narrow notion of capital which corresponds not as much with total wealth but with taxable capital generating income. We also choose a low elasticity of substitution of labor. This elasticity value is not that different from those estimated for example by Castañeda et al. (2000).

With respect to the mobility parameter, we choose a conservative assessment of the costs of moving. Specifically, if 20% of the installed capital were to be uninstalled in one country, only 18% could be successfully used in the other. The degree of intertemporal substitutability of consumption that we impose is quite standard, just a little bit less than log. We choose the size of government in the baseline model economy to be half way between the U.S. (outlays are 27% of GDP, 15% of expenditures and 12% of transfers) and Europe (we choose 18% of expenditures and 22% of transfers totaling 40% of GDP).

Some of these considerations are related to the fact that under certain parameter configurations, it is very hard to compute the equilibrium of the model.\(^3\) Table 2 has the

\(^3\)In particular this is the case when labor is quite elastic. Note also that we abstract in this paper from consumption taxation. In models of this type with homogeneous agents, and absence of tax collection considerations, labor taxes and consumption taxes are quite similar along some dimensions and this
actual values of the parameters.

Table 1: Calibration Targets for the baseline economy (annual base).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate of return</td>
<td>0.07</td>
</tr>
<tr>
<td>Investment to GDP Ratio</td>
<td>0.15</td>
</tr>
<tr>
<td>Capital Share of output</td>
<td>0.30</td>
</tr>
<tr>
<td>Fraction of time worked</td>
<td>0.40</td>
</tr>
<tr>
<td>Elasticity of hours worked</td>
<td>0.75</td>
</tr>
<tr>
<td>Government purchases to GDP Ratio</td>
<td>0.165</td>
</tr>
<tr>
<td>Government transfers to GDP Ratio</td>
<td>0.17</td>
</tr>
<tr>
<td>Length of Period</td>
<td>Two years</td>
</tr>
<tr>
<td>Loss of Capital if Moved</td>
<td>2% of amount moved if 20% moved</td>
</tr>
</tbody>
</table>

Table 2: Calibration values for the baseline economy (annual base).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discount rate</td>
<td>β</td>
</tr>
<tr>
<td>Utility parameter</td>
<td>σ</td>
</tr>
<tr>
<td>Utility parameter</td>
<td>σ</td>
</tr>
<tr>
<td>Utility parameter</td>
<td>α</td>
</tr>
<tr>
<td>Capital share</td>
<td>η</td>
</tr>
<tr>
<td>Depreciation rate</td>
<td>δ</td>
</tr>
<tr>
<td>Capital mobility parameter</td>
<td>γ</td>
</tr>
</tbody>
</table>

Obviously we will conduct some sensitivity analysis for all the non standard dimensions of the calibrated model—specifically, the degree of capital mobility, the length of the period, the elasticity of labor and the composition of government expenditures—we will conduct a sensitivity analysis.

6 Properties of the baseline model and sensitivity analysis

We now turn to the discussion of the properties of the model in general in what we call the baseline model economy and of how the answers depend on some of the parameters.

creates computational problems: under some circumstances governments want to have huge consumption taxes and huge labor subsidies, while under not so different considerations, governments wants to do just the opposite. We leave for the future the explicit study of these issues.
6.1 The baseline model economy

We now report the properties of the steady state of a world with two identical countries as defined in the previous section. We therefore are not yet looking at the situation of the U.S. versus Europe. The main statistics of this economy are those defined as targets in the previous section. Here we are only looking at the properties of their tax structure.

The first row of table 3 reports the steady state result for the baseline economy. As we can see, labor income is taxed at a lower rate than capital income. This is especially noteworthy given that capital income is taxed twice. Once with corporate taxes when it is generated (source principle) and then with capital taxes when it is paid to consumers (residence principle). Even though it is taxed at a lower rate, the share size of labor income (the size of the tax base) ensures that a larger fraction of revenues comes from labor taxes.

