A Markov Voting Model with Farsighted Agents

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Abstract

In this paper I present a model of how individuals make voting decisions in the setting of a continuing program. In this setting, any policy enacted today becomes tomorrow’s status quo, and thus leads to a future stream of legislation which is to some extent dependent upon it. Decision-making is modeled as a Markov process in which agents value a given policy according to both the utility received from that policy and from the future policies it likely leads to. Alternatives to replace the status quo arise probabilistically, where the likelihood of a policy defeating the status quo at time $t + 1$ is weighted by the players’ time $t$ valuations over these alternatives. One implication is that when policy selection is viewed as a dynamic process in which agents are aware that any current policy will eventually be replaced by a new alternative, a player’s “value” of a given policy, or expected sum of utility once that policy is enacted, is not necessarily monotone in the utility received from that policy. Another implication is that the distribution of other players’ payoffs is integral to every player’s value of a given policy.
1 Introduction

Formal modeling has perhaps had its most important impact on the study of legislative politics. Beginning with the field of social choice theory a half-century ago, scholars sought a formal means of directly aggregating individual preferences into collective outcomes. However, the social choice theoretic technique proved to have limited predictive power; the most well-known social choice theoretic results tell us that there is no normatively appealing means of aggregating individual preferences, and that, generically, any given policy can defeat any other via an amendment agenda. These “impossibility” and “chaos” theorems led many to believe that the direct aggregation of preferences into outcomes was not a promising approach to the study of collective choice. As a solution to this dilemma, Shepsle (1979) presented the idea of a “structure-induced equilibrium”, in which institutional detail is combined with social choice theory to yield core-based predictions. Shepsle’s argument is that, for a model to have predictive power, some specific institutional form must generally be assumed. While the notion of SIE has today been discarded in favor of other equilibrium concepts such as Nash, Shepsle’s argument in favor of institutions foreshadowed the course of formal modeling.

Institutional models today are generally non-cooperative and game-theoretic, where the game form (or institutional venue) is assumed exogenous. One of the advantages of these models is that they have strong predictive power, as Nash equilibria virtually always exist. However the game theoretic approach can also prove problematic, particularly when institutions are regarded as solutions to the so-called chaos problem. First, the predictions of many models are not robust to slight institutional changes; Nash equilibria in particular are highly sensitive to specific institutional detail. And second, when strict institutional assumptions
are made, they can affect outcomes in extreme, and sometimes quite unrealistic, ways. Diermeier and Krehbiel (2002) argue that it is through the comparison of institutional models that the link between institutions and outcomes should be drawn. Institutional models in isolation should serve a more methodological, and less predictive, role.

This paper models the process of legislative bargaining by recognizing the fact that individuals not only have immediate tastes over policies, but also preferences over future turns of events, and that the behavior of a voter may depend on both. It examines how individuals evaluate policies in a setting of repeated interaction, when they are aware that any policy enacted today will become tomorrow’s status quo, and will thus lead to a future stream of legislation which is to some extent dependent upon it. The focus is on continuing programs, in which policies remain in effect until new legislation is enacted. Examples of such programs include entitlements, regulation, and both distributive and redistributive programs.1 Formal models to date have not been able to make compelling predictions in the environment of a continuing program. I will demonstrate that modeling farsightedness is not only a more realistic approach to the study of legislative behavior, but that this model also yields compelling predictions in a variety of legislative environments. Furthermore, it does so in an “institution-free” way, by keeping the level of institutional detail to a minimum. The paper focuses on two main questions. First, is there a way of evaluating policies in terms of what they are likely to produce over time? And second, what do these individual-level evaluations imply about the types of outcomes likely to emerge when programs are continuing?

The formal setup of the model is that of an infinite-horizon continuing pro-

1See Baron (1996) for a more detailed discussion of continuing programs.
gram which legislators vote on in discrete time. For every potential status quo, there exists a stationary density from which alternatives to replace that status quo are drawn, and this density is known by all individuals. The modus operandi is a nonstationary Markov voting model, in which past events affect how individuals view their current choices in the following way. Players’ time $t$ valuations determine the likelihood a policy emerges as a time $t+1$ outcome, and these likelihoods in turn determine the players’ time $t+1$ valuations. Thus, individuals vote retrospectively, according to the expected utility they believe a policy will yield based on their most current information about that policy. The focus of the analysis is to prove the existence of, and numerically compute, those valuations which are self-generating, ex ante. These functions are of interest because, in these instances, the value every player assigns to a policy equals the true expected value of that policy, given the valuations of the other players. Using these equilibrium valuations, we can then calculate a fixed density over observed outcomes.

I show in this setting that, in the absence of a game form, players are not indifferent between different policies which provide them with the same level of utility. This is because the space of alternatives which defeat each policy, and which each policy defeats, are substantively different. In the setting of a continuing program in which status quos are endogenously determined, we can expect a certain path dependence to be observed in policy outcomes, and this path dependence cannot be captured in a simple one-shot model of voting. In this model, it is precisely the probabilistic path which a policy leads to that defines that policy. One consequence is that players are induced into taking the payoffs of others into account when voting, not because of a behavioral assumption such as altruism, inequality aversion, or sophisticated voting, but because they know that the behavior of
others in large part determines which policies are enacted in the future.

