Introduction to Econometrics
Fall 2004
Assignment 6

Today’s Date: 11/2/2004
Due Date: 11/8/2004

Please show all of your work and clearly indicate your final response to each question.

1. (Review) A random variable $Y$ is normally distributed with mean $\mu$ and variance $\sigma_y^2$ in a population. You have access to a random sample of size $N$ from the population. Denote the sample draws by $\{y_1, \ldots, y_N\}$.

1. Define the least squares estimator of $\mu$ as

$$\hat{\mu} = \arg\min_{\mu} \sum_{i=1}^{N} (y_i - \mu)^2.$$  

Show that this estimator of $\mu$ is unbiased. Derive the standard error associated with this estimator.

2. Define the maximum likelihood estimator of $\mu$, which you can denote by $\hat{\mu}$. How are the $\hat{\mu}$ and $\hat{\mu}$ related?

3. Add regressors to this model. Instead of every population member have a common mean value of $Y$, let the mean for individual $i$ be given by $X_i\beta$, where $X_i$ is a $1 \times K$ vector of observed characteristics and $\beta$ is a $K \times 1$ (unknown parameter vector). Compare the least squares and maximum likelihood estimators of $\beta$ in this case.

2. A dependent variable $y$ has a uniform distribution on the interval $[0, a]$, so that the density (p.d.f.) of $y$ is given by

$$f(y) = \frac{1}{a} \mathbf{1}[y \in [0, a]].$$

You have access to a random sample of 5 draws from this distribution, and your goal is to estimate the unknown parameter $a$. The observations from the random sample are $\{2, 1, 5, 3, 8\}$.

1. Derive the maximum likelihood estimator of $a$, which we will denote by $\hat{a}$. What is the maximum likelihood estimate of $a$ from this sample?
2. Consider an alternative estimator for \( a \), which is

\[
\tilde{a} = 2\bar{y}_N,
\]

where \( \bar{y}_N \) is the sample mean based on \( N \) observations. Find the value of \( \tilde{a} \) from this sample.

3. Determine whether \( \tilde{a} \) and \( \hat{a} \) are unbiased. Based on this criterion alone, which one would you favor?

4. Assuming that the variance of \( \hat{a} \) is a decreasing function of sample size (which it is), are both estimators consistent?

3. Let the difference in utility between choosing an action \( a \) and not choosing it to individual \( i \) be given by

\[
U^*_i = \beta x_i + \varepsilon_i,
\]

where the random variable \( \varepsilon_i \) is uniformly distributed on the interval \([-0.5, 0.5]\) for all individuals \( i \), \( x_i \) denotes an observable characteristic individual \( i \), and \( \beta \) is an unknown parameter. The individual chooses the action \( a \) if the net utility from doing so is positive, that is,

\[
d_i = \begin{cases} 
1 & \text{if and only if } U^*_i > 0 \\
0 & \text{if and only if } U^*_i \leq 0
\end{cases}.
\]

1. Derive the probability that individual \( i \) with characteristics \( x_i \) will choose action \( a \).

2. Say that you have access to a random sample of observations on \( \{d_i, x_i\}_{i=1}^N \). Derive the maximum likelihood estimator of \( \beta \).