6.2 Sensitivity analysis

Some of the findings of how the tax structure changes when the fundamentals change are quite interesting. We report the findings of the sensitivity analysis in Table 3 where we look at alternative values for the degree of capital mobility, the length of the period, the leisure-consumption elasticity and the size of the government.

6.2.1 The degree of capital mobility

We look at values of the mobility parameter of $\gamma = 0$ and $\gamma = .02$, compared to a value of .01 for the baseline model economy. We see that small changes to the cost have large implications for the corporate tax. Remember that the mobility cost is small in the baseline model: the cost of moving 20% of capital is 2% of this capital, which leaves 19.6% of useful capital. The qualitative properties of these implications are the obvious ones; the easier it is to take the capital abroad, the lower the tax rate. Note also that, to offset partially the larger corporate taxes, capital income taxes are lower the less mobile is capital. The balanced-budget requirement imposes a logical adjustment on the labor income tax.

What is remarkable is that in a sense the parameter $\gamma$ plays absolutely no role since
there is no shifting of capital in equilibrium. Moreover, the marginal cost of transfering the first unit of capital is zero in. Why then the importance of this parameter? The reason lies on the fact that governments are not competitive agents and that when analyzing the effects of their policies, they go beyond the first derivatives. In response to their actions, not only the private sector changes their behavior (not only locally), but also there is a response from the other government.

Note also, that perfect mobility of capital, ($\gamma = 0$), does not lead to zero profit taxes. The reason is that, although taxing profits repels capital, however, the movement of capital to the other country reduces the return on capital in the other country which prevents a large outflow of capital. This would not be the case if the country were a small open economy. Under this circumstances, the actions of the government would not be restricted by the understanding that there is a limit to capital flight.

<table>
<thead>
<tr>
<th>Table 3: Baseline Economy and Sensitivity Analysis.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Departure from Baseline</td>
</tr>
<tr>
<td>--------------------------</td>
</tr>
<tr>
<td>Baseline</td>
</tr>
<tr>
<td>Capital mobility</td>
</tr>
<tr>
<td>$\gamma = 0.00$</td>
</tr>
<tr>
<td>$\gamma = 0.02$</td>
</tr>
<tr>
<td>Length of period</td>
</tr>
<tr>
<td>One and a half years</td>
</tr>
<tr>
<td>Four years</td>
</tr>
<tr>
<td>Elasticity of labor</td>
</tr>
<tr>
<td>$\sigma = 1.7$</td>
</tr>
<tr>
<td>$\sigma = 2.3$</td>
</tr>
<tr>
<td>Composition of expenditures</td>
</tr>
<tr>
<td>$\gamma = 0.135, \gamma = 0.200$</td>
</tr>
<tr>
<td>$\gamma = 0.195, \gamma = 0.140$</td>
</tr>
</tbody>
</table>
6.2.2 The length of the period

The second experiment relates to the length of the period which is a form of commitment: the longer the period, the later in the future the new capital income taxes will be implemented. As can be seen from the table, a decrease in the political cycle or length of governmental commitment induces a very large increase in the capital income tax rate. The response of the other taxes is to increase just to be able to pay for the expenditures.

6.2.3 The elasticity of substitution of hours worked

The third experiment consists of changing the value of the parameter $\sigma$. In changing $\sigma$, we also change the parameter $\alpha$ so that the steady state hours worked do not change under constant taxes. A more elastic labor supply (smaller values of $\sigma$) increases the corporate tax rate. This is because, when labor is very elastic, high labor taxes induce a large capital outflow due to the reduction in the productivity of capital (following the reduction in the supply of labor). Corporate taxes also induce an outflow of capital. However, because the impact of labor taxes increases when labor is elastic, the government prefers higher corporate taxes in this case. In setting the capital income tax rate, the government anticipates that future corporate taxes will be higher and to prevent an excessive taxation of capital it decreases $\theta$. The increase in corporate taxes is almost compensated by the reduction in capital income taxes so that the whole taxation of capital does not change substantially.