A closely related paper is that of Kalandrakis (2002), in which the author analyzes an infinitely repeated divide-the-dollar game with an endogenous reversion point, where the status quo in any round is determined by the bargaining outcome of the previous round. He finds that the Markov Perfect Nash equilibrium (in stage-undominated vote strategies) of the game is characterized by a situation in which the proposer in each round allocates himself the entire dollar, and this allocation is approved by a majority of players. This result is interesting because it negates two common theories about the outcomes of infinite-horizon bargaining games—namely, chaos and centrality. Under the characterized equilibrium, only a finite number of outcomes are ever achieved with positive probability and once the steady-state distribution is reached, every subsequent proposal allocates everything to the proposer. Furthermore, the methodology used (MPNESUV) is appealing from a game-theoretic point of view.

However, a distressing aspect of this result is that it seems both unrealistic and driven by the fact that in every stage, a subset of players is totally disenfranchised. The author addresses this point in his paper by stating that the result may generate intuition for the argument that, in actuality, budgets are deliberated under an exogenous reversion point. While this may be the case, it may also be the case that the primitives of the model are incorrect; in reality, budget deliberations may be history-dependent, policies to replace the status quo may arise probabilistically rather than deterministically, and deliberators may tremble when casting votes. Thus, while Kalandrakis’ model tells us that only the most extreme policy allocations are ever generated with positive probability, and McKelvey (1979) tells us that collective preference can be manipulated to generate virtually any policy out-
come, this model demonstrates that there exists a methodological middle ground. By taking the agenda-setting process to be exogenous and probabilistic, and by ex ante evaluating policies on the basis of what they are likely to produce over time, a unique distribution over observed outcomes can be found.

The paper proceeds as follows: Section 2 describes the notation used and presents the Markov model. Section 3 proves some analytic results. Under very general conditions I prove that there exists a unique self-generating value function when players vote according to a stationary rule. Then I show that in the absence of stationarity there exists a self-generating value function in certain settings. Section 4 provides some simple examples of the model in the setting of a unidimensional, finite policy space. Section 5 discusses the specific applications of the model in greater detail and presents numerical results concerning these applications in complex policy spaces. Section 6 concludes.

2 A Model of Farsighted Valuations

2.1 Notation

I assume a set $N = \{1, 2, ..., n\}$ of voters, a compact set $X \subset \mathbb{R}^m$ of alternatives, or policies, and, for each $i \in N$, voter preferences are represented by a real-valued utility function, $u : X \to \mathbb{R}_+$. When the set $X$ is infinite, also assume that these utility functions are differentiable, and that their derivatives are uniformly bounded by some constant $U$. A nonempty subset $C \subset N$ is called a coalition. I will restrict attention to simple and anonymous games, so that given a collection of coalitions $W$ with $C \in W$, then $C \subseteq C'$ implies $C' \in W$. Anonymity implies that the voting rules considered here are $q$-rules, such that for some fixed integer
$q > N/2$, $W = \{C \subseteq N : |C| \geq q\}$. The collection $W$ can be considered the set of winning or decisive coalitions.

### 2.2 The Markov model, in finite and continuous policy spaces

The model is a Markov process. First, assume $X$ is finite. Players’ beliefs at time $t$ are represented by a vector of continuous value functions $v_t : X \rightarrow \mathbb{R}$. Player $i$’s value function at time $t$, $v_{it}$ is the $i^{th}$ element of vector $v_t$. We can interpret $\mathbb{R}^X$ as the set of all functions from $X$ into the real line. Then $v_{it} \in \mathbb{R}^X$ and $v_i \in \prod_{i \in N} \mathbb{R}^X$. The function $v_{it}$ is represented by

$$v_{i0}(x) = u_i(x)$$

and

$$v_{it+1}(x) = u_i(x) + \delta \sum_{y \in X} \left[ v_{it}(y)p(v_i(x), v_i(y)) + v_{it}(x)(1 - p(v_i(x), v_i(y))) \right] Q(y).$$

The function $v_{i0}$ equals the utility player $i$ receives from alternative $x$. The probability of transitioning from state $x$ to state $y$ at time $t + 1$, given the two states are paired against each other, is represented by $p(v_t(x), v_t(y)) \in [0, 1]$. $Q(y)$ is the probability mass from which alternatives $y$ to replace the status quo are drawn. $\delta \in [0, 1]$ is a discount factor.

In the infinite case, Equation 2 is written

$$v_{it+1}(x) = u_i(x) + \delta \int_{y \in X} v_{it}(y)p(v_t(x), v_t(y)) + v_{it}(x)(1 - p(v_t(x), v_t(y))) \ dQ(y)$$

(3)
and $Q(y)$ is instead a density. In this case, $Q$ is assumed to have full support, and to be continuous and differentiable in $y$. Let $\mathcal{V}$ be the space of continuous, real-valued functions. Then $v_{it} \in \mathcal{V}$ and $v_t \in \mathcal{V}^n$.

Note that there are two types of transitions playing into the above equation, $p$ and $Q$. I will refer to these as “transition probabilities” and “transition densities” (or “masses”, when $X$ is finite), respectively. Intuitively, alternatives to replace the status quo arise probabilistically, picked from a stationary transition density $Q$. Since $Q$ is exogenous, the model assumes that legislators do not explicitly set the agenda themselves, but have fixed beliefs over the types of alternatives which will be added to the agenda. These beliefs could be uniform over all alternatives (uninformative), or could be generated by fixed external pressures from special interests, constituencies, or some function of the ideal points of the legislators themselves. However, once such an alternative is picked, it must then be pitted against the status quo, and will defeat the status quo with some transition probability $p$.

