1 A Model of Minimum Wage Effects on Labor Market Careers

Minimum wages can have complex effects on the labor market experiences of individuals through participation decisions, unemployment durations, and wage offers. Furthermore, the nature and quantitative effects of minimum wages may be expected to vary systematically over the life cycle. In the United States, where minimum wage levels are relatively low by the standards of most developed countries, the majority of individuals paid the minimum wage are young labor market participants. For this reason, and because of the nature of the data available to us, we will focus on the effects of the minimum wage on the labor market experiences of individuals between the ages of 16 and 24, inclusive.

The model we will use to theoretically and empirically investigate minimum wage effects on labor market experiences views individuals and firms operating in stochastic environments that are governed by probabilistic laws that don’t change over time. This is quite an abstraction from reality, but for purposes of examining minimum wage effects may not be too limiting for a number of reasons. As mentioned above, minimum wage impacts on labor market outcomes are likely to be concentrated in the first few years of labor market activity. Over such a relatively short period, the probabilistic structure of the labor market is not likely to change dramatically. Pragmatically speaking, the data to which we have access [drawn from the Current Population Survey] are essentially static. To estimate a dynamic model using such data requires us to assume that the labor market environment does not change over time. In particular, under the assumption of stationarity the rational behavior by labor market participants and firms can be characterized in terms of decision rules which do not change over time. If we were to allow the model to be nonstationarity, we are typically forced to specify all the conditions that individuals and firms will face over the infinite past and future if we are describe their optimizing decisions at any point in time. Clearly we don’t have the data to begin to attempt such a modeling effort.

Why attempt to build such a model in the first place? The point which we will make repeatedly in this book is so as to comprehensively summarize the impact of minimum wages on labor market outcomes and the welfare of labor market participants and firms. To make welfare statements, it is necessary to endow each set of agents in the economy with their own set of objectives. Subject to technological and budget constraints, and given the optimizing decisions of other agents in the market, each individual or
firm acts so as to maximize the value of their objective function. We will view minimum wages as constraining the actions of all labor market participants, both individuals and firms, though as we shall see the minimum wage “constraint” may, under certain circumstances increase the welfare of either labor market participants or firms. After going through the model in some detail, it hopefully will be clear what is behind this seemingly paradoxical result.

The objective function with which we endow labor market participants is one of expected wealth maximization. Thus, individuals are posited to care only about their (discounted) average earnings over their labor market careers; in particular, the variance of income flows does not favorably or unfavorably affect welfare. This is a strong assumption, and is made primarily for reasons of tractability. However, it is easy to show that when consumption decisions can be “decoupled” from earnings, as when there exist perfect capital markets for borrowing and lending, expected wealth maximization behavior in the labor market is consistent with individuals being risk-averse in terms of consumption levels. While young labor market participants may not have access to “perfect” capital markets, transfers between parents and children may serve the same role. In many cases, there is no strong reason to expect that young labor market participants will behave other than to maximize expected wealth.

Perhaps less controversially, firms will be assumed to behave so as to maximize expected profits. There will be no other “general equilibrium” links between searchers and firms in our model, for example through the mechanism of public ownership. While the welfare analyses we will perform do not explicitly concern the impact of minimum wages on the profit levels of firms, it is at least the case that in any equilibrium all firms will earn positive profits. We have not attempted to further link the behavior and welfare levels of these two sets of agents [as have Lundjist and Sargent (1997), for example] because we only have access to “supply side” data. We have thus deliberately kept simple our model of firm behavior and the link between firms and searchers.

1.1 Characterization of the Labor Market Career

We will model labor market events as happening continuously in time. By this we mean that there are no natural times, say weeks or months, at which labor market events always take place. Technically speaking, we view the labor market as a continuous time point process, which means that at any point in time an unemployed individual can receive a job offer. Furthermore,
at any point during an employment contract the contract can be exogenously terminated. The labor market process characterizing unemployed search is in actually a “marked” point process, in that when a potential worker and potential employer encounter one another, the total value of the match is also revealed.

Say that an individual begins her labor market career at time 0. Assuming that she will continue to participate in the market as an unemployed searcher or worker over her entire life, her labor market career can be completely characterized by the time at which she meets prospective employers and the value of the match associated with each contact, as well as the time at which employment matches she has accepted were (exogenously) terminated. For example, since she begins her labor market career in the unemployed search state at time 0, the first fifteen “events” in her labor market career might be given by the values in the following table.

<table>
<thead>
<tr>
<th>Event Number</th>
<th>State</th>
<th>Time of Event</th>
<th>Match Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>n</td>
<td>.891</td>
<td>6.243</td>
</tr>
<tr>
<td>2</td>
<td>n</td>
<td>3.168</td>
<td>4.329</td>
</tr>
<tr>
<td>3</td>
<td>n</td>
<td>15.554</td>
<td>3.871</td>
</tr>
<tr>
<td>4</td>
<td>n</td>
<td>15.558</td>
<td>10.918</td>
</tr>
<tr>
<td>5</td>
<td>e</td>
<td>38.921</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>n</td>
<td>44.236</td>
<td>7.891</td>
</tr>
<tr>
<td>7</td>
<td>n</td>
<td>56.793</td>
<td>12.119</td>
</tr>
<tr>
<td>8</td>
<td>e</td>
<td>157.421</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>n</td>
<td>164.772</td>
<td>10.145</td>
</tr>
<tr>
<td>10</td>
<td>e</td>
<td>322.510</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>:</td>
<td>:</td>
<td>:</td>
</tr>
</tbody>
</table>

The interpretation of the figures in Table 2.1 is as follows. The individual initiated search at time 0, at which point she occupied the nonemployed state (n). At time .891 she encountered her first potential employer. [Time units are arbitrary here, but it may help to think of the unit of time as the week.] When she met her first employer, the productivity of the potential match was revealed to be 6.243; this is to be thought of as the “flow” value of
productivity that will occur if she actually works at this employer. The value of this first potential match was insufficient to result in an employment contract so that the individual remained in the nonemployed state. The second potential employer was encountered at time 3.168 and the potential value of that match was 4.329, which was also unacceptable. Only the fourth potential match resulted in an employment contract. This employment contract began at time 15.558 and had a total flow value associated with it of 10.918. This match surplus is divided between the worker and the firm according to an idealized bargaining process which will be described in detail below. Whatever the division, it was sufficient to induce both parties to begin the employment contract, which was determined (exogenously) at time 38.291. This left the individual back into the nonemployment state, and she again began to search for an acceptable employment contract. After one unsuccessful encounter at time 44.236, she found another acceptable match with a value of 12.119 at time 56.793. This match was eventually terminated at time 157.421, and the nonemployed search process was repeated.

There are a number of things to note about this idealization of the labor market career. First is the implicit assumption that the only way an employment match can end is through an exogenous termination. This is clearly counterfactual, and can fairly readily be dispensed with if we allow for on-the-job search. If we posit that employed individuals encounter potential firms and can possibly be “bid away” by them if the match value at the new employer exceeds the match value at the current one, we will have a model in which jobs can end either by exogenous termination [i.e., the firm goes out of business due to a drop in demand for its product] or by a “voluntary” separation in which the worker accepts employment at a firm making her a better offer. In this book we don’t pursue such an extension for two reasons. First, and most importantly, the CPS data really don’t offer us the possibility of examining job-to-job transitions in sufficient detail to make the estimation of such a model feasible. Secondly, allowing for on-the-job search will add another layer of complexity to our model and to some extent make less clear the role of minimum wages in determining labor market outcomes. For these reasons we ignore on-the-job search in what follows.\(^1\)\(^2\)

Another important point to note is that the events described in Table 2.1 are not strictly exogenous. Some events are conditionally exogenous; for example, given the decision to begin the labor market career at time 0 the

\(^1\) For an analysis of on-the-job search in a bargaining-search model similar to the one developed here, see Flinn (1999).

\(^2\) In ignoring on-the-job search, we will tend to overstate the effect of minimum wages on lifetime welfare levels. We will return to this point in the conclusion.
arrival of the first possible job match at time .891 and the total flow value of that match, 6.243, are determined strictly randomly. However, the decision to reject that possible match is a behavioral one made by the searcher and the first firm contacted. This led the individual to continue in the unemployed state and to continue on to meet her second potential match at a time which was, once again, determined randomly. Thus the labor market career is a sequence of exogenous events followed by decisions which lead to further exogenous events [conditional on the previous exogenous events and past decisions], and so on. The method of dynamic programming (DP) allows us to formalize this process.

Finally, note that the only decision explicitly made by the searcher in this simple conceptualization of the labor market career is whether or not to accept a particular employment contract. Under the assumption that the environment is constant, the decision of whether or not to begin a particular employment “match” will be made by comparing the value of the match, which will be denoted by $\theta$, with a constant, $\theta^*$. An employment contract will be initiated whenever a match value $\theta \geq \theta^*$. The parameter $\theta^*$ will be a function of all of the parameters that characterize the economic environment. From the partial labor market history contained in Table 2.1 it is clear that $7.891 < \theta^* \leq 10.145$.