6.2.4 Composition of the government outlays

The last change is with respect to the composition of government outlays. In the first of the two experiments the countries have much larger transfers and consequently smaller government expenditures, while in the second experiment the structure of outlays is the opposite. As shown in the bottom section of Table 3, the change in the composition of government expenditures has only a marginal impact on the equilibrium tax structure.

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4^This property of the model is also present in other analysis of government policy (see Krusell & Ríos-Rull (1999) and Klein & Ríos-Rull (1999)).
7 U.S. versus Europe

As discussed in section 2, capital income taxes are higher in the U.S. than in continental Europe. In this section we ask whether the different taxation structure between the U.S. and Europe can be explained by some economic differences between the two sets of countries or regions. To be able to do this, we need a theory of what makes Europe different from the U.S. A theory that has widespread support does not exist. Hence, we base our discussion on a small set of observables that can be taken to be exogenous and that are clearly different between the two sets of countries (and one also consistent with the differences between the U.K. and continental Europe). These differences are that the U.S. has a smaller government (relative to the size of the economy) and a larger economy (compared to each of the European countries). Specifically, we modify the baseline model to incorporate these features that differentiate the two areas.

Before proceeding, we would like to point out a special theoretical feature of the model. A steady state equilibrium requires that the capital income tax rate is the same in the two countries, otherwise the citizens of the two countries would save at different rates. If they save at different rates, the world distribution of capital will change, meaning that the equilibrium cannot be a steady state. However, when countries are heterogeneous, their governments may have different incentives to tax capital. These incentives depend on how the ownership of world capital is distributed, that is, they depend on $A_1$ and $A_2$. This implies that in a steady state equilibrium the world distribution of wealth must be such that the two governments have the same incentive to tax capital. This also implies that the world distribution of wealth in a steady state equilibrium may differ from the distribution of capital among the productive units of the two countries, that is, $K \neq A$. Therefore, in searching for a steady state equilibrium we search for the steady state ratio $A_1/K_1$ or $A_2/K_2$ for which the two countries tax capital income at the same rate.

In what follows there is the analysis of the steady state equilibria associated with each of the two types of heterogeneity between the U.S. and Europe.
7.1 Government size

The public sector is larger in Europe than in the U.S. Conservative measures of the size of the government in each country yield that government taxation is about 27 percent of GDP in the U.S. and about 40 percent of GDP in the three major countries in continental Europe. Therefore, we assume that in country 1 government expenditures are 27 percent while in country 2 they are 40 percent. The equilibrium tax rates are reported in Table 4.

The country with a larger government sector (country 2) has both higher corporate taxes and much higher labor taxes. What is important is that the second country has a larger fraction of revenue deriving from labor taxes. Therefore, countries with larger government sectors, like the European countries, tend to tax labor more than capital.

Table 4: Differences in Government Size: $\frac{G_1}{T_1} = .27$, $\frac{G_2}{T_2} = .40$.

<table>
<thead>
<tr>
<th>Capital Tax, $\theta$</th>
<th>Corporate Tax, $\zeta$</th>
<th>Labor Tax, $\tau$</th>
<th>Capital Taxes</th>
<th>Total Revenue</th>
</tr>
</thead>
<tbody>
<tr>
<td>Country 1</td>
<td>51.0%</td>
<td>20.3%</td>
<td>26.2%</td>
<td>32.0%</td>
</tr>
<tr>
<td>Country 2</td>
<td>51.0%</td>
<td>32.4%</td>
<td>42.8%</td>
<td>25.0%</td>
</tr>
</tbody>
</table>

The reason why a larger government leads to a higher proportion of revenues raised from the taxation of labor income is related to the different tax base for capital and labor taxes. Suppose we are in a steady state equilibrium characterized by a certain fraction of revenues raised from the taxation of capital and the other from the taxation of labor. Starting from this steady state equilibrium assume that government expenditures increase. To finance the higher expenditures the government needs to increase capital and/or labor taxes. However, because the tax base for labor taxes is larger than the tax base for capital taxes, a one percent increase in the labor tax rate increases tax revenues more than a one percent increase in the capital tax rate (corporate or income). In fact, the labor tax base is about four times larger than the capital tax base. Therefore, labor taxes are more efficient for increasing tax revenues and countries with larger public expenditures
use labor taxes proportionally more than capital taxes.