Transition probabilities are possibly nonstationary because legislators retrospectively update the values that they assign to policies, and vote according to these updated values. A legislator could have initially assigned a very high value to policy $x$. However if $x$ is continually defeated by policies which the legislator dislikes, then the value he assigns to $x$ will be brought down in subsequent rounds, and this will be reflected in how he votes.

The main focus of the following analysis is to prove the existence of, and numerically compute, value functions which are self-generating. These functions are of interest because they represent equilibria in beliefs. When a player behaves according to such a function, the value he assigns to a policy equals the true
future expected value of that policy. When transition probabilities are nonstationary, then voters retrospectively update how they vote, based on the values they placed on alternatives in the last period. Self-generating value functions are particularly meaningful in this setting because in these instances retrospective voting and prospective voting are the same. The vote strategies of players generate value functions which generate the same vote strategies. Thus, beliefs and behavior are entirely consistent with each other.

**Assumption 1 Transition probability assumption**

For all $x, y \in X$, $p(v_t(x), v_t(y))$, or the probability of transitioning from policy $x$ to policy $y$ at time $t + 1$, given $x$ and $y$ are put to a vote and given value function $v_t$, can be written as the probability of victory of $y$ over $x$:

$$p(v_t(x), v_t(y)) = \sum_{C \in W} \prod_{i \in C} p_i(v_t(x), v_t(y)) \prod_{i \notin C} (1 - p_i(v_t(x), v_t(y)))$$ (4)

where $p_i(v_t(x), v_t(y)) \in [0, 1]$ represents Player $i$’s probability of voting for $y$ over $x$ given value function $v_t$. It is assumed that $p_i$ is independent of $p_j$ for all $i, j \in N$, that $p_i(v_t(x), v_t(y)) + p_i(v_t(y), v_t(x)) = 1$, and that $p_i$ is increasing in $v_{it}(y) - v_{it}(x)$. Since $p$ is a function of $v$, which is indexed by time, the transition functions are possibly nonstationary. An example of a nonstationary transition function to be discussed later is one in which players vote probabilistically, according to a logistic function. In this case, $p_i(v_t(x), v_t(y)) = \frac{e^{\lambda v_{it}(y)}}{e^{\lambda v_{it}(x)} + e^{\lambda v_{it}(y)}}$ for some $\lambda \in \mathbb{R}_+$, where at time $t + 1$ players vote according to their time $t$ values.

**Assumption 2 Continuity and differentiability of transition probabilities**

For the remainder of the formal analysis, it is assumed that for all $i$, $p_i(v_t(x), v_t(y))$ is continuous and differentiable in both of its arguments. While this assumption is
made solely to simplify the analysis, the reader should note that it is always possible to approximate a discontinuous function with a continuous and differentiable one. For example, the logistic vote function converges to the deterministic case as $\lambda$ is driven to infinity.

3 Analytic Results

At any given time $t$, $v_t$ is a function of $v_{t-1}$. Let this function be called $g$, so that $v_t(\cdot) = g(v_{t-1}(\cdot))$. The following proposition proves that if $X$ is finite, then there exists a value function which generates itself.

**Proposition 1** If $X$ is finite, then there exists a self-generating value function.

**Proof:** Since $\delta < 1$ and $u_i$ is real-valued for all $i \in N$, the upper bound any individual’s value function could take is $\frac{1}{1-\delta} \max_{x \in X} u_i(x)$, and the lower bound is zero. Thus, for every $v_t \in \prod_{i \in N}^\mathbb{R}^X$, $v_t \in \prod_{i \in N}^\mathbb{R}^X \left[0, \frac{1}{1-\delta} \max_{x \in X} u_i(x)\right]^X$, and so the set of value functions is bounded. Furthermore, the set of value functions is convex, since the convex combination of two bounded functions taking $X$ to $\mathbb{R}$ is itself bounded. Last, the set of value functions is closed, trivially. It follows that the set of value functions taking $X$ into the real numbers $\mathbb{R}$ is a nonempty, closed, bounded and convex subset of a finite-dimensional vector space, $\mathbb{R}^X$.

The mapping $g : \mathbb{R}^X \to \mathbb{R}^X$, such that $g(v_t) = v_{t+1}$ (see Equation 2) is single-valued by definition, and is continuous by the continuity of every $p_i(v_t(x), v_t(y))$. By the Kakutani fixed-point theorem, there exists a $v_t \in \mathbb{R}^X$ such that $g(v_t) = v_t$. Thus, there exists a self-generating value function. □

Now consider the case where $X$ is infinite. The following proposition proves that
if transition probabilities are stationary then there exists a unique value function which generates itself. Moreover, the Markov process will limit to this function.

**Proposition 2** If transition probabilities, \( p \), are stationary, then \( g \) is a contraction mapping.

**Proof:** Endow \( \mathcal{V} \) with the following metric and the topology induced by it: \( \rho(v_i, w_i) = \sup_{x \in X} |v_i(x) - w_i(x)| \) and for \( v, w \in \mathcal{V}^n \), \( \rho(v, w) = \max_{i \in N} \rho(v_i, w_i) \). We must show that for any two vectors of functions \( v = (v_1, ..., v_n), w = (w_1, ..., w_n) \in \mathcal{V}^n \), \( \rho(g(v), g(w)) < \gamma \rho(v, w) \), for a \( \gamma \in [0, 1) \). Redefine the domain of \( p \) so that \( p : \{ \prod_{i \in N} \mathbb{R}^X \times \prod_{i \in N} \mathbb{R}^X \times X \times X \} \rightarrow [0, 1] \). Then stationarity in \( p \) implies that for all \( v, w \in \mathcal{V}^n \) and all \( x, y \in X \), \( p(v(x), v(y), x, y) = p(w(x), w(y), x, y) = p^*(x, y) \). Choose any \( v, w \in \mathcal{V}^n \) and let \( \xi = \rho(v, w) \). Then for every \( i \in N \),

\[
\rho(g(v_i), g(w_i)) = \delta \sup_{x \in X} \int_{y \in X} (v_i(y) - w_i(y)) p^*(x, y) + (v_i(x) - w_i(x)) p^*(y, x) dQ(y).
\]