There are two other labor market decisions which will be considered. One is the manner in which the surplus value of the match will be divided between the worker and the firm. Under our assumption of Nash bargaining, this division will be accomplished in a completely mechanistic manner. The other decision, which is particularly relevant for young [and old] individuals is the decision of whether or not to participate [i.e., undertake search] in the labor market. For the moment, we will ignore this decision and assume that all individuals in the population we are considering are labor market participants. We shall return to a consideration of the participation decision at the end of this chapter.

### 1.2 The Stationary Labor Market Environment

The formal characterization of the environment within which firms and individuals interact begins with the notion of a counting process. Say that an individual begins a spell of search at time 0, as in the example above. If she were to search from time 0 until time $t$, $t > 0$, we can define the number of potential employers she encounters by the variable $N(t)$. Looking again at the example in Table 2.1, consider the first search spell. The timing of the first four encounters with potential employers which occurred during this
spell are given by the first four entries in the third column of the table. The first search spell was ended by the individual after the fourth contact because the match value was sufficiently high. If this search spell was not voluntarily “truncated” by the individual at this point, we could have imagined it continuing indefinitely, with one chance meeting with a potential employer followed by another. Figure 2.1 plots the number of firms encountered as of time \( t, N(t) \), for this case, where we have followed the individual through her first 10 encounters in such a spell [which include the first four recorded in Table 2.1]. Note that that the value of \( N(t) \) is 0 until the first encounter, which occurs at time .891. At this point it jumps to 1, since there has been 1 contact up through \( t = .891 \). Then the value of \( N \) jumps to 2 at the time of the second contact, or \( N(3.167) = 1 \) but \( N(3.168) = 2 \). Thus \( N \) takes values in the set of nonnegative integers, is monotonically increasing, and, whenever changed, is incremented by 1.

The Poisson process is a particular type of counting process. There are a number of different yet equivalent ways to characterize this process; for the purpose of defining the dynamic programming problem, the most useful one is probably the following.

**Definition 1** \( \{N(t), t \geq 0\} \) is a Poisson process with parameter \( \lambda (> 0) \) if

1. \( N(0) = 0 \)
2. \( \{N(t), t \geq 0\} \) has stationary, independent increments\(^{3}\)
3. \( P(N(t + \varepsilon) - N(t) \geq 2) = o(\varepsilon) \) for all \( t \geq 0 \)
4. \( P(N(t + \varepsilon) - N(t) = 1) = \lambda \varepsilon + o(\varepsilon) \), for all \( t \geq 0 \)

The notation \( o(\varepsilon) \) which appears in items (3) and (4) of the definition has the following interpretation. A function \( f(\varepsilon) \) has the property of being \( o(\varepsilon) \) if

\[
\lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{\varepsilon} = 0. \tag{2}
\]

The practical import of this is the following. Consider the search interval of \([0, \varepsilon]\). When \( \varepsilon \) is “large,” say equal to 1 time unit, in principle a countably

\(^{3}\)A stochastic process \( X \) has independent increments if the change in the process over the interval \([t_1, t_2]\) is independent of the change in the process over the interval \([t_3, t_4]\) for any \( 0 < t_1 < t_2 < t_3 < t_4 \). A stochastic process has stationary independent increments if in addition the probability distribution of \( N(t_2 + \varepsilon) - N(t_1 + \varepsilon) \) is the same as the probability distribution of \( N(t_2) - N(t_1) \) for all \( 0 < t_1 < t_2 \) and \( \varepsilon > 0 \).
infinite number of contacts with potential employers can be made, although the probability of making a very large number of contacts will in general (depending on $\lambda$) be quite small. As we “shrink” the time interval by reducing the value of $\varepsilon$, the probability of encountering more than 1 potential employer becomes negligibly small. Of course, when $\varepsilon = 0$, the probability of making any contact is 0. Thus items (3) and (4) mean that for $\varepsilon > 0$ but arbitrarily small, the probability of two or more events is arbitrarily close to 0 and the probability of one contact is approximately proportional to the length of the arbitrarily small interval. Below we shall refer to $\lambda \varepsilon$ as the probability of experiencing a contact in the small interval $\varepsilon$, and the reader should bear in mind that this interpretation is only “approximately” valid.

The parameter $\lambda$ is referred to as the rate of the Poisson process. It has this interpretation since

$$EN(t) = \lambda t$$

$$\Rightarrow E(N(t)/t) = \lambda,$$ 

so that the average number of contacts over any period of length $t$ is $\lambda$. For example, if $\lambda = .2$, the average number of contacts per period is .2. Conversely, the average wait between contacts is the reciprocal of .2, or 5 time periods in this case.

Note that under the assumption that the searcher-employer contact process is Poisson, every time a new nonemployed search process is begun it has the same probabilistic properties as the first search spell. In this sense, each search spell can without loss of generality be considered to “restart” at time 0. A stochastic process with this property is referred to as a renewal process, since whenever a particular state, in this case nonemployment, is revisited, the “clock” is reset.

The labor market model we develop actually is built upon two separate and independent counting processes, the first being the nonemployed searcher-potential employer contact process and the second being the match dissolution process. Recall that we will be assuming that employment matches dissolve exogenously. This dissolution process will also be Poisson, with a rate parameter given by $\eta$, $\eta > 0$. Let an employment contract between a searcher and a firm begin at time 0 [in inessential normalization]. Let $D(t)$ be the number of times the process would have exogenously dissolved as of time $t$; the times at which the contract would have been terminated may correspond to the times at which adverse demand shocks would have put the firm out of business. This termination process has the same properties as the searcher-firm contact process accept the parameter
characterizing the process may be different. Clearly, we have defined the two processes so as to be conceptually and probabilistically independent of one another.

The final component of the stationary labor market environment is the matching productivity distribution, \( G(\theta) \). To ensure stationarity of the environment, we make the assumption that this matching distribution does not change over calendar time or as a function of the elapsed time searching during the spell. All contacts with potential employers are assumed to be independent and identically distributed [i.i.d.] draws from this distribution.

In summary, the labor market environment is described physically by the contact rate between firms and searchers, \( \lambda \), the dissolution rate of on-going employment contracts, \( \eta \), and the time-invariant matching distribution \( G \). In order to completely characterize labor market equilibrium, we shall need a few other parameters but they are best described in the course of examining the decision-theoretic aspects of the model.

1.3 A Simple Decision-Theoretic Model

The individuals in our model are posited to be taking actions so as to maximize their expected lifetime wealth. They can only maximize expected wealth because of “search frictions,” in this case, search frictions refer to the fact that individuals do not know the location of the firms with employment vacancies, and even more importantly, do not know the identity of the firm with which they could achieve the highest productivity level. Prior to actually contacting a given firm an ascertaining their productivity level there, all potential employers look alike to the individual. In an expectational sense, then, the individual is initially indifferent with respect to the identity of the firm which is contacted.\(^4\) Firms are only differentiated after contact.

At each moment in time, the individual makes decisions which act to maximize expected wealth given her current choices and her knowledge of the parameters which characterize the labor market environment. The method used to investigate her decision is dynamic programming (DP). To illustrate the DP approach to a continuous time model like ours, we will begin by assuming that workers recieve the entire value of the match. In this case, the matching distribution \( G \) is synonymous with a wage offer distribution.

\(^4\)Under the informational assumptions of this model, jobs, or more properly potential jobs, are a pure search goods. Once a potential employer is contacted, both the individual and the firm will learn the total value of the match as well as the share of that value accruing to each.
\( G(w) \) [since \( w = \theta \) for any \( \theta \) draw].

At any arbitrary instant in time, say that the individual is currently unemployed. There will be a value associated with being in that state at that moment in time. This value consists of the sum of a "flow" value which is the "current period return" [a period should be thought of as an instant, in this case] and the discounted value of next period's problem given the current state and action taken. If the current state is denoted by \( s \), the current choice or action by \( a \), and next period's state is given by \( s' \), then

\[
V(s) = \max_a R(s, a) + \beta E[V(s')|s, a],
\]

where \( R(s, a) \) denotes the current period return to the action \( a \) taken when the state is \( s \), \( \beta \) is a discount factor which is greater than or equal to 0 and less than 1, \( s' \) denotes next period's state, and the \( E[V(s')|s, a] \) is the conditional expectation of next period's decision problem. In computing the expectation, the conditioning on \( s \) and \( a \) reflects the fact that in general the probability distribution of \( s \) is not independent of the current state and the action taken \( a \).

Consider first the value of being in state \( n \), that is, nonemployed and searching. Assume that the flow value of searching is given by the constant scalar \( b \). This flow value of nonemployment can reflect unemployment benefits, direct costs of search activity, and the value of other activities undertaken while the individual searches. Our strong stationarity assumptions require that \( b \) must be time invariant and that time spent in search does not change the parameters characterizing the labor market environment of the individual. These are both clearly counterfactual, especially for young labor market participants. In the case of unemployment benefits, for example, there are limitations on the length of time they can be received. Individuals typically face an unemployment benefit function which has the property that some amount \( B \) is received for a period \( t \) after which nothing is received. In such a case, clearly \( b \) is not time-invariant. Also note that young labor market searchers are often simultaneously investing in human capital. This investment in human capital would be expected to impact several parameters characterizing their labor market environment, such as the productivity

---

5 The discount factor \( \beta \) is restricted to be less than 1 to reflect the fact that individuals prefer rewards sooner rather than later. That is, a dollar received next period is only “worth” \( \beta \) whereas a dollar received this period is worth 1 [given our assumption of wealth maximization, dollars and utils are equivalent]. Thus when \( \beta = .5 \), two dollars received next period has the same valuation as 1 dollar received today.