The last column of Table 4 reports the ratio between the capital used in the country and the assets of the country. If this ratio is smaller than one then the country is a net exporter of capital. This is the case for country 1: because of the higher corporate taxes, some of the domestic assets are invested abroad.

7.2 The size of the economy: population and productivity

During much of its recent history, the U.S. has had a centralized tax authority while Europe consists of a relative large number of countries. Therefore, we consider the case in which country 1 (the U.S.) has a larger population than country 2 (individual European country). Table 5 shows the results when the population of country 1 is four times the population of country 2 (country 1 has 4/5 of the world population). This is the population of the United States relative to the average population of the three largest countries in continental Europe individually considered (France, Germany and Italy).

Table 5: Differences in Population Size. Country 1 has 4/5 of the world’s population.

<table>
<thead>
<tr>
<th></th>
<th>Capital Tax, $\theta$</th>
<th>Corporate Tax, $\zeta$</th>
<th>Labor Tax, $\tau$</th>
<th>Capital Taxes</th>
<th>Total Revenue</th>
</tr>
</thead>
<tbody>
<tr>
<td>Country 1</td>
<td>44.6%</td>
<td>45.0%</td>
<td>31.8%</td>
<td>33.5%</td>
<td>94.4%</td>
</tr>
<tr>
<td>Country 2</td>
<td>44.6%</td>
<td>23.2%</td>
<td>38.1%</td>
<td>20.3%</td>
<td>123.6%</td>
</tr>
</tbody>
</table>

As explained above, the capital income tax rates are the same in both countries. However, corporate taxes are much higher in the larger country. This is consistent with the notion that if a country is large, then it is less concerned about capital flight. The rest of the world is small and capital outflow is limited by the decreasing returns abroad induced by these flows.

To capture the productivity differential (output per hour) between the U.S. and Eu-
Table 6: **Differences in Total Factor Productivity:** $\lambda_1 = 1$, $\lambda_2 = .9$.

<table>
<thead>
<tr>
<th></th>
<th>Capital Tax, $\theta$</th>
<th>Corporate Tax, $\zeta$</th>
<th>Labor Tax, $\tau$</th>
<th>Capital Taxes</th>
<th>Total Revenue</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Country 1</strong></td>
<td>54.0%</td>
<td>26.0%</td>
<td>33.6%</td>
<td>29.8%</td>
<td>98.7%</td>
</tr>
<tr>
<td><strong>Country 2</strong></td>
<td>54.0%</td>
<td>24.8%</td>
<td>34.1%</td>
<td>28.8%</td>
<td>101.5%</td>
</tr>
</tbody>
</table>

europe, we assume that the total factor productivity of country 2 is 10 percent smaller than country 1, that is, $\lambda_1 = 1.0$ and $\lambda_2 = 0.9$. Table 6 shows that the more productive country (country 1) taxes more heavily corporate profits. This is because the country is less concerned about capital flight as in the case of a country that has a larger population. As a result of the higher incentive to tax corporate profits, the fraction of revenues deriving from the whole taxation of capital is higher in the more productive country.

### 7.3 All differences combined

We now consider the case in which all the above differences are combined together. In particular, the second country has a larger government, a smaller population and a lower productivity. Ideally we would like to calibrate the model to the U.S. and Europe. This is a not trivial task for two reasons. First, Europe is not composed of homogeneous countries. Second, the U.S. and Europe also interact with other countries with different degrees of integration. Therefore, in assigning the parameter values, some abstraction and compromise have to be made. Taking into consideration these difficulties, we make the same calibration choices we made above in considering each of the individual differences. More specifically: (i) Government expenditures are 27 percent of GDP in country 1 and 40 percent in country 2. As claimed above, these numbers are approximately the averages for the United States and the largest countries in continental Europe (France, Germany, Italy); (ii) The population of country 2 is 25% of the population of country 1 ($\mu = 0.25$); (iii) Total factor productivity of country 2 is 90% the total factor productivity of country 1 ($\lambda_2 = 0.9$).
Table 7: Countries heterogeneous in government spending, population and productivity. Model calibrated to the U.S. and Europe.