However, we know that \( p^*(x, y) \in [0, 1] \), so the maximum value the argument of the integral could take for any given \( y \) is \( \max\{|v_i(y) - w_i(y)|, |v_i(x) - w_i(x)|\} \). Our worst case scenario is that \( Q \) assigns all of its weight to the policy which maximizes this argument. This implies that the maximum value \( \rho(g(v), g(w)) \) could take is \( \delta \sup_{y \in X} |v_i(y) - w_i(y)| \) which is equal to \( \delta \xi \) which is strictly less than \( \gamma \xi \) for \( \gamma \in (\delta, 1) \). Since \( \delta \in [0, 1) \), we know such a \( \gamma \) exists. Thus \( g \) is a contraction mapping. \( \square \)

While it is straightforward to prove convergence when transition probabilities are stationary, the task becomes more difficult when voters retrospectively update their voting strategies. The following assumption on \( p \) is needed to prove the
existence of a self-generating value function when transitions are nonstationary and \( X \) is infinite. Also, a definition is needed.

**Assumption 3**  *Multiplicative separability and boundedness of derivative of* \( p \)

Let \( \rho(v(r), v(s)) \) represent the following metric:

\[
\rho(v(r), v(s)) = \max_{i \in N} |v_i(r) - v_i(s)|
\]

and assume that for all \( x, y \in X \) and all \( v \in \mathcal{V}^n \), \( |\frac{\partial}{\partial x} p(v(x), v(y))| \leq \max_{i \in N} |v'_i(x)\) * \( B| \), where \( B \in \mathbb{R} \) is a constant and

\[
|B| * \max_{r, s \in X} \rho(v(r), v(s)) < \frac{1 - \delta}{\delta}.
\]

While this assumption may seem strange, many commonly used vote functions satisfy it. Consider the example where individuals vote probabilistically, according to a logistic function.

**Example 1**  *The implications of Assumption 3 under probabilistic voting*

Recall that in this case, \( p_i(v(x), v(y)) = \frac{e^{\lambda v_i(y)}}{e^{\lambda v_i(x)} + e^{\lambda v_i(y)}} \). Thus, \( \frac{\partial}{\partial x} p_i(v(x), v(y)) = v'_i(x)h(v_i(x), v_i(y)) \), where \( h(v_i(x), v_i(y)) = -\lambda e^{\lambda v_i(x) + v_i(y)} / (e^{\lambda v_i(x)} + e^{\lambda v_i(y)})^2 \).

By Assumption 1 we get

\[
\frac{\partial}{\partial x} p(v(x), v(y)) =
\]

\[
\sum_{C \in \mathcal{W}} \left[ \left( \sum_{i \in C} v'_i(x)h(v_i(x), v_i(y)) \prod_{j \in C \setminus i} p_j(v(x), v(y)) \prod_{j \not\in C} (1 - p_j(v(x), v(y))) \right) - \left( \sum_{i \in C} v'_i(x)h(v_i(x), v_i(y)) \prod_{j \in C} p_j(v(x), v(y)) \prod_{j \not\in C \cup i} (1 - p_j(v(x), v(y))) \right) \right]
\]

\[
= \sum_{i \in N} v'_i(x)h(v_i(x), v_i(y)) Z_i(\{p_j(v(x), v(y))\}_{j \in N \setminus \{i\}}) \tag{5}
\]
where

\[
Z_i(\{p_j(v(x), v(y))\}_{j \in N \setminus \{i\}}) = \\
\sum_{C \in W} \left[ 1_{\{i \in C\}} \prod_{j \in C \setminus \{i\}} p_j(v(x), v(y)) \prod_{j \notin C} (1 - p_j(v(x), v(y))) \right] - \left[ 1_{\{i \in C\}} \prod_{j \in C} p_j(v(x), v(y)) \prod_{j \notin C \cup \{i\}} (1 - p_j(v(x), v(y))) \right].
\] (6)

Let the function being summed over in Equation 6 be called \(Z_{iC}\), so that \(Z_i = \sum_{C \in W} Z_{iC}\). First note that we are considering only \(q\)-rules. For any \(k \in \{q, \ldots, n-1\}\), \(|\{C \subset N : |C| = k \text{ and } i \notin C\}| = \binom{n-1}{n-k-1} = |\{C \subset N : |C| = k+1 \text{ and } i \in C\}|.