6 See van den Berg (1990) for a nice theoretical and econometric analysis of a single-spell search model with this type of declining unemployment benefit.
distribution \( G \) and the rates of match arrivals and dissolutions of employment contracts. In the conclusion, we will have something to say regarding how the model could be modified to allow for human capital accumulation and certain forms of nonstationarity.

Our model is set in continuous time, and therefore has no natural “periods” which can be defined which delineate the “current” from the “future.” Our strategy will be to assume that there does exist a “decision period” of duration \( \varepsilon \) over which new actions are precluded. That is, the individual will take an action given the state \( s \) and will reap the reward from this action over the period \( \varepsilon \). At the conclusion of this decision period, she will take a new action given the state to which the system has then evolved. The decision period \( \varepsilon \) is in the end just an artifice, for we shall define behavior and the value of the problem in the limiting case corresponding to \( \varepsilon \to 0 \).

Define the value of the unemployed search problem as

\[
V_n = \max\{0, \frac{b \varepsilon}{1 + \rho \varepsilon} + \frac{\lambda \varepsilon}{1 + \rho \varepsilon} \int \max[V_n, V_e(w)] dG(w) \} + \frac{(1 - \lambda \varepsilon)}{1 + \rho \varepsilon} V_n + \frac{\alpha(\varepsilon)}{1 + \rho \varepsilon}.
\]

(6)

The first term on the right hand side [RHS] in the maximization operator corresponds to the value of not participating at all in the market - this value we have normalized to 0. Our current discussion is predicated on the assumption that all individuals are active labor market participants, which means that the second term in the max operator must be greater than 0.

In terms of the correspondence between [5] and [6], note that the relevant discount factor is \( \beta \varepsilon = 1/(1 + \rho \varepsilon) \), where \( \rho \) is the discount rate. The interpretation of the term \( \frac{b \varepsilon}{1 + \rho \varepsilon} \) is as follows. Over the short period \( \varepsilon \) the individual receives \( b \) per instant. Thus the total amount received at the end of this period is \( b \varepsilon \). However, since this amount is paid at the end of the period \( \varepsilon \), its beginning of period value must be suitably discounted. Applying the discount factor, the current period return of the action is \( \beta \varepsilon b \varepsilon \).

Now turn to the following term. Assume that the searcher obtains exactly one job offer at wage \( w \). Her choice then will be either to accept employment at that wage or to continue searching. The value of accepting a wage offer of \( w \) is given by \( V_e(w) \) and will be discussed shortly. Given the receipt of the offer \( w \), the individual will choose which ever option produces the highest return \( V_n \) or \( V_e(w) \), so that the value of getting an offer of \( w \) is given by \( \max[V_n, V_e(w)] \). The expected value of getting an offer is then the expectation of \( \max[V_n, V_e(w)] \) taken with respect to the distribution of all possible wage offers, which is given by \( G(w) \) in this case. Given the receipt
of an offer, the discounted expected value is $\beta \varepsilon \int \max[V_n, V_e(w)] dG(w)$. The approximate probability of getting one offer is $\lambda \varepsilon$.

Finally, if no offer is received the individual will simply continue to search. This is the true because in a stationarity environment, if a certain action was optimal at some arbitrary time when the individual faces choices in the set $C$, then the same decision will be made at any other time when the individual faces the same choices.\(^7\) Thus the value of not receiving an offer by the end of the period is $\beta \varepsilon V_n$, and the approximate likelihood of this event is $(1 - \lambda \varepsilon)$. Thus, in terms of our DP decomposition given in [5], the current period return is $\beta \varepsilon b \varepsilon$, and the discounted expected value of future choices given search is

$$\beta \varepsilon \lambda \varepsilon \int \max[V_n, V_e(w)] dG(w) + \beta \varepsilon (1 - \lambda \varepsilon) V_n + \beta \varepsilon o(\varepsilon).$$

Before we can analyze the problem facing the nonemployed searcher any further it is necessary to examine the situation of an employed individual be paid an instantaneous wage of $w$. Since we have precluded on-the-job search, and since it was initially optimal to accept a wage of $w$, an employed individual who has accepted a wage of $w$ will never quit and enter either the state of nonemployed search or nonparticipation. Thus, she will simply remain at her job until such time as the employment match is exogenously terminated. Formally,

$$V_e(w) = \frac{w \varepsilon}{1 + \rho \varepsilon} + \frac{\eta \varepsilon}{1 + \rho \varepsilon} V_n + \frac{1 - \eta \varepsilon}{1 + \rho \varepsilon} V_e(w) + \frac{o(\varepsilon)}{1 + \rho \varepsilon}, \quad (7)$$

where the current period return is now given by $\beta \varepsilon w \varepsilon$ and the expected future value of continuing to work at the job during this “period” is

$$\beta \varepsilon \eta \varepsilon V_n + \beta \varepsilon (1 - \eta \varepsilon) V_e(w) + \beta \varepsilon o(\varepsilon).$$

This is term is composed of the discounted value of being dismissed during the period and being in the unemployment state at the end of it times the probability of being dismissed $[\eta \varepsilon]$ plus the discounted value of ending the period in the same job times the probability of not being dismissed $[1 - \eta \varepsilon]$ plus the discounted value of the remainder term $o(\varepsilon)$, which reflects the value and probabilities of all other events which could occur in an interval of length $\varepsilon$.

\(^7\) We have already used this invariance property implicitly when we argued that given it was optimal to search at one point in time it will never be optimal to exit the labor market in the future. Obviously, if employment conditions, such as the wage offer distribution for example, were allowed to vary over time this would not be true in general.
We can determine the value of employment as follows. Multiply both sides of [7] by $1 + \rho \varepsilon$ to get

$$V_e(w)(1 + \rho \varepsilon) = w\varepsilon + \eta \varepsilon V_n + (1 - \eta \varepsilon)V_e(w) + o(\varepsilon)$$

$$\Rightarrow V_e(w)(\rho + \eta)\varepsilon = w\varepsilon + \eta \varepsilon V_n + o(\varepsilon)$$

$$\Rightarrow V_e(w) = \frac{w + \eta V_n}{\rho + \eta} + \frac{o(\varepsilon)}{\varepsilon},$$

where the last line is obtained after dividing both sides of line 2 by $\varepsilon$. Now taking limits, we have

$$\lim_{\varepsilon \to 0} V_e(w) = \frac{w + \eta V_n}{\rho + \eta} + \lim_{\varepsilon \to 0} \frac{o(\varepsilon)}{\varepsilon}$$

$$= \frac{w + \eta V_n}{\rho + \eta},$$

by the definition of the term $o(\varepsilon)$.

With this definition of $V_e(w)$, we can return to our consideration of $V_n$. First note that

$$\max[V_n, V_e(w)] = \max[V_n, \frac{w + \eta V_n}{\rho + \eta}]$$

$$= \frac{1}{\rho + \eta} \max[V_n(\rho + \eta), w + \eta V_n]$$

$$= \frac{\eta V_n}{\rho + \eta} + \max[\rho V_n, w]$$

$$= \frac{\eta V_n}{\rho + \eta} + \rho V_n + \frac{\max[0, w - \rho V_n]}{\rho + \eta}$$

$$= V_n + \frac{\max[0, w - \rho V_n]}{\rho + \eta}. \tag{8}$$

This is an important result, for it shows that for a given wage offer $w$, the option of accepting the employment match exceeds the value of the option of continuing to search when the wage offer $w$ exceeds the scalar value $\rho V_n$. This is an important enough result to warrant the following terminology.

**Definition 2** The reservation wage $w^*$ is equal to $\rho V_n$ and has the property that any wage offer $w \geq w^*$ will be accepted and any $w < w^*$ will be rejected.

The reservation wage $w^*$ completely summarizes the one decision rule utilized in this simple search model. Since the value of search $V_n$ will depend
on all of the parameters which characterize the labor market environment, so does the reservation wage. Below we shall discuss the manner in which we can solve for \( w^* \).