<table>
<thead>
<tr>
<th>Capital Tax, $\theta$</th>
<th>Corporate Tax, $\zeta$</th>
<th>Labor Tax, $\tau$</th>
<th>Capital Taxes Total Revenue</th>
</tr>
</thead>
<tbody>
<tr>
<td>Country 1</td>
<td>36.5%</td>
<td>42.5%</td>
<td>25.2%</td>
</tr>
<tr>
<td>Country 2</td>
<td>36.5%</td>
<td>25.4%</td>
<td>48.8%</td>
</tr>
</tbody>
</table>

The results are reported in Table 7. As expected from the previous analysis, the combination of the three features that differentiate the U.S. from the European countries induce a much higher taxation of labor income in country 2. In this country (Europe) the fraction of revenue deriving from the taxation of labor is more than twice the fraction of country 1 (the U.S.).

There is another feature of the model that we would like to emphasize. As a result of the higher taxation of labor, hours worked are smaller in the second country. In the calibrated model, agents in country 1 work 5% more than agents in country 2. This is consistent with the data on working hours in the U.S. and the major countries in continental Europe. Finally, we observe in the last column of Table 7 that the first country is a net exporter of capital. This is also consistent with the fact that the United States is a net exporter of capital (at least until the last few years).

8 Conclusions

This paper has studied the optimal taxation policies in a two-country model with international mobility of capital. In this framework policy makers are benevolent but they cannot commit to future policies, a feature that the model accounts for generating time-consistent equilibrium policies. The analysis of several differences among countries allowed us to study the possible factors that may account for the different taxation structure between the U.S. and continental Europe. We found that differences in public spending and
the size of the economy (population and productivity) may account for the higher capital taxation (and lower labor taxes) of the U.S. relative to continental Europe. Differences in public spending can also explain why the taxation structure in the United Kingdom is more similar to the United States than to the other European countries. An additional feature of our model is the fact that hours worked are lower in the country identified with Europe.
A Computational procedure

The computational procedure combines the basic linear-quadratic approximation method with a policy iteration procedure that follows the steps used in section 4 to define a policy equilibrium. The details of these steps are as follows.

1. We approximate the utility functions of the representative agents in both countries with a second order Taylor expansion around the steady state values.

2. We guess policy rules $\varphi = (\varphi_1, \varphi_2)$ for the current corporate taxes and for the next period capital taxes as linear functions of the states.

3. Given the policy rules, the agents’ problem is well defined and we can derive the households’ value functions $V_i(S, a; \varphi)$, as defined in (4), through value function iteration. Given the quadratic form of the approximated return functions, $V_i(S, a; \varphi)$ is also quadratic.

4. Given the functions $V_i(S, a; \varphi)$, we construct the (quadratic) functions $\hat{V}_i(S, \pi, a; \varphi)$ as defined in (29). After imposing the aggregate consistency conditions $a = A$, we take the first order conditions with respect to $\pi_1$ for country 1 and with respect to $\pi_2$ for country 2. This gives the (linear) reaction functions for each of the two countries.

5. By solving the two reaction functions with respect to $\pi_1$ and $\pi_2$, we find the current policy rules as functions of the states, that is, $\pi = \pi(S; \varphi)$.

6. The functions $\pi(S; \varphi)$ are then used to update the guess for the policy functions $\varphi$. The procedure is then restarted from step 3 until policy convergence, that is, $\varphi(S) = \pi(S; \varphi)$. 
References