Pick any \(C \in W\) such that \(i \notin C\). Let \(|C| = k\). Then

\[
Z_{iC} = - \prod_{j \in C} p_j(v(x), v(y)) \prod_{j \notin C \cup \{i\}} (1 - p_j(v(x), v(y))).
\]

Now pick \(C' \in W\) such that \(C' = C \cup \{i\}\). Then

\[
Z_{iC'} = \prod_{j \in C' \setminus \{i\}} p_j(v(x), v(y)) \prod_{j \notin C'} (1 - p_j(v(x), v(y))) = -Z_{iC}.
\]

Since we know that the number of winning coalitions of size \(k\) that player \(i\) is not a member of equals the number of winning coalitions of size \(k+1\) that \(i\) is a member of, it follows that all of the \(Z_{iC}\) terms cancel out except those corresponding to minimal winning coalitions that \(i\) is a member of. Let \(C_i^M\) be the set of minimal winning coalitions that \(i\) is a member of. Thus,

\[
Z_i(\{p_j(v(x), v(y))\}_{j \in N \setminus \{i\}}) = \sum_{C \in C_i^M} \prod_{j \in C \setminus \{i\}} p_j(v(x), v(y)) \prod_{j \notin C} (1 - p_j(v(x), v(y))).
\]
The function $Z_i$ can be thought of as the probability that Player $i$’s vote is pivotal in determining the winning outcome, and thus $Z_i \in [0, 1]$. Let $\overline{Z} = \max_{k \in N} Z_k(\{p_k(v(x), v(y))\}_{i \in N \setminus \{k\}})$. Using Equation 5 we get

$$|\frac{\partial}{\partial x} p(v(x), v(y))| \leq n\overline{Z} \max_{i \in N} |v'_i(x)| \max_{j \in N} |h(v_j(x), v_j(y))|.$$ 

Using the assumption of probabilistic voting and assumption that utility is positive and bounded, we get

$$B = (\lambda/4)\overline{Z}n.$$ 

Since utility is positive and bounded we also know that for all $v$,

$$\max_{r, s \in X} \rho(v(r), v(s)) \leq \frac{1}{1 - \delta} \max_{i \in N} \max_{x \in X} \{u_i(x)\} = \frac{u}{1 - \delta}.$$ 

Thus, $|B| \ast \max_{r, s \in X} \rho(v(r), v(s)) < \frac{4\Delta^2}{\delta}$ for

$$\lambda \in [0, \frac{4(1 - \delta)^2}{\delta n u \overline{Z}}).$$ 

If individuals are allowed to vote deterministically, then $\overline{Z}$ is maximized at one. However, if individuals vote probabilistically and never with probability one for any alternative over another, then $\overline{Z}$ approaches zero as the number of players becomes large. If $\overline{Z}$ approaches zero faster than $n$ approaches infinity (i.e. $\overline{Z} \sim o(\frac{1}{n})$), then Assumption 3 would impose basically no restriction on $\lambda$ as $n$ becomes large.

**Definition:** A set of real-valued functions $\mathcal{V}^* \subset \mathcal{V}$ is *equicontinuous* if for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\rho(s, t) < \delta \text{ and } f_i \in \mathcal{V}^* \Rightarrow |f_i(s) - f_i(t)| < \epsilon.$$
For the purposes of the following proofs, we are concerned in particular with a set $\mathcal{B}_M^n \subset \mathcal{V}^n$ of vectors of differentiable functions taking $X$ to $\mathcal{R}$ whose derivatives are uniformly bounded by the constant $M$. This set is equicontinuous; let $M$ be a bound for the derivatives of the functions in $\mathcal{B}_M$, and recall that for $f \in \mathcal{V}^n$, $\rho(f(s), f(t)) = \max_{i \in N} |f_i(s) - f_i(t)|$. Then, by an extension of the Mean Value Theorem, $\rho(s, t) < \delta$ implies that $\rho(f(s), f(t)) = \max_i |\nabla f_i(\theta)|\rho(s, t) \leq M\delta$, for some $\theta$ on the line segment between $s$ and $t$. Thus, given $\epsilon > 0$, the choice $\delta = \epsilon/(M + 1)$ demonstrates that $\mathcal{B}_M$, and thus $\mathcal{B}_M^n$, is equicontinuous.

**Lemma 1** If Assumption 3 holds, then the set of value functions can be restricted to a closed, bounded, and equicontinuous subset of $\mathcal{V}^n$.

**Proof:** Boundedness is attained because $\delta < 1$. Let $\mathcal{B}_M^n$ be the set of vectors of differentiable functions whose derivatives are uniformly bounded by the constant $M$. The set $\mathcal{B}_M^n$ is closed. I will show that there exists an $M \in \mathbb{R}_+$ such that for any $t$, if $v_t \in \mathcal{B}_M^n$, then $v_{t+1} \in \mathcal{B}_M^n$.

Differentiating Equation 3 and using Assumption 3, we get:

$$v'_{it+1}(x) = u'_t(x) + \delta v'_t(x)(1 - \int_{y \in X} p(v_t(x), v_t(y))dQ(y)) + \delta B \int_{y \in X} (v_{it}(y) - v_{it}(x)) \max_{j \in N} v'_j(x) dQ(y).$$  \hspace{1cm} (7)

The absolute value of the sum of the last two terms of this equation is weakly less than
\[
\delta \max_{j \in \mathcal{N}} |v'_{j,t}(x)| \left[ 1 - \max_{x,y \in X} p(v_t(x), v_t(y)) \right] + B \int_{y \in X} (v_{it}(y) - v_{it}(x)) dQ(y) \]

which is weakly less than

\[
\delta \max_{j \in \mathcal{N}} |v'_{j,t}(x)| \left[ 1 + |B \max_{r,s \in X} \rho(v(r), v(s))| \right].
\]

By Assumption 3, this can be rewritten

\[
\gamma \max_{j \in \mathcal{N}} |v'_{j,t}(x)|
\]

where \( \gamma \in [0, 1) \). Plugging back into Equation 7 we get

\[
|v'_{it+1}(x)| \leq |u'_{i}(x)| + \gamma \max_{j \in \mathcal{N}} |v'_{j,t}(x)|.
\]