Using [8] we can rewrite [6] as

\[
V_n = \frac{b \varepsilon}{1 + \rho \varepsilon} + \frac{\lambda \varepsilon}{1 + \rho \varepsilon} \int [V_n + \frac{\max[0, w - \rho V_n]}{\rho + \eta}] dG(w)
+ \frac{(1 - \lambda \varepsilon)}{1 + \rho \varepsilon} V_n + \frac{o(\varepsilon)}{1 + \rho \varepsilon}
\]

\[
= \frac{b \varepsilon}{1 + \rho \varepsilon} + \frac{V_n}{1 + \rho \varepsilon} + \frac{\lambda \varepsilon}{1 + \rho \varepsilon} \int \frac{\max[0, w - \rho V_n]}{\rho + \eta} dG(w) + \frac{o(\varepsilon)}{1 + \rho \varepsilon}.
\]

Then

\[
V_n(1 + \rho \varepsilon) = b \varepsilon + V_n + \frac{\lambda \varepsilon}{\rho + \eta} \int_{\rho V_n} \frac{[w - \rho V_n]}{\rho + \eta} dG(w) + o(\varepsilon)
\]

\[
\Rightarrow \quad \rho V_n = b + \frac{\lambda}{\rho + \eta} \int [w - \rho V_n] dG(w),
\]

where the final term is obtained after dividing the first by \( \varepsilon \) and taking limits. Since \( w^* \equiv \rho V_n \), we can rewrite [9] as

\[
w^* = b + \frac{\lambda}{\rho + \eta} \int [w - w^*] dG(w)
\]

Clearly [10] cannot in general be manipulated so as to yield a closed-form solution for \( w^* \), however it is not difficult to establish that there exists a unique solution \( w^* \) to this equation which is relatively straightforward to compute. If we partially differentiate both sides of [10] with respect to \( w^* \), we see that the derivative of the left hand side (LHS) is simply 1, while the partial derivative of the RHS is

\[
\frac{\partial \text{RHS}(10)}{\partial w^*} = \frac{\lambda}{\rho + \eta} [-(w^* - w^*)g(w^*) - \int_{w^*} dG(w)]
\]

\[
= -\frac{\lambda}{\rho + \eta} \tilde{G}(w^*) < 0,
\]

where \( \tilde{G}(x) \), called the survivor function, is defined as \( 1 - G(x) \). The distribution \( G \) has been assumed to be everywhere differentiable on its support, with an associated probability density function given by \( g(w) \). We will assume that the support\(^8 \) of the distribution \( G \) is the positive real line.

\(^8\) Assuming that the distribution function \( G \) is everywhere differentiable, the support of the distribution is defined as the subset of the real line \( S \subseteq R \) such that \( g(s) > 0 \) for all \( s \in S \). In our discussion we assume that wage offers are always strictly positive, so that \( S = R_+ \), the positive real line.
Proposition 3 There exists a unique reservation wage $w^*$ if and only if

$$b + \frac{\lambda}{\rho + \eta} E(w) > 0. \quad (11)$$

Proof. Since we have assumed that the lowest possible wage offer is bounded from below by 0, at a reservation wage of 0 the LHS of [10] is equal to 0 while the RHS of the equation is equal to $b + \frac{\lambda}{\rho + \eta} E(w)$, where $E(w)$ denotes the expected value of the wage offer distribution $G$. Since the LHS of the [10] is an increasing function of $w^*$ and the RHS is a decreasing function of $w^*$, by Brouwer’s fixed point theorem there exists one and only one solution $w^* > 0$ to [10].

Regarding the proposition we have just stated and proved, it is natural to ask why we restricted the reservation wage to be strictly greater than 0. This restriction follows from two considerations. First, since the wage offer distribution was defined so that the lowest possible wage offer was bounded from below by 0, a reservation wage of 0 or less is not unique in terms of the behavioral decision. That is, a reservation wage of -2 implies that all wage offers will be accepted, just as does a reservation wage of -3, so that there is no behavioral difference between $w^* = -2$ and $w^* = -3$. Second, in terms of the original value of nonemployment equation [6], recall that the value of nonemployment was the greater of the value of 0, which was associated with nonparticipation, and the value of unemployed search. Therefore, the value of nonemployment is bounded from below by 0, and since the reservation wage $w^* \equiv \rho V_n$, the reservation wage itself is bounded from below by 0. For these two reasons we require $w^* > 0$.

We now illustrate the manner in which the reservation wage is computed. We consider an example labor market where $b = -1$, $\lambda = .2$, $\rho = .005$, $\eta = .02$, and we assume that the wage offer distribution facing the individual is log normal with parameters $\mu$ and $\sigma$ ($\sigma > 0$). For our example we have set $\mu = 1$ and $\sigma = 1$. Since the expected value of a log normally distributed random variable is given by $E(w; \mu, \sigma) = \exp[\mu + \frac{1}{2}\sigma^2]$, in this case we have $E(w) = 4.482$. Checking the condition required for the existence of a unique,

\footnote{Discussion of Brouwer’s fixed point theorem}

\footnote{The log normal density is given by}

$$g(w; \mu, \sigma) = \frac{1}{w\sqrt{2\pi}\sigma} \exp\left\{ -\frac{1}{2} \left( \frac{\ln(w) - \mu}{\sigma} \right)^2 \right\}.$$
positive reservation wage, we find that

$$-1 + \frac{2}{0.05 + 0.02} \cdot 4.482 = 34.856 > 0,$$

so that a unique \( w^* > 0 \) exists for this example.

Figure 2.2 plots the LHS and RHS of [10] as a function of \( w^* \). As we know, the \( LHS(w^*) = w^* \), while \( RHS(w^*) \) is a monotone decreasing function. In the case of our example, the two lines intersect at the point \( w^* = 7.439 \), which is the value which completely characterizes rational labor market behavior in this model. Note that \( w^* \) is appreciably greater than the mean wage offer in this model. In particular, we might ask what is the probability that a wage offer will be accepted? Clearly this probability that an offer is accepted is the probability that a random draw from the distribution \( G \) exceeds the reservation wage, or \( \tilde{G}(w^*; \mu, \sigma) = \tilde{G}(7.439; 1, 1) = 0.157 \). Thus most offers in this case are rejected. The reason for the “choosiness” of the searcher we see in this example is due partially to the relatively low rate of discounting [with \( \rho \) “small” the value of waiting for a good offer increases], the relatively low rate of exogenous terminations [when \( \eta \) is small it is more worthwhile to wait for a better offer since it will be kept longer, on average], and the relatively high rate of offer arrivals [\( \lambda \)]. We now turn to a short consideration of how comparative statics exercises can be conducted with this type of model.

Let us rewrite [10] in slightly different terms as

$$w^* = Q(w^*; \omega),$$

$$\Rightarrow 0 = w^* - Q(w^*; \omega)$$

where \( Q \) is the RHS of [10] and \( \omega \) is a vector containing all of the parameters which characterize the labor market in this model. In general \( Q \) is a differentiable function of all of the elements of \( \omega \), which means that we can totally differentiate [12] as follows

$$0 = \left( 1 - \frac{\partial Q(w^*; \omega)}{\partial w^*} \right) dw^* - \frac{\partial Q(w^*; \omega)}{\partial \omega_i} d\omega_i$$

$$\Rightarrow \frac{dw^*}{d\omega_i} = \frac{\frac{\partial Q(w^*; \omega)}{\partial \omega_i}}{1 - \frac{\partial Q(w^*; \omega)}{\partial w^*}} \tag{13}$$

where \( \omega_i \) denotes the \( i \)th element of the parameter vector \( \omega \). Because \( \frac{\partial Q(x; \omega)}{\partial x} \) is negative, the denominator of [13] is always positive, so that

$$\text{sgn} \left( \frac{dw^*}{d\omega_i} \right) = \text{sgn} \left( \frac{\partial Q(w^*; \omega)}{\partial \omega_i} \right),$$

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where $\text{sgn}(X)$ denotes the sign of the expression $X$.

For example, consider the flow cost of job search, $b$. Since $\partial Q(x; \omega)/\partial b = -1$, an increase in $b$ results in a decrease in the reservation wage, a result which is intuitive. Similarly, since

$$\frac{\partial Q(x; \omega)}{\partial \lambda} = \frac{1}{\rho + \eta} \int_{w^*}^{w^*} [w - w^*] dG(w) > 0,$$

an increase in $\lambda$ increases the reservation wage. From inspection of [10] it is obvious that $\partial w^*/\partial \rho$ and $\partial w^*/\partial \eta$ are both negative. Less obvious is the fact that, under the log normality assumption regarding $G$, $\partial w^*/\partial \mu$ and $\partial w^*/\partial \sigma$ are both positive.\(^{11}\) In the following chapter we will conduct a number of comparative statics exercises using the complete model we develop in the following sections.

The simple search model described here has been described by Rothschild (????) as being one of “partial-partial” equilibrium. In fact, it is just a model of individual choice - in this case the choices are limited to one to accept an offered wage and whether or not to begin the search process. Such a model is inadequate for studying minimum wage effects on labor market outcomes. Imagine that a minimum wage is imposed by the government, and that the imposition of this minimum wage, $m$, has no effect on any other parameters of the model. Let $w^*(\omega)$ be the original reservation wage optimally chosen for the labor market environment. If the minimum wage is set at a value no greater than the reservation wage, i.e., $m \leq w^*(\omega)$, then there is no effect on choices or outcomes. If $m > w^*(\omega)$, then clearly the individual is worse off than before. The reason is that certain wage offers which were previously acceptable, those $w \in [w^*(\omega), m)$, are now precluded. Since the individual was free to choose a reservation wage equal to $m$ previously but chose not to, she cannot be better off under this law. Thus in a “partial-partial” equilibrium search model the imposition of minimum wages cannot be beneficial for labor market participants.

