[1] Consider a model of a family farm. Individual effort $e$ is aggregated over $n$ individuals to produce output $f(\sum_{i=1}^n e_i)$. Individual consumption is $c$, and $\sum_{i=1}^n c_i = f(\sum_{i=1}^n e_i)$. Individuals have identical preferences given by $u(c) - v(e)$, where $u(c)$ is the utility from consumption and $v(e)$ is the disutility from effort. Make the usual assumptions on $u$, $v$ and $f$.

(i) Write down the conditions characterizing the symmetric first best allocation (studied in class). Call these $c^*$, $e^*$, and let $E^* \equiv ne^*$.

(ii) Now suppose that $e_i$ is chosen independently by agent $i$, under the assumption that the total output will be equally divided. Write down the conditions characterizing a symmetric (interior) equilibrium, and let $\hat{c}$, $\hat{e}$, $\hat{E}$, be the outcome. Compare these values with $c^*$, $e^*$, and $E^*$. Why do we get underproduction relative to the first best?

(iii) Find out what happens to $\hat{E}$ in this model when $n$ is reduced (go ahead, take derivatives even though $n$ is an integer). Under what conditions might $\hat{E}$ even go up? explain why this is possible here, even though in the first-best problem it is impossible for $dE^*/dn$ to be negative (unless the derivative includes changed taxation levels).

[2] Here is a variation on problem [1]. Suppose that payments are made according to work points. That is, at the end of the production period, each person accumulates points equal to $e_i$, the effort that she put in. Then form the share

$$s_i \equiv \frac{e_i}{\sum_j e_j}$$

which is the fraction of workpoints contributed by person $i$. Divide aggregate output according to these shares.

(i) Suppose that $e_i$ is chosen independently by agent $i$, under the assumption that the total output will be divided according to workpoints. Write down the conditions characterizing a symmetric (interior) equilibrium, and let $\tilde{c}$, $\tilde{e}$, $\tilde{E}$, be the outcome. Compare these values with $c^*$, $e^*$, and $E^*$. Why do we get overproduction relative to the first best?

(ii) Now the idea is to combine the underproduction from equal division with the overproduction from workpoints. Can you construct a mix of the two systems (some part of output divided equally, some part according to workpoints) such that the overall equilibrium implements exactly the first best? What does the mix depend on? [For more on this topic, see Amartya Sen, “Labor Allocation in a Cooperative Enterprise,” Review of Economic Studies 1964.

[3] Our analysis also throws some light on the celebrated “neutrality problem” in the provision of public goods. Suppose that a community of $n$ individuals produces two goods. Individual $i$ has resources $w_i$, which must be divided between a private good $c_i$ and contributions to a public good $r_i$. The contributions together produce a pure public good $g$, according to the production function $g(r)$, where $r$ is the sum of all the individual contributions $r_i$. 

Each person has an identical utility function \( u(c) + v(g) \), where these functions have all the usual properties (smoothness, strict concavity, unbounded steepness at zero).

Contributions are made selfishly: each individual takes as given the sum of all other contributions and maximizes with respect to his own.

[A] Prove that every individual who makes a positive contribution must get the same utility in equilibrium, regardless of wealth. Explain this result verbally.

[B] Now suppose that there are only two individuals. Take both \( u \) and \( v \) to be logarithmic, and \( g(r) = \sqrt{r} \). Find a critical ratio of wealth levels such that if individual wealths are less dispersed than this ratio, then the conditions of [A] hold.

[C] Go back to the general case — but with only two agents — and assume that both make positive contributions. Prove that a small transfer of wealth from one agent to another has no effect on the utility (common utility, by part [A]) of the two individuals.

[D] Mancur Olson has argued that higher inequality may sometimes be better for the provision of public goods, because a greater portion of the marginal gains from the public good is internalized by the rich. Assuming two individuals and using parts [A] and [B], evaluate the Olson argument.

[E] Discuss whether the results in part [C] are robust with respect to dropping the additive specification of inputs in the production function for \( g \).

[4] Here is an application of these broad ideas to demography. Consider an economy with a rural sector and an industrial sector. Assume that all job seekers come from the rural sector. In the rural sector, there are \( N \) identical families presently engaged in making fertility decisions. If there are \( n \) children in a family assume that the present value cost to the family of rearing them is \( C(n) \). Define \( C''(n) \equiv C(n+1) - C(n) \), and assume that \( C''(n) \) is increasing in \( n \).

At the time these children come of age, assume that there will be exactly \( N \) urban jobs available, one for each family. [But, of course, there may be more than \( N \) (former) children seeking these jobs.] If a job is obtained, assume that it carries a lifetime income \( w \). If not, then assume that the lifetime income is \( v \), where \( w > v \).

Each family takes as given the probability of each of its children getting an urban job, assumes it cannot influence this probability (\( N \) is “large”), and chooses the number of children to maximize expected lifetime income.

Describe the decision-making process of the family. Show that the aggregate outcome might well be Pareto-inferior for all the families, while for each individual family, the decisions are perfectly rational. How does the Pareto-optimal number of children compare to the family equilibrium?