Recall that the derivative of \( u_i \) is uniformly bounded by a constant \( U \), for all \( i \in \mathcal{N} \). Let \( K \) be any constant such that \( K > \frac{U}{1 - \gamma} \). Then if \( v_{jt} \in \mathcal{B}_K \), we get

\[
|v'_{it+1}(x)| \leq |u'_{i}(x)| + \gamma K < U + \gamma \frac{U}{1 - \gamma} = \frac{U}{1 - \gamma} < K.
\]

Thus, \( v_{it+1} \in \mathcal{B}_K \). It follows that the set of value functions can be restricted to \( \mathcal{B}^n_K \), a closed, bounded and equicontinuous subset of \( \mathcal{Y}^n \). □

Using this lemma, we can now establish the existence of a self-generating value function.

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**Proposition 3** There exists a self-generating value function when transition probabilities satisfy Assumption 3.

**Proof:** The Heine-Borel theorem in a function space tells us that a subset \( \mathcal{V}^* \subset \mathcal{V} \) is compact if and only if it is closed, bounded, and equicontinuous. Lemma 1 proves that the set of value functions can be restricted to the compact set \( \mathcal{B}_M^n \). Since the function \( g : \mathcal{B}_M^n \to \mathcal{B}_M^n \) such that \( g(v_t) = v_{t+1} \) is continuous, we need only convexity of the set of value functions to prove that there exists a self-generating value function.

Take the convex combination of any two value functions, \( v, w \in \mathcal{B}_M^n \), so that for any \( \gamma \in [0, 1] \), \( \gamma v(x) + (1 - \gamma)w(x) = z(x) \). Clearly \( z \) is continuous, since \( v \) and \( w \) are continuous. Furthermore, \( z'(x) = \gamma v'(x) + (1 - \gamma)w'(x) \leq M \). Thus, \( z \) is differentiable, and the derivative of \( z \) is bounded by the constant \( M \). It follows that \( z \in \mathcal{B}_M^n \), and that \( \mathcal{B}_M^n \) is convex. By Brouwer’s fixed-point theorem, there exists a \( v \) such that \( g(v) = v \). □

What do value functions imply about the types of policies likely to be observed? The following definitions are useful when examining the relationship between individual values over alternatives and policy outcomes.

**Definition:** The *conditional transition measure* \( f_{v_{t+1}}(Z|x) \) represents the relative likelihood of transitioning to a policy in the set \( Z \subset X \) at time \( t+1 \), given a status quo \( x \):

\[
f_{v_{t+1}}(Z|x) = 1_{\{x \notin Z\}} \int_{y \in Z} p(v_t(x), v_t(y))dQ(y) + 1_{\{x \in Z\}} \int_{y \in X} (1-p(v_t(x), v_t(y)))dQ(y).
\]  

(8)
Definition: The marginal transition measure $f_{X}^{t+1}(x)$ represents the relative likelihood of $x$ being the status quo at time $t + 1$, and the marginal transition measure $f_{Y}^{t+1}(y)$ represents the relative likelihood of transitioning to policy $y$ at time $t + 1$. These measures are defined recursively, with

$$
 f_{X}^{t+1}(x) = \frac{1}{\int_{y \in X} dy} 
$$

$$
 f_{Y}^{t+1}(y) = \int_{x \in X} f_{X}^{t+1}(x) \cdot f_{t+1}(y|x) dx 
$$

and

$$
 f_{X}^{t+1}(x) = f_{Y}^{t+1}(x). 
$$

Definition: The transition measure $f_{t+1}^{t+1}(x, y)$ represents the relative likelihood of transitioning from policy $x$ to policy $y$ at time $t + 1$, and is simply the product of the conditional and partial transition measures:

$$
 f_{t+1}^{t+1}(x, y) = f_{t+1}(y|x) \cdot f_{t+1}^{t+1}(x) 
$$

$$
 = f_{t+1}(y|x) \cdot f_{t+1}(x) 
$$

Of most interest to us is the marginal transition measure, $f_{X}^{t+1}(x)$, or the likelihood policy $x$ is the observed status quo at time $t + 1$. In the numerical simulations of Section 5, I calculate both the limiting value functions of players and the limiting density over outcomes that these value functions generate, as represented by the marginal transition measure.

4 One-dimensional Examples

In this section I will present some simple examples of how this model performs in a finite one-dimensional spatial setting, when voters have single-peaked pref-
erences. In all of the examples I will assume that $\delta = .9$ and that voting is deterministic, with

$$p_i(v_i(x), v_i(y)) = \begin{cases} 1 & \text{if } v_{ii}(y) > v_{ii}(x) \\ 0 & \text{if } v_{ii}(y) < v_{ii}(x) \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Example 2 When value functions are not monotone in utility.

