[1] If $\delta$ is close enough to zero then every subgame perfect equilibrium must involve the play of a static Nash equilibrium after every $t$-history, if the number of actions is finite. To see this, fix any action profile $a$ that is not a Nash equilibrium. Let $d(a) \equiv \max_i [d_i(a) - f_i(a)]$, where $d_i(a)$ is the maximum payoff to player $i$ assuming all others are playing $a_{-i}$. Then $d(a) > 0$, because $a$ isn’t a Nash profile. Let $d \equiv \min d(a)$, where the minimum is taken over all profiles $a$ that are not one-shot Nash. Because there are only finitely many action profiles, $d(a)$ must be positive. Let $M$ and $m$ be the maximum and minimum payoffs available to anyone in the game. Now pick $\delta$ close enough to zero such that

$$d > \frac{\delta}{1 - \delta} (M - m).$$

It is easy to see that for such $\delta$ there is no other subgame perfect equilibrium than the play of a one-shot Nash in every period.

This conclusion may be false if there are infinitely many actions available to each player. Take this two-person game in which each $A_i = [0, 1]$, and the payoff to player $i$ from $(a_i, a_j)$ is $2a_j - a_i^2$. Then each $i$'s dominant strategy is to set $a_i = 0$, but of course, it is easy to check that all symmetric actions $(a, a)$ yield payoffs that are increasing in $a$ (on $[0, 1]$). For any discount factor, some $a > 0$ can be supported if

$$2a(1 - \delta) \leq 2a - a^2.$$

It is easy to see that no matter how close $\delta$ is to zero, there is some positive value of $a$ (depending on $\delta$, of course), that can be supported.

[2] (a) Not exactly correct. If the worst punishment to player $i$ does not entail her playing a best response (statically), then player $i$’s continuation payoff must be strictly higher than the worst punishment. Let $p^i$ denote the vector that punishes player $i$ and let $(a, p', \hat{p}^1, \ldots, \hat{p}^n)$ be any supporter of $p^i$. Then by the condition of support,

$$p^i = (1 - \delta)f_i(a) + \delta p'_i,$$

while

$$p^i_i \geq (1 - \delta)d_i(a) + \delta \hat{p}^i_i$$

Combining the two and using the fact that $d_i(a) > f_i(a)$, we see that $p'_i > \hat{p}^i_i \geq p^i_i$.

(b) The player can always play a static best response to the action profile at any date, for every history. This will guarantee her at least her security level in every period and therefore over her lifetime.

[3] (a) Take any $p \in F^*$. Then there exist convex weights $\lambda_1, \ldots, \lambda_m$ (where $m$ is finite) and $m$ action profiles $a^1, \ldots, a^m$ such that

$$p = \sum_{k=1}^m \lambda_k f(a^k).$$
Because any system of weights can be approximated by a set of rational weights, for every \( \epsilon' > 0 \) there exists an integer \( N \) and numbers \( s_1, \ldots, s_m \) adding to \( N \) such that

\[
| \left( \frac{s_1}{N}, \ldots, \frac{s_m}{N} \right) - \lambda | < \epsilon'.
\]

Consequently, for every \( \epsilon'' > 0 \) if I define

\[
p'' = \sum_{k=1}^{m} \frac{s_k}{N} f(a^k),
\]

then

(1) \[ |p'' - p| < \epsilon''. \]

Now consider the repeated game, and play the action profiles \( a^1, \ldots, a^m \) in sequence, playing \( a^k \) \( s_k \)-many times, for a total of \( N \) plays. Repeat this cycle forever. For each \( \delta \), let \( p(\delta) \) be the normalized lifetime payoff thus generated. It is easy to see that

(2) \[ p(\delta) \to p'' \text{ as } \delta \to 1. \]

Combining (1) and (2), we are done.

[4] Properties of the “support mapping” \( \phi \).