For minimum wage laws to possibly have beneficial effects on a search model, their imposition must change the labor market environment in some positive way for the individual. For minimum wages to alter the labor market environment requires, at the minimum, a partial equilibrium model of the interaction between the supply and demand sides of the market. The bargaining framework developed we describe below provides an appropriate framework.

\(^{11}\) Provide discussion of these derivations.
1.4 Nash-Bargained Employment Contracts

When a potential worker meets a potential employer, the (flow) value of the match, $\theta$, is assumed to be immediately observed. If the two parties enter into an employment contract, how is the match value $\theta$ to be divided between the worker and the firm? It is important to realize that the productivity value $\theta$ is specific to the match and not attributable to either the worker or the firm. In that sense, both have a valid claim to it. How is it to be divided then?

There are a variety of forms of “bargaining power” that we might consider to defining the surplus division problem. First is the notion that each individual should receive at least in compensation what he or she could earn in the next best activity available to them. Clearly, under the assumptions which we have made about the search technology and labor market environment, a searcher who does not receive an acceptable job offer will optimally continue to search. Thus the value of the next best option available to a potential employee bargaining over her share of $\theta$ is $V_n$. For the moment assume that the next best option available to a potential employer has a value of $V_f$.

Bargaining results in a division of the flow value of the output produced. The firm agrees to pay the individual an “instanteous” wage rate of $w$, and the firm’s instantaneous profit rate is then $\theta - w$. Given her wage payment $w$, the individual is indifferent concerning the actual value of the match. In other words, the value of an employment pair $\{w, \theta\}$ to the individual is given by $V_e(w, \theta) = V_e(w)$. Thus the “surplus value” of the employment contact which pays $w$ to the individual is $V_e(w) - V_n$. This difference is the rent which accrues to the worker from the employment contract.

The value of the employment match to the firm requires a little more discussion. We think of firms as operating constant returns to scale production technologies. In this sense, the productivity of each worker is independent of every other in the firm. We also will assume that it is costless for firms to create vacancies. Thus, whenever a firm meets a potential employee, there is no fixed cost, for example, to create a possible position for the applicant. We will also assume that labor is the only factor of production, or at least that all other factors are fixed with total (instantaneous) cost equal to $F$. At any point in time $t$, firm $i$ will have a set of $N_i(t)$ employees, where

---

\footnote{While these assumptions are probably not very good descriptions of the situations facing firms in practice, they are required if we are to estimate our model using data which essentially only contains information on individual workers and searchers.}
The set of match values associated with these \( N_i(t) \) employees is given by \( \Theta_i(t) \), where \( \Theta_i(t) = \{ \theta_1, ..., \theta_{N_i(t)} \} \) when \( N_i(t) > 0 \) and is otherwise equal to the null set \( \emptyset \). Associated with this set of match values is a corresponding set of instantaneous wages paid these \( N_i(t) \) workers, denoted by \( W_i(t) \). Then the revenues of firm \( i \) at any moment \( t \) will be

\[
R_i(t) = \sum_{j \in \Theta_i(t)} \theta_j.
\]

Then the firm \( i \)'s instantaneous profit at time \( t \) is

\[
\pi_i(t) = \sum_{j \in E_i(t)} (\theta_j - w_j) - F. \tag{14}
\]

There are a few points to note concerning the instantaneous profit function [14]. Say that at time \( t \) when the firm had the set of employees described by the sets \( \Theta_i(t) \) and \( W_i(t) \), a new potential employee was contacted. Since the profit function is linear in \( \theta \) and \( w \), and since other costs are fixed, the decision of whether or not to employ this individual, and if so, at what wage, is independent of the match values and wages paid all current employees. Furthermore, since contact rate with other potential employees and the level of the fixed cost does not depend on characteristics of the sets \( \Theta_i(t) \) and \( W_i(t) \), the value of not entering into a contact with the potential employee at time \( t \) is zero to the firm. This is due to the fact that each contact is essentially unique, as far as the firm is concerned. Therefore under these assumptions concerning the firm’s production and search technology, \( V_f = 0 \).

Let the firm’s value of an employment contract in which an employee has an instantaneous output level of \( \theta \) and an instantaneous wage rate of \( w \) be given by \( Q(\theta, w) \). Then the solution to the generalized Nash bargaining problem is given by

\[
w^*(\theta; \omega, \alpha) = \arg \max_w (V_e(w) - V_n)^\alpha (Q(\theta, w) - 0)^{1-\alpha}, \tag{15}
\]

where \( \alpha \in [0, 1] \) is termed the bargaining power parameter [in this case, it measures the bargaining power of the individual while \( 1 - \alpha \) is the bargaining power of the firm], and where \( \omega \) contains all of the parameters describing the

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13 At any instant in time the firm may have no employees, though in the steady state the average number of employees per firm is strictly positive. Firms are assumed to be infinitely-lived as are searchers in this model. While firm profits may be negative at any moment in time, which would be the case if they had no employees and hence no production yet were paying a fixed cost \( F \), it is assumed that expected steady state profit levels are strictly nonnegative.
labor market environment with the exception of $\alpha$. Note that $V_e(w) - V_n$ measures the gain from participating in the employment contract paying a wage of $w$ with respect to the next best alternative, which is to continue searching. As we argued in the last paragraph, the next best alternative to the firm with respect to engaging in this particular contract is equal to 0. Therefore the surplus value to the firm associated with a particular employment contract is equal to $\tilde{V}_f(\theta, w)$.

To determine $\tilde{V}_f(\theta, w)$, we will assume that firms have the same discount rate as individuals, $\rho$. If an employment contract lasts for duration $t$, the ex post value of the contract is

$$\int_0^t (\theta - w) \exp(-\rho u) du = \frac{(\theta - w)}{\rho} (1 - \exp(-\rho t))$$

Since the probability density function of completed employment contracts is given by $\eta \exp(-\eta t)$, the expected value of an employment contract $\{\theta, w\}$ is given by

$$\tilde{V}_f(\theta, w) = E_t \left( \frac{\theta - w}{\rho} (1 - \exp(-\rho t)) \right)$$

$$= \frac{(\theta - w)}{\rho} \int (1 - \exp(-\rho t)) \eta \exp(-\eta t) dt$$

$$= \frac{(\theta - w)}{\rho} \left[ 1 - \eta \int \exp(-(\rho + \eta) t) dt \right]$$

$$= \frac{(\theta - w)}{\rho} \left[ 1 - \eta \frac{\rho}{\rho + \eta} \right]$$

$$= \frac{\theta - w}{\rho + \eta}.$$

We will use this expression for $\tilde{V}_f(\theta, w)$ in explicitly solving the bargaining problem.

In concluding this section, it is appropriate to say a few words about $\alpha$, a parameter which will figure prominently in the theoretical, econometric, and empirical work which follows. As we stated previously, there are essentially two aspects of bargaining advantage. One is the value of the next best option available to each of the two bargainers. The “threat point” of the firm, $V_f$, has been fixed at 0, while the individuals is set at $V_n$. Clearly, the larger is the value of $V_n$ the higher the wage payment required for the individual to enter into the employment contract.

The second aspect of bargaining advantage is the parameter $\alpha$. When $\alpha$ is equal to 1, the individual is assumed to have all of the bargaining power.
and extracts all of the “rents” from the match. The case $\alpha = 1$ corresponds exactly to the standard search problem we considered in the previous section, for in this case the matching distribution $G$ and the wage offer distribution are identical. At the other extreme, when $\alpha = 0$, firms possess all of the bargaining power. In this case, the wage payment is independent of the match value $\theta$, so that all employees are paid the same wage. If all employees are paid the same wage, then there is no motivation for search, and all offers will be accepted. If the value of nonparticipation is 0, and assuming that the instantaneous return associated with the search state $b < 0$, then it is not difficult to show that the common wage paid all workers will be $\bar{w} = -(\eta + \rho) b$.\(^{14}\) In this case, firms will earn instantaneous profit of $\theta - \bar{w}$ on all matches $\theta \geq \bar{w}$.

In general, the bargaining power parameter $\alpha$ is not equal to 0 or 1. It then merely indicates the relative “strength” of the two parties in bargaining, conditional on their threat points. This parameter is admittedly difficult to interpret. We think of it as comprising a type of summary statistic of the labor market “position” of a particular group. For example, the match value distribution for low-skilled workers may be stochastically dominated by the match value distribution for high-skilled workers, but in addition low-skilled workers may be at an additional disadvantage due to their having little bargaining power. Their low bargaining power may derive from there being many substitutes for them in the production process, or to the relative number of low-skilled workers to the number of positions for this type of worker. In this sense, the parameter cannot be really thought of as “primitive” since significant policy changes - such as a doubling of the minimum wage - may result in participation effects or substitution responses by firms which change the labor market “position” of the group, and hence change the bargaining power parameter. Thus comparative statics exercises and policy experiments performed with the estimates obtained from this model will only be valid locally, that is, for small changes in policy variables. This limitation is not too disturbing, since it probably applies to all empirical research which has been conducted in this area. We shall often return to reconsider the interpretation of $\alpha$ in the sequel.