Let $N = 3$, and $X = \{1, \frac{3}{4}, \frac{1}{4}\}$. Let the utility of the three players be defined by the functions $u_i(x) = 1 - |p_i - x|$, where $p_i$ is Player $i$’s ideal point and $p_1 = 1$, $p_2 = \frac{1}{2}$, and $p_3 = 0$. Finally, $Q(1) = \frac{1}{5}$, $Q(\frac{3}{4}) = \frac{3}{5}$, and $Q(\frac{1}{4}) = \frac{3}{5}$. The following table depicts a fixed-point vector of value functions (for readability, decimal approximations are given, with the actual values subscripted in parentheses):

<table>
<thead>
<tr>
<th>$i$</th>
<th>$p_i$</th>
<th>$v_i(1)$</th>
<th>$v_i(\frac{3}{4})$</th>
<th>$v_i(\frac{1}{4})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>4.5122(\frac{6}{11})</td>
<td>4.5652(\frac{14}{11})</td>
<td>3.4783(\frac{6}{11})</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{2}$</td>
<td>7.1951(\frac{2}{1})</td>
<td>7.5000(\frac{2}{1})</td>
<td>7.5000(\frac{2}{1})</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>5.4878(\frac{1}{1})</td>
<td>5.4348(\frac{1}{1})</td>
<td>6.5127(\frac{1}{1})</td>
</tr>
</tbody>
</table>

As is consistent with a traditional spatial model, the median voter (with ideal point $\frac{1}{2}$) is indifferent between $x_2 = \frac{3}{4}$ and $x_3 = \frac{1}{4}$, and strictly prefers both of these policies to $x_1 = 1$. However, Player 1 (with ideal point 1) strictly prefers $x_2 = \frac{3}{4}$ to his own ideal point. This is because when $x_3 = \frac{1}{4}$ is given an advantage over the other two policies by being chosen more often from density $Q$, the policy which makes Player 1 best off is not his ideal point, but the policy closest to his ideal point which defeats $x_3$, his least favorite policy. Thus, it is in Player 1’s best interest to concede some utility in the current round to reap higher rewards.
in future rounds. Finally, Player 3 (with ideal point 0) strictly prefers \( x_1 = 1 \) to \( x_2 = \frac{3}{4} \), even though \( x_2 \) is closer to his ideal point. Loosely speaking, this is because at \( x_1 \) there is a 60-percent chance of transitioning to \( x_3 \), Player 3’s favorite policy, while at \( x_2 \), this chance drops to 30-percent.

While this example is not surprising, it provides a clear picture of how this model works, and demonstrates that the predictions that this model yields are often quite intuitive. In the next example I will add a fourth policy to the same three-player setting considered above, and show that the independence of irrelevant alternatives property fails to hold.

**Example 3 Failure of IIA.**

Consider the same setting and players as above, but now let \( X = \{ 1, \frac{3}{4}, \frac{1}{3}, \frac{3}{5} \} \). Let \( Q(1) = \frac{1}{10}, Q\left(\frac{3}{4}\right) = \frac{1}{10}, Q\left(\frac{1}{3}\right) = \frac{2}{10}, \) and \( Q\left(\frac{3}{5}\right) = \frac{1}{2} \). The following table depicts a fixed-point vector of value functions:

<table>
<thead>
<tr>
<th>( i )</th>
<th>( p_i )</th>
<th>( v_i(1) )</th>
<th>( v_i\left(\frac{3}{4}\right) )</th>
<th>( v_i\left(\frac{1}{3}\right) )</th>
<th>( v_i\left(\frac{3}{5}\right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>6.0119, ( \frac{20294}{3000} )</td>
<td>5.9303, ( \frac{14043}{2200} )</td>
<td>5.1491, ( \frac{4299}{1000} )</td>
<td>6.0000</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{2} )</td>
<td>8.3199, ( \frac{20031}{3006} )</td>
<td>8.5946, ( \frac{1415}{244} )</td>
<td>8.5946, ( \frac{414}{61} )</td>
<td>9.0000</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>3.9881, ( \frac{13438}{2006} )</td>
<td>4.0697, ( \frac{9057}{1508} )</td>
<td>4.8509, ( \frac{1457}{144} )</td>
<td>4.0000</td>
</tr>
</tbody>
</table>

Note that the relative frequencies of our initial three policies (as defined by \( Q \)) are the same as in Example 1, only halved. However, the presence of a Condorcet winner makes the long-run behavior of the three initial policies very similar, as they are all likely to be defeated by the Condorcet winner. Thus, starting utility differentiates these policies more than their long-run behavior does. Consequently the players’ rankings over the initial three alternatives changes with the addition
of this fourth policy, and becomes monotone with respect to starting utility. For example, before \( x_4 \) was added to the policy space, Player 1 ranked the alternatives \((x_2, x_1, x_3)\), from highest to lowest. After the addition of \( x_4 \), he ranks the initial three policies \((x_1, x_2, x_3)\). Also note that we can directly verify that \( x_4 = \frac{3}{5} \) is a Condorcet winner; at a fixed point, for any Condorcet winner \( c \), \( v_{it}(c) = v_{io}(c) + \delta v_{it}(c) \), which implies that \( v_{it}(c) = v_{io}(c)/(1 - \delta) \).

5 Numerical Results

What follows is a look at several numerical simulations of this model in a continuous policy space. The first setting is that of a three-player constant sum game and the second setting is that of a three-player, 2-dimensional spatial model where players have Euclidean preferences. The simulations were run by discretizing the policy space and then iterating the Markov process until it converged numerically. The graphs that follow depict both the limiting value functions of the players and the limiting marginal transition measure \( f_X(x) \) over alternatives. \( f_X(x) \) represents the likelihood that any given alternative is a future observed policy. In all of the simulations it is assumed that players vote deterministically, as in the previous section.

Example 4 Three players divide a dollar when \( Q(y) \) is uniform.