[a] Let \( p \in \phi(E) \), then it has supporter \( (a, p', p_1, \ldots, p_n) \), where \( (p', p_1, \ldots, p_n) \in E \). If \( E \subset E' \), then \( (p', p_1, \ldots, p_n) \) lie in \( E' \) as well. It follows that \( p \in \phi(E') \).

[b] Let \( p^{(m)} \) be a sequence of payoff vectors in \( \phi(E) \) converging to \( p \). We need to show that \( p \in \phi(E) \). Attached to each \( p^{(m)} \) is a supporter \( (a^{(m)}, p'^{(m)}, p^{1(m)}, \ldots, p^{n(m)}) \). All \( a^{(m)} \)'s lie in a compact set, and so does the rest of the supporter. Extract convergent subsequence such that \( (a^{(m)}, p'^{(m)}, p^{1(m)}, \ldots, p^{n(m)}) \to (a, p', p^{1}, \ldots, p^n) \). By compactness of \( A \) and \( E \), this last collection is itself a valid supporter. We have to show that it supports \( p \). The only step to take care of here is the use of the maximum theorem, to argue that the “maximum deviation function” \( d_i(a) \) is continuous in \( A \).

[5] In a homogeneous Bertrand game the one-shot NE gives everyone their security level, so it must be the worst punishment. Can’t say the same for Cournot.

[6] [a1] I will need to add a condition to Conditions 1 and 2:

Condition 3. For every symmetric action \( a \) for the others, \( f(0, a) \) is bounded in \( a \) (for instance, can set \( f(0, a) = 0 \)).

For each person, write the action set as \([0, \infty)\). Fix some symmetric action \( a \) for the other players and look at one player’s best response. Condition 1 tells us that this player’s payoff is quasiconcave in his own actions and condition 2 tells us that the best payoff is well-defined. By quasiconcavity, the set of best responses \( A(a) \) to \( a \) is convex-valued. By continuity of payoffs, \( A(a) \) is upperhemicontinuous.

Now I claim that for large \( a \) we have \( a' < a \) for all \( a' \in A(a) \). Suppose on the contrary that there is \( a_m \to \infty \) and \( a'_m \in A(a_m) \) for each \( a_m \) such that \( a'_m \geq a_m \). So there is a sequence \( \lambda_m \in (0, 1] \) such that \( a_m = \lambda_m a'_m \) for all \( a_m \). By quasiconcavity, \( f(a_m) \geq \)
min\{f(0,am), f(a'_m, am)\}. But the former term is bounded by Condition 3, and the latter term is bounded below by condition 2. This contradicts the fact that \( f(am) \to -\infty \) as \( m \to \infty \).

This proves, by a slight variation on the intermediate value theorem, that there exists \( a^* \) such that \( a^* \in A(a^*) \). Clearly, \( a^* \) is a strongly symmetric equilibrium, which proves [a1].

[a2] This means that the set of strongly symmetric perfect equilibrium payoffs \( V \) is nonempty. Simply repeat \( a^* \) regardless of history. Now define \( d \equiv \inf_a d(a) > -\infty \) by assumption. \( d \) is like the strongly symmetric security level. Lifetime payoffs can’t be pushed below this. Therefore the infimum of payoffs in \( V \) is at least as great as \( d \).

Of course, the supremum is bounded because all one-shot payoffs are bounded by assumption, so in particular the symmetric equilibrium payoff is bounded above.

[a3] Let \( p^m \) be a sequence of strongly symmetric equilibrium payoffs in \( V \) converging down to the infimum payoff \( p \). For each such \( p^m \) let \( a^m(t) \) be an action path supporting \( p^m \) using strongly symmetric action profiles at any date. Let \( M \) be the maximum strongly symmetric payoff in the game. Then, because infimum payoffs in \( V \) are bounded below, say by \( d \), it must be the case that

\[
(1 - \delta)f(a^m(t)) + \delta M \geq d
\]

for every date \( t \). But this means that there exists an upper bound \( \bar{a} \) such that \( a^m(t) \leq \bar{a} \) for every index \( m \) and every date \( t \).