1.5 The Search-Bargaining Model without Minimum Wages

We are now ready to combine the search and bargaining aspects of the model. Since $V_\epsilon(w) = (w + \eta V_\epsilon) / (\rho + \eta)$, the individual’s surplus with respect to

\(^{14}\)This result will be derived in the course of working through the labor market example contained in the following section.
the alternative of continued search is

\[ V_e(w) - V_n = \frac{w + \eta V_n}{\rho + \eta} - V_n = \frac{w - \rho V_n}{\rho + \eta}, \]

so that the solution to the bargaining problem is given by

\[ w(\theta, V_n) = \underset{w}{\arg\max} \left( \frac{w - \rho V_n}{\rho + \eta} \right)^\alpha \left( \frac{\theta - w}{\rho + \eta} \right)^{1-\alpha} \]

\[ = \underset{w}{\arg\max} [w - \rho V_n]^\alpha [\theta - w]^{1-\alpha} \]

\[ = \alpha \theta + (1 - \alpha) \rho V_n, \]

so that the wage is a weighted average of the match value and the reservation wage, \( \rho V_n \).

We can now move onto computing the value of nonemployment. Instead of writing the value of employment as a function of the wage, we write it as a function of the “primitive parameters,” \( \theta \) and \( V_n \). Then rewriting [9], we have

\[ \rho V_n = -c + \lambda \int_{\rho V_n} [V_e(w(\theta, V_n)) - V_n] dG(\theta). \]

Since

\[ V_e(w(\theta, V_n)) = \frac{\alpha \theta + (1 - \alpha) \rho V_n + \eta V_n}{\rho + \eta} = \frac{\alpha \theta - \alpha \rho V_n}{\rho + \eta} + V_n, \]

we have

\[ V_e(w(\theta, V_n)) - V_n = \frac{\alpha \theta - \alpha \rho V_n}{\rho + \eta}. \quad (16) \]

Then the final (implicit) expression for the value of search is

\[ \rho V_n = -c + \frac{\alpha \lambda}{\rho + \eta} \int_{\rho V_n} [\theta - \rho V_n] dG(\theta). \quad (17) \]

Note that this expression is identical to the expression for the reservation value in a model with no bargaining when \( \theta \) is the payment to the individual except for the presence of the factor \( \alpha \). This is not unexpected, since when
\( \alpha = 1 \), the entire match value is transferred to the worker, and thus search over \( \theta \) is the same as search over \( w \).

Now we can summarize the important properties of the model. The critical “match” value \( \theta^* \) is equal to \( \rho V_n \), which is defined by [17]. Since at this match value the wage payment is equal to \( w^* \equiv w(\theta^*, V_n) = \alpha \theta^* + (1 - \alpha) \theta^* = \theta^* \), the reservation wage is identical to the reservation match value. Then the probability that a random encounter generates an acceptable match is given by \( \tilde{G}(\theta^*) \), where \( \tilde{G} \) denotes the survivor function. The rate of leaving unemployment is \( \lambda \tilde{G}(\theta^*) \). As we can see from [17], since \( \theta^* \) is a decreasing function of \( \alpha \), rates of unemployment are higher when searchers have more bargaining power.

The observed wage density is a simple mapping from the matching density. Since

\[
\begin{align*}
   w(\theta, V_n) &= \alpha \theta + (1 - \alpha) \theta^* \\
   \Rightarrow \tilde{\theta}(w, V_n) &= \frac{w - (1 - \alpha) \theta^*}{\alpha},
\end{align*}
\]

then the probability density function of observed wages, \( h(w) \), is given by

\[
h(w) = \begin{cases} 
   \frac{\alpha^{-1} g(\tilde{\theta}(w, V_n))}{\tilde{G}(\theta^*)} & w \geq \theta^* \\
   0 & w < \theta^* 
\end{cases}.
\]

**1.5.1 An Example**

In order to fix ideas, we conclude this section with a detailed example of the computation of decision rules and the characterization of equilibrium for one particular labor market environment. We have chosen to work with very simple functional forms so as to make the computational steps as clear as possible. In the empirical work reported on latter, functional forms which produce results more in line with the data will be employed.

We will assume that the matching distribution \( G(\theta) \) is uniform, with the support of the distribution given by the interval \([0, 10]\). Thus

\[
G(\theta) = \begin{cases} 
   0 & \theta < 0 \\
   \theta/10 & 0 \leq \theta \leq 10 \\
   1 & 10 < \theta
\end{cases}
\]

and

\[
g(\theta) = \begin{cases} 
   0 & \theta < 0 \text{ or } 10 < \theta \\
   1/10 & 0 \leq \theta \leq 10
\end{cases}.
\]
We have also assumed that \( \lambda = .5, \eta = .02, \) and \( \rho = .01. \) These values imply that on average offers arrive to unemployed searchers every two time periods, jobs last for 50 periods on average, and searchers are “strongly” forward looking [i.e., the value of \( \rho \) is close to 0]. We will characterize the equilibrium of the model for values of \( \alpha = 1, 0, \) for a generic value \( \alpha \in (0, 1). \)

\( \alpha = 1 \) As we saw above, when \( \alpha = 1 \) the matching distribution and the wage offer distribution are identical, that is, there really is no bargaining problem per se. To determine the decision rule used by an agent in this environment, we use

\[
\begin{align*}
\Rightarrow w^* &= b + \frac{\lambda}{\rho + \eta} \int_{w^*}^{10} [w - w^*] dG(w) \\
\Rightarrow w^* &= b + \delta \int_{w^*}^{10} [w - w^*] \frac{1}{10} dw \\
\Rightarrow w^* &= b + \frac{\delta}{10} \left( \frac{w^2}{2} - w^* \left( w \mid w^* \right) \right) \\
\Rightarrow w^* &= b + \frac{\delta}{10} \left( 50 - \frac{(w^*)^2}{2} - 10w^* + (w^*)^2 \right) \\
\Rightarrow w^* &= (b + 5\delta) - \delta w^* + \frac{\delta}{20}(w^*)^2 \\
\Rightarrow 0 &= \frac{\delta}{20}(w^*)^2 - (1 + \delta)w^* + (b + 5\delta),
\end{align*}
\]

where \( \delta \equiv \lambda / (\rho + \eta). \) The quadratic equation [19] has two positive, real roots. One root is 6.918 and the other is 14.282. Since the second root cannot correspond to a welfare-maximizing strategy [it implies that all offers would be rejected], we have determined that optimal policy is to set \( w^* = 6.918 \) in this case. Note that this implies that a large proportion of offers will be rejected, \( G(w^*) = .692. \) An average spell of unemployed search will last \( 1 / (\lambda G(w^*)) = 6.488 \) periods and, as noted above, the average employment spell lasts 50 periods. From [18], the accepted wage density is given by

\[
h(w) = \begin{cases} 
\frac{1}{10} & 6.918 \leq w \leq 10 \\
\frac{3.98}{3.98} & 0 \leq w < 6.918 \\
0 & \text{else}
\end{cases}
\]

It could be a wealth-maximizing strategy to reject all offers under certain, rather peculiar, circumstances. Say that the largest possible wage offer was 10, as in our example, and that the instantaneous return associated with search, \( b, \) was 20. Then it would be optimal to reject all offers and to always remain in the search state. When the wage offer distribution has unbounded support, for any finite value of \( b \) there will exist a reservation wage policy which has the property that some offers will be accepted.
\(\alpha = 0\) In this case, all the rents accrue to the firm. How are wages determined?

We have assumed that the value of nonparticipation is 0. Then firms will set a common wage, \(\bar{w}\), so that labor market participants are indifferent between participation and nonparticipation, which implies that \(V_n = 0\). Now let us rewrite the value of search in a slightly different way. We can break up the labor market career into a sequence of search and employment spells, which we might think of as a “cycle.” When the individual enters the market, say that her first cycle is composed of a search spell of duration \(t_n\) and an employment spell of duration \(t_e\). Of course, these are random variables, of which due account will be taken. Then the value of can be defined as

\[
V_n = E_{t_n,t_e}[b \int_0^{t_n} \exp(-\rho u) du + \bar{w} \int_{t_n}^{t_n+t_e} \exp(-\rho u) du + \exp(-\rho(t_n+t_e))V_n],
\]

so that when \(V_n = 0\) we have

\[
0 = E_{t_n,t_e}[b \int_0^{t_n} \exp(-\rho u) du + \bar{w} \int_{t_n}^{t_n+t_e} \exp(-\rho u) du].
\] (20)

Now \(t_n\) and \(t_e\) are independently distributed random variables, so that the joint probability density function of the two is given by

\[
f_{T_n,T_e}(t_n,t_e) = f_{T_n}(t_n)f_{T_e}(t_e) = \lambda \tilde{G}(w) \exp(-\lambda \tilde{G}(w) t_n) \times \eta \exp(-\eta t_e).
\]