Below is a graph of Player 1’s value function. The setting is a divide-the-dollar game in which players have linear preferences. The 2-dimensional unit simplex is pictured, and Player 1’s ideal point (the policy \( x = (1, 0, 0) \)) is at the top of the simplex. The bottom of the simplex denotes those policies which give Player 1 no
portion of the dollar. The darkest policies yield the highest values. The shading then moves to white, which denotes the policies which yield the lowest values. It is apparent that the policies which Player 1 values most are not Player 1’s ideal point, but rather those which divide the dollar about equally between himself and one other player, or \((1/2, 1/2, 0)\) and \((1/2, 0, 1/2)\). In social choice theory the set of policies which divide the dollar evenly between all members of a minimal winning coalition, in this case \(\{(1/2, 1/2, 0), (0, 1/2, 1/2), (1/2, 0, 1/2)\}\), is referred to as the von Neumann-Morgenstern stable set.

![Diagram of Player 1’s value function with uniform Q.](image)

**Figure 1:** Player 1’s value function with uniform \(Q\).

Figure 2 depicts \(f_X(x)\), the density over observed policy outcomes when value functions and behavior are consistent with each other. This, and all subsequent pictures of \(f_X(x)\), was generated by drawing approximately 200,000 policies from the density \(f_X(x)\) and then plotting their frequencies. The darkest policies are those which are most frequently observed. In this example, only a small subset of the total policy space is ever observed with positive probability. In particular,
Figure 2: $f_X(x)$, the density over observed outcomes when $Q$ is uniform.

the points in the stable set appear to constitute a majority-rule top cycle set with respect to players’ value functions. Figure 1 demonstrates this— since the setting is symmetric, it is clear that each of Player 1’s most-preferred policies is also the most-preferred policy of another player.

In the next series of pictures the same setting is considered, however $Q$ is no longer uniform. In these examples, $Q$ draws heavily from the “corners” of the simplex, or from those policies which give most of the dollar to a single player. Figure 3 shows that when alternatives to replace the status quo are picked from this different density, Player 1’s ranking over alternatives changes. In this case, holding the portion of the dollar given to Player 2 or 3 fixed, Player 1’s value is now monotone in his utility. This is because $Q$ weights most heavily those policies which are the least likely to majority-defeat any other policy. Thus, every policy is likely to remain in effect for a relatively long time once enacted, and so the utility a policy yields is a close proxy for what it is likely to yield over
Figure 3: Player 1’s value function when $Q$ draws heavily from the “corners”.

time. Note that value is not perfectly monotone in utility. It is clear that Player 1’s preferences have become concavified; holding his utility constant, the policies which he prefers most are those which give the remainder of the dollar to only one other player. Again, this is because these policies are the most stable to being overturned by a coalition consisting of Players 2 and 3.

Figure 4 depicts the density over observed outcomes generated when $Q$ draws heavily from the corners of the simplex. As in Figure 2, the most observed policies lie close to the stable set. However, these policies no longer constitute a top cycle set. Indeed, many policies will be observed with positive probability. Policies near the centroid are the least likely to be observed. This is due to both the fact that they are rarely drawn from density $Q$ to replace the status quo, and because they are easily overturned by policies on the edges of the simplex.

The last series of pictures depict a 2-dimensional spatial model, where the ideal points of the three players are no longer symmetric, but are located at $(0, \frac{1}{2})$, $1$\%
Figure 4: $f_X(x)$, the density over observed outcomes when $Q$ draws heavily from the “corners”.

Figure 5: Player 1’s value function with uniform $Q$ and circular preferences.

$(0, 0)$, and $(1, 0)$. The policy space has been restricted to the Pareto set, and $Q(y)$ is assumed to be uniform. In this setting, the stable set approximately equals \{(0, .19), (.28, .36), (.19, 0)\}. Pictured is the value function of Player 1, whose ideal point is located at $(0, \frac{1}{2})$. In Figure 5, we can see that Player 1’s most valued-alternative is approximately $(0, .25)$, closer to the alternative in the stable
set corresponding to a coalition between himself and the player whose ideal point is \((0, 0)\) than to his own ideal point. Figure 6 shows us that the most observed outcome is approximately \((0, .22)\), close to the alternative in the stable set corresponding to a coalition between the two players whose ideal points are closest to each other. This alternative is essentially a core.

6 Conclusions

This paper presents a model of how individuals value policies in the environment of a continuing program, in which any policy enacted today will become tomorrow’s status quo. Thus, any policy enacted today will lead to a future stream of policies that are to some extent dependent upon it. The first task of the paper is to present a formal means of evaluating policies in terms of what they are likely to produce over time. Policy evaluation is modeled as a Markov voting model in which past events affect how individuals view their current choices. The second task of this paper is to examine what these individual-level evaluations imply about the types of outcomes likely to emerge when programs are continuing.
In the formal analysis, I prove the existence of a set of individual valuations over alternatives which is internally consistent. These valuations are of interest because internal consistency implies that the value every player assigns to a policy equals the true expected value of that policy. Using these equilibrium valuations, we can then calculate a fixed density over observed outcomes. In the analytic and numerical examples which follow, I then calculate both what these valuations look like in different settings, and which types of policies are likely to emerge over time. These examples appear to generate a different intuition for why minimal winning coalitions often emerge. In this scenario, policies which benefit a bare majority actually leave players best off, from an expected utility standpoint.

Perhaps most interestingly, I show in this setting that, in the absence of a game form, players are not indifferent between different policies which provide them with the same level of utility. This is because the space of alternatives which defeat each policy, and which each policy defeats, are substantively different. This model demonstrates that in dynamic environments, the set of alternatives which can and cannot defeat a policy have as much impact on individual decision-making as the substance of the policy itself.

**References**