This bound allows us to extract a convergent subsequence of \( m \) — call it \( m' \) — such that for every \( t \),

\[
a^{m'}(t) \to a(t).
\]

It is very easy to show that that the simple strategy profile defined as follows:

“Start up \( \{a(t)\} \). If there are any deviations, start it up again,”

is a simple penal code that supports the infimum punishment.

[a4] Finally, prove the compactness of \( V \). Take any payoff sequence \( p^m \) each of which lies in \( V \), converging to some \( p \). Each can be supported by some action path \( a^m(t) \), with the threat of starting up the simple penal code of [a3] in case there is any deviation. Take a convergent subsequence of \( a^m(t) \), call the pointwise limit path \( a(t) \), and show that it supports \( p \) with the threat of retreating to \( g(t) \) if there is any deviation.

(b) Part [a]. Fix some strongly symmetric equilibrium \( \hat{\sigma} \) with payoff \( v^* \). Because the continuation payoff can be no more than \( v^* \), the first period action along this equilibrium must satisfy

\[
f(a) \geq \frac{-\delta v^* + v^*}{1 - \delta}.
\]

Using Condition 1, it is easy to see that there exists \( a_* \) such that \( f(a_*) = \frac{-\delta v^* + v^*}{1 - \delta} \). By Condition 2, it follows that \( d(a_*) \leq d(a) \). Now, because \( \hat{\sigma} \) is an equilibrium, it must be the case that

\[
v_*(1 - \delta)d(a) + \delta v_*(1 - \delta)d(a) + \delta v_*
\]
so that the proposed strategy is immune to deviation in Phase I. If there are no deviations, we apply some SGPE creating \( v^* \), so it follows that this entire strategy as described constitutes a SGPE.

Part [b]. Let \( \tilde{\sigma} \) be a strongly symmetric equilibrium which attains the equilibrium payoff \( v^* \). Let \( a \equiv a(\tilde{\sigma}) \) be the path generated. Then \( a \) has symmetric actions \( a(t) \) at each date, and

\[
v^* = (1 - \delta) \sum_{t=0}^{\infty} \delta^t f(a_t).
\]

Clearly, for the above equality to hold, there must exist some date \( T \) such that \( f(a_T) \geq v^* \). Using Condition 1, pick \( a^* \geq a_T \) such that \( f(a^*) = v^* \). By Condition 2, \( d(a^*) \leq d(a_T) \). Now consider the strategy profile that dictates the play of \( a^* \) forever, switching to Phase I if there are any deviations. Because \( \tilde{\sigma} \) is an equilibrium, because \( v^* \) is the worst strongly symmetric continuation payoff, and because \( v^* \) is the largest continuation payoff along the equilibrium path at any date, we know that

\[
v^* \geq (1 - \delta) d(a_T) + \delta v^*.
\]

Because \( d(a_T) \geq d(a^*) \),

\[
v^* \geq (1 - \delta) d(a^*) + \delta v^*
\]
as well, and we are done.

(c) We know that in the punishment phase,

\[
v_* \geq (1 - \delta) d(a_*) + \delta v_* \tag{3}
\]

while along the equilibrium path,

\[
v_* = (1 - \delta) f(a_*) + \delta v^*. \tag{4}
\]

Suppose that strict inequality were to hold in (3), so that there exists a number \( v < v_* \) such that

\[
v \geq (1 - \delta) d(a_*) + \delta v. \tag{5}
\]

Using Condition 1, pick \( a \geq a_* \) such that

\[
v = (1 - \delta) f(a) + \delta v^*. \tag{6}
\]

[To see that this is possible, use Condition 1, (4), and the fact that \( v < v_* \).] Note that \( d(a) \leq d(a_*) \), by Condition 2. Using this information in (5), we may conclude that

\[
v \geq (1 - \delta) d(a) + \delta v. \tag{7}
\]

Combining (6) and (7), we see from standard arguments (check) that \( v \) must be a strongly symmetric equilibrium payoff, which contradicts the definition of \( v_* \).

[7] and [8] read the references!