Then we can write [20] as

\[
\int \int \{ [b \int_0^{t_n} \exp(-\rho u) du + \bar{w} \int_{t_n}^{t_n+t_e} \exp(-\rho u) du]
\times \lambda \tilde{G}(w) \exp(-\lambda \tilde{G}(w) t_n) \times \eta \exp(-\eta t_e) dt_e dt_n
\]

\[
= \int \int \left\{ \left[ \frac{b}{\rho} (1 - \exp(-\rho t_n)) \right] + \left[ \frac{\bar{w}}{\rho} (\exp(-\rho t_n) - \exp(-\rho(t_n + t_e))) \right] \right\} dt_e dt_n
\]

\[
= \frac{b}{\rho} + \frac{\bar{w} - b}{\rho} \int \exp(-\rho t_n) \lambda \tilde{G}(w) \exp(-\lambda \tilde{G}(w) t_n) dt_n
\]

\[
- \frac{\bar{w}}{\rho} \int \exp(-\rho t_n) \lambda \tilde{G}(w) \exp(-\lambda \tilde{G}(w) t_n) dt_n \int \exp(-\rho t_e) \eta \exp(-\eta t_e) dt_e.
\] (21)

In performing the required integration, the following two results will prove very useful.
Remark 4 Let $Y$ be a continuously distributed random variable which only takes nonnegative values, and let $\tilde{G}(\cdot)$ be the survivor function associated with $Y$. Then

$$\int_0^\infty \tilde{G}(y) \, dy = E(Y).$$

Remark 5 Let $Y$ be a random variable which is exponentially distributed with parameter $\varsigma$. Then the survivor function of $Y$ is given by

$$\tilde{G}(y) = \exp(-\varsigma y),$$

and $E(Y) = 1/\varsigma$.

Then it follows that

$$\int \exp(-\rho t_n) \lambda \tilde{G}(\varpi) \exp(-\lambda \tilde{G}(\varpi) t_n) \, dt_n$$

$$= \lambda \tilde{G}(\varpi) \int \exp(-(\rho + \lambda \tilde{G}(\varpi)) t_n) \, dt_n$$

$$= \frac{\lambda \tilde{G}(\varpi)}{\rho + \lambda \tilde{G}(\varpi)},$$

since the integrand $\exp(-(\rho + \lambda \tilde{G}(\varpi)) t_n)$ is recognizable as the survivor function of an exponentially distributed random variable with parameter $\rho + \lambda \tilde{G}(\varpi)$. It therefore follows that the integral is equal to the mean value of this random variable, which is $1/(\rho + \lambda \tilde{G}(\varpi))$. Similarly, the product of integrals which appears in [21] is equal to

$$\frac{\lambda \tilde{G}(\varpi)}{\rho + \lambda \tilde{G}(\varpi)} \times \frac{\eta}{\rho + \eta}.$$

Then [21] is given by

$$0 = \frac{b}{\rho} + \frac{\varpi - b}{\rho} \frac{\lambda \tilde{G}(\varpi)}{\rho + \lambda \tilde{G}(\varpi)} - \frac{\varpi}{\rho + \lambda \tilde{G}(\varpi)} \frac{\eta}{\rho + \eta}$$

$$\Rightarrow 0 = b(\rho + \lambda \tilde{G}(\varpi) - \lambda \tilde{G}(\varpi)) + \varpi(\lambda \tilde{G}(\varpi) - \lambda \tilde{G}(\varpi) \frac{\eta}{\rho + \eta})$$

$$\Rightarrow 0 = \rho b + \varpi \lambda \tilde{G}(\varpi) \frac{\rho}{\rho + \eta}$$

$$\Rightarrow \varpi = -\frac{b(\rho + \eta)}{\lambda \tilde{G}(\varpi)}. \quad (22)$$
Even given that $b < 0$, there may be multiple or zero solutions to [22]. We will not attempt to provide a systematic characterization of the solutions to [22] since the case of $\alpha = 0$ is a pathological one which will not be of interest to us in the econometric and empirical analysis. However, it is clear that if there is more than one solution to [22], the lowest when will be the equilibrium value. For our example there are two solutions, the smallest of which is $\bar{w} = .060$. At this wage, the probability of employment contracts being formed is extremely high since $p(w \geq .060) = G(.060) = .940$. The average duration of search per spell is correspondingly short at 2.012. Since $\alpha = 0$ implies that workers receive no rents, this illustrates that high steady state employment rates are by no means synonymous with “good times” on the supply side of the market.

$\alpha \in (0, 1)$ This case is very much similar to that of $\alpha = 1$ which we have already examined. In particular, to determine the reservation wage associated with a reservation value $\alpha \in (0, 1)$, we merely modify the quadratic equation [19] by replacing the quantity $\delta \ (\equiv \lambda/(\rho + \eta))$ with the quantity $\delta' \ (\equiv (\alpha \lambda)/(\rho + \eta))$. It should be intuitive that the reservation wage $w^*_\alpha$ is an increasing function of $\alpha$ - this follows since the value of search should be positively related to the share of the match value one receives. The probability of an acceptable match value is a decreasing function of the reservation wage, so this probability is declining in $\alpha$. The average duration of search is an increasing function of the reservation wage, and hence is an increasing function of $\alpha$.

In Table 2.2 we present some illustrative calculations of these quantities for selected values of $\alpha$. We have already discussed the case in which $\alpha = 0$. When the worker has a small amount of bargaining power, e.g. $\alpha = .25$, the situation changes markedly. The reservation match value jumps to 4.758 and the probability that a random encounter with a perspective employer results in an employment contract declines considerably. The average duration of search increases by 80 percent. Increments of .25 in the bargaining power parameter result in monotonic changes in all of the characteristics computed, though the changes become increasingly less dramatic. For example, the difference between the $w^*_{\alpha=.75}$ and $w^*_{\alpha=1}$ is less than 6 percent. When workers have all of the bargaining power, the probability of an acceptable match is less than 70 percent, and search spells are on average 3 times longer than when workers have no bargaining power.
We have also computed the accepted wage distributions associated with these example labor markets; the probability density functions of accepted wages are plotted in Figure 2.3. In Figure 2.3.a we have plotted the p.d.f. of the matching parameter \( \theta \), which is uniform on the interval \([0,10]\). Figure 2.3.b is the one figure that is not a plot of a p.d.f. Instead, it contains a plot of the equilibrium wage function for each value of \( \alpha \) that we have examined (with the exception of \( \alpha = 0 \)). There are three interesting points to note concerning this diagram. First, for each value of \( \alpha \) the function “begins” at the reservation value \( w_\alpha^* = \theta_\alpha^* \). As noted above, this point is an increasing function of \( \alpha \). Second, the slope of the equilibrium wage function is equal to \( \alpha \). When \( \alpha = .25 \) the line is relatively flat, whereas it achieves its maximum slope when the workers get all the rents (i.e., \( w = \theta \)). Third, the function “ends” at different values of \( w \) which depend on \( \alpha \). In our example, the maximum value \( \theta \) attains is 10. Since the equilibrium wage function is \( w(\theta, w_\alpha^*, \alpha) = \alpha \theta + (1 - \alpha)w_\alpha^* \), and because \( w_\alpha^* < 10 \), a wage of \( w \) can only be observed for the case of \( \alpha = 1 \). In this example, the maximum wage offer is \( 10\alpha + (1 - \alpha)w_\alpha^* \), and it is easily seen that this value is an increasing function of \( \alpha \).\(^{16}\) Thus the wage functions differ in terms of the minimum and maximum observed wage as well as the slope of the wage function.

The probability density function of observed wages is determined by the p.d.f. of match values \( g \) and the equilibrium of the model as seen in [18]. Given our assumption that \( G \) is uniform, the accepted wage distributions will also be uniform for all values of \( \alpha \), with the exception of the limiting case of \( \alpha = 0 \) [when the wage distribution is distribution has all of its mass concentrated at the value \( \overline{\theta} \)]. The uniform accepted wage distributions will only differ in their lower and upper bounds. For example, consider the

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\(^{16}\)The partial derivative of \( 10\alpha + (1 - \alpha)w_\alpha^* \) with respect to \( \alpha \) is \( 10 - w_\alpha^* + (1 - \alpha)[\partial w_\alpha^*/\partial \alpha] \). Since \( w_\alpha^* < 10 \) and \( \partial w_\alpha^*/\partial \alpha \) is positive, the partial derivative is strictly positive.
case in which $\alpha = .5$. From [18] we know that for any possible wage rate acceptable when $\alpha = .5$, the value of the p.d.f. must be

$$\frac{.5^{-1}(.)}{.408} = .490.$$ 

Now the largest wage observable in equilibrium is 7.962 [$= 10(.5) + .5(5.924)$] and the smallest is the reservation wage of 5.924, so under our uniformity assumption that density can also be computed directly from $1/(7.962 - 5.924) = .490$. Similar calculations apply for all the other p.d.f.s which were graphically presented in Figures 2.3.c-f. It is interesting to note that both the mean and dispersion in the accepted wage distribution are increasing functions of $\alpha$ in this example. We will see to what extent this is a general result in the following chapter.

1.6 Bargaining with a Minimum Wage Constraint

The introduction of minimum wages into the search-bargaining framework is accomplished in a very straightforward manner. We assume that the labor market environment is exactly as described above, with the exception that a “side constraint” is introduced into the worker-firm bargaining problem [15]. This constraint is that any employment contract must yield a wage payment of at least $m$ to the worker no matter what the value of $\theta$. The minimum wage is assumed to be set by the government and applies to all potential matches. This assumption represents the U.S. case relatively well, since the minimum wage applies to virtually all employment contracts in the labor force.\textsuperscript{17} More controversially, we assume that the only compensation provided by the firm is the wage. Thus there are no other forms of compensation the firm can adjust so as to “undo” the minimum wage payment requirement.\textsuperscript{18}

The modified bargaining problem we develop is then represented by

$$w^*(\theta, m; \omega, \alpha) = \arg \max_{w\geq m} (V_e(w) - V_n)^{\alpha} \left( \frac{\theta - w}{\rho + \eta} \right)^{1-\alpha}$$

\textsuperscript{17} There are some provisions of federal and state minimum wage statutes which allow lower wage payments than $m$ to be paid to certain classes of workers. However, the class of workers to which these provisions apply is very small, and there is little empirical evidence that employers take advantage of them to any appreciable extent [see, e.g., Card ...].

\textsuperscript{18} A number of researchers have investigated the manner in which minimum wage laws could be “defeated” when compensation includes more than only wage payments. For example, Lazear (19??) investigates how the requirement of paying a high minimum wage may result in the firm providing less on-the-job training to young employees. Others (e.g., ?? and ??) have looked at the manner in which nonpecuniary compensation is reduced in response to minimum wage increases.
As should be clear, any \( m \leq \theta^* \) has no effect on the behavior of applicants or firms and thus would be a meaningless, or “slack,” constraint. Thus we consider only the effects of an imposition of \( m > \theta^* \).

The first thing to note concerning the effect of the constraint on behavior is that, since the value of the employment contract to the firm is proportional to \( \theta - w \), no employment contract will be formed for which the match value \( \theta < m \), since in this case the firm would lose money. If there is no opportunity cost to the firm of forming match, then any match for which it doesn’t lose money will be acceptable to it. Since \( \theta^* < m \), this implies that fewer encounters between searchers and firms will result in employment contracts, which will result in an increase in the unemployment rate. This loss of employment effect is consistent with that predicted by the simplest static competitive labor market models. We shall see later the extent to which this prediction is robust with respect to alterations in modelling assumptions.

The addition of the minimum wage “side constraint” on the solution to the bargaining problem is relatively intuitive. Under the “constrained” Nash-bargaining problem, there will exist a value of search which we denote \( \hat{V}_n \) [Note that this value is not equal to \( V_n \) - it will be defined below]. If we ignore the minimum wage constraint and solve [23] using \( \hat{V}_n \), we will get the wage offer function

\[
\hat{w}(\theta, \hat{V}_n) = \alpha \theta + (1 - \alpha)\rho \hat{V}_n.
\] (24)

Under this division of the match value, the worker would receive a wage of \( m \) when \( \theta = \hat{\theta} \), where

\[
\hat{\theta}(m, \hat{V}_n) = \frac{m - (1 - \alpha)\rho \hat{V}_n}{\alpha}.
\]

Then if \( \hat{\theta} \leq m \), all “feasible” matches would generate wage offers at least as large as \( m \). When \( \theta > m \), this is not the case. When \( \theta \) belongs to the set \([m, \hat{\theta}]\), the offer according to [24] is less than \( m \). However, when confronted with the choice of giving some of its surplus to the worker versus a return of 0, the firm pays the wage of \( m \) for all \( \theta \in [m, \hat{\theta}] \). Wages for acceptable \( \theta \) outside of this set [i.e., when \( \theta \geq \hat{\theta} \)] are determined according to [24].

We can now consider the individual’s search problem given this wage
offer function. Using the $\varepsilon$ interval formulation,

$$
\tilde{V}_n = \frac{b \varepsilon}{1 + \rho \varepsilon} + \frac{\lambda \varepsilon}{1 + \rho \varepsilon} \left\{ \int_{m}^{\hat{\theta}(m, \tilde{V}_n)} \left[ \frac{m + \eta \tilde{V}_n}{\rho + \eta} \right] dG(\theta) \right\}
$$

$$
+ \int_{\hat{\theta}(m, \tilde{V}_n)}^{\theta} \left[ \frac{\alpha (\theta - \rho \tilde{V}_n)}{\rho + \eta} \right] dG(\theta) + \tilde{V}_n G(m) \right\}
$$

$$
+ \frac{1 - \lambda \varepsilon}{1 + \rho \varepsilon} \tilde{V}_n + \frac{o(\varepsilon)}{1 + \rho \varepsilon}.
$$

The first integral which appears in (25) is

$$
\int_{m}^{\hat{\theta}(m, \tilde{V}_n)} \left[ \frac{m + \eta \tilde{V}_n}{\rho + \eta} \right] dG(\theta) = \left[ \frac{m + \eta \tilde{V}_n}{\rho + \eta} \right] \int_{m}^{\hat{\theta}(m, \tilde{V}_n)} dG(\theta)
$$

$$
= \left[ G(\hat{\theta}(m, \tilde{V}_n)) - G(m) \right] \left[ \frac{m + \eta \tilde{V}_n}{\rho + \eta} \right],
$$

and represents the value of encountering a match value in the interval $[m, \hat{\theta})$, which is $V_e(m) = (m + \eta \tilde{V}_n)/(\rho + \eta)$, multiplied by the probability of encountering a match in this interval, which is $G(\hat{\theta}(m, \tilde{V}_n)) - G(m)$. The second integral contains an integrand which is the product of the value of employment given $\theta > \hat{\theta}(m, \tilde{V}_n)$ and the density of $\theta$. Taking limits after collecting terms, we have

$$
\rho \tilde{V}_n = b + \frac{\lambda}{\rho + \eta} \left\{ \left[ G(\hat{\theta}(m, \tilde{V}_n)) - G(m) \right] [m - \rho \tilde{V}_n] \right\}
$$

$$
+ \alpha \int_{\hat{\theta}(m, \tilde{V}_n)}^{\theta} [\theta - \rho \tilde{V}_n] dG(\theta). \right\}
$$

It is important to note a fundamental difference between the value $\rho \tilde{V}_n$, which we might want to refer to as the “implicit” reservation wage in the presence of a binding minimum wage constraint, and $\rho V_n$, the corresponding “explicit” reservation wage [and match] value when no minimum wage constraint is binding. The value of $\rho V_n$ is an acceptance value, that is, it completely characterizes the decision of whether or not an employment contract is struck. When a binding minimum wage is present, employment contacts are formed whenever $\theta \geq m$. The value of $\rho \tilde{V}_n$ is only instrumental in determining the equilibrium wage contract and wage distribution. Put another way, when there is no binding minimum wage constraint, the smallest possible observed wage will be equal to $\rho V_n$, and the distribution of wages
will be continuous as long as $G$ itself is. When there is a binding minimum wage, the smallest observed wage will be given by $m$, and we know that $\rho \tilde{V}_n < m$. The observed wage distribution will consist of a mass point at the minimum wage, the size of which is given by $G(\hat{\theta}(m, \tilde{V}_n)) - G(m)$, while the distribution of wages immediately above $m$ will be continuous [once again, as long as $G$ is]. In the presence of a binding minimum wage, the observed wage distribution is given by

$$p(w) = \begin{cases} \frac{\alpha^{-1}G(\hat{\theta}(w, \tilde{V}_n))}{G(m)} & w > m \\ \frac{G(\hat{\theta}(m, \tilde{V}_n)) - G(m)}{G(m)} & w = m \\ 0 & w < m \end{cases}$$

We shall analyze the solution to the constrained search-bargaining model further in the following chapter. For the present, we only provide an illustrative example.

1.6.1 An Example (Continued)

Let us consider the effect of the imposition of a minimum wage in our previous labor market example. As we will see in the next chapter, the imposition of a minimum wage may have large impacts on labor market outcomes, and the nature of the impact will very much depend on the level of the bargaining power parameter $\alpha$. We will postpone the explicit discussion of welfare issues until later. For now, the reader might want to think of the analysis as pertaining to a situation in which a local labor market characterized by $b = -1$, $\lambda = .5$, $\eta = .02$, $\rho = .01$, and $\theta$ distributed uniformly on the interval $[0, 10]$ is inhabited by groups of individuals with different levels of bargaining power, $\alpha \in \{0, .25, .5, .75, 1\}$. As is true in the U.S., at least over the past several decades, one minimum wage is applied to all labor market participants, independently of the characteristics of their local labor market. We will describe the effects of imposing a minimum wage $m = 6$ in this example on the labor market outcomes for individuals with varying levels of bargaining power.

Using the simple form of $G$ in this example, we can solve for $\rho \tilde{V}_n$ in [26] as follows. Write

$$x = b + \delta \left\{ \frac{1}{10} \hat{\theta}(m, x) - m \right\} [m - \rho \tilde{V}_n]$$

$$+ \frac{\alpha}{10} \int_{\hat{\theta}(m, x)}^{10} [\theta - \rho \tilde{V}_n] d\theta,$$
where $x \equiv \rho \tilde{V}_n$, 

TO BE COMPLETED ...