
First I go over what we did in class, and then I will indicate the very simple step that I could not finish during the lectures.

Consider a country which has pegged its exchange rate at some value $e$. (Think of $e$ as the number of dollars required to buy one unit of the domestic currency.) We shall assume that the exchange rate is overvalued, in the following sense: suppose that there is some random variable $\theta$ (the state) on $[0, 1]$ which determines the “true” exchange rate $f(\theta)$ were the currency to be allowed to float at $\theta$. Then $e$ always exceeds $f(\theta)$ for all $\theta \in [0, 1]$.

But $\theta$ also influences the exchange rate: which is to say that $f(\theta)$ varies with $\theta$. Arrange so that $f(\theta)$ is strictly increasing in $\theta$. So the idea is that $\theta$ is some “fundamental” which influences the country’s capacity to export or import, or to attract investment; the higher being $\theta$, the more favorable the climate.

Now there is a bunch of speculators (of total measure 1), each of whom can sell one unit of the local currency. If they do, they pay a transactions cost $t$. If the government holds the peg, the exchange rate stays where it is, and the payoff to selling is $-t$. If the government abandons the peg, then the speculators make a profit of $e - f(\theta)$, so their net payoff is $e - f(\theta) - t$.

What about the government’s decisions? It has only one decision to make: whether to abandon or to retain the peg. We assume that it will abandon the peg if the measure of speculators exceeds $a(\theta)$, where $a$ is increasing in $\theta$ (that is, if the basic health of the economy is better, the government is more reluctant to abandon). We will assume that there is some positive value of $\theta$, call it $\bar{\theta}$, such that below $\bar{\theta}$ the situation is so bad that the government will abandon the peg anyway. In other words we are assuming that $a(\theta) = 0$ for $\theta \in [0, \bar{\theta}]$. Then it rises but always stays less than one by assumption.

Consider, now, a second threshold for $\theta$ which we’ll call $\bar{\theta}$: this is the point above which no one wants to sell the currency even though she feels that the government will abandon the peg for sure. In other words, $\bar{\theta}$ solves the equation

\[ e - f(\bar{\theta}) - t = 0. \]  

We will assume that such a $\bar{\theta}$, strictly less than one, indeed exists. But we also suppose that there is a gap between $\bar{\theta}$ and $\bar{\theta}$: that $\bar{\theta} < \bar{\theta}$.

[If there were no such gap, there wouldn’t be a coordination problem to start with.]

Now we are ready to begin our discussion of this model. First assume that the realization of $\theta$ is perfectly observed by all agents, and that this information is common knowledge. Then there are obviously three cases to consider.

---

1See Morris and Shin (1998) for a very simple account of how to derive $a(\theta)$ from a somewhat more basic starting point.
Case 1. \( \theta \leq \bar{\theta} \). In this case, the government will abandon the peg for sure. The economy is not viable, all speculators must sell, and a currency crisis occurs.

Case 2. \( \theta \geq \bar{\theta} \). In this case no speculator will attack the currency, and the peg will hold for sure.

Case 3. \( \bar{\theta} < \theta < \bar{\theta} \). Obviously this is the interesting case, in which multiple equilibria obtain. There is an equilibrium in which no one attacks, and the government maintains the peg. There is another equilibrium in which everyone attacks and the government abandons the peg. This is a prototype of the so-called “second-generation” financial crises models, in which expectations — over and above fundamentals — play an important role.

So much for this standard model. Now we drop the assumption of common knowledge of realizations (but of course we maintain the assumption of common knowledge of the information structure that I am going to write down).

Suppose that \( \theta \) is distributed uniformly on \([0, 1]\): its value will be known perfectly at the time the government decides whether or not to hold the peg or to abandon it. Initially, however, the realization of \( \theta \) is noisy in the following sense: each individual sees a signal \( x \) which is distributed uniformly on \([\theta - \epsilon, \theta + \epsilon]\), for some tiny \( \epsilon > 0 \) (where \( \theta \) is the true realization). Conditional on the realization of \( \theta \), this additional noise is iid across agents.

Proposition 1. There is a unique value of the signal \( x \) such that an agent attacks the currency if \( x < x^* \) and does not attack if \( x > x^* \).

This is an extraordinary result in the sense that a tiny amount of noise refines the equilibrium map considerably. Notice that as \( \epsilon \to 0 \), we are practically at the common knowledge limit (or are we? the question of what sort of convergence is taking place is delicate and important here), yet there is no “zone” of multiple equilibria! The equilibrium is unique.

What is central to the argument is the “infection” created by the lack of common knowledge (of realizations). To see this, we work through a proof of Proposition 1, with some informal discussion.

Start by looking at the point \( \bar{\theta} - \epsilon \). Suppose that someone receives a signal \( x \) of this value or less. What is she to conclude? She doesn’t know what everyone else has seen, but she does know that the signal is distributed around the truth with support of size \( 2\epsilon \). This means that the true realization cannot exceed \( \bar{\theta} \), so the government will abandon the peg for sure. So she will sell. That is, we’ve shown that for all

\[
x \leq x_0 \equiv \bar{\theta} - \epsilon,
\]

it is dominant to sell.

Now pick someone who has a signal just bigger than \( x_0 \). What does he conclude? Suppose, for now, he makes the assumption that only those with signals less than \( x_0 \) are selling; no one else is. Now what is the chance — given his own signal \( x \) — that someone else has received a signal not exceeding \( x_0 \)? To see this, first note that the true \( \theta \) must lie in \([x - \epsilon, x + \epsilon]\). For each such \( \theta \) the chances that the signal for someone else is below \( x_0 \) is \((1/2\epsilon)[x_0 - (\theta - \epsilon)]\), so the overall chances are just these probabilities integrated over all conceivable values of \( \theta \), which yields \((1/2\epsilon)(x_0 - (x - \epsilon))\). So the “infection” spreads: if \( x \) is close to \( x_0 \), these chances are close to 1/2. In this region, moreover, it is well known that the government’s threshold is very low: close to zero sellers (and certainly well less than half the
population) will cause an abandonment of the peg. Knowing this, such an \( x \) must sell. Knowing that all with signals less than \( x_0 \) must sell, we have deduced something stronger: that some signals above \( x_0 \) must create sales as well.

So let us proceed recursively: Suppose we have satisfied ourselves that for some index \( n \), everyone sells if the signal is no bigger than \( x_n \) (we already know this for \( x_0 \)). We define \( x_{n+1} \) as the largest value of the signal for which people will want to sell, knowing that all below \( x_n \) are selling.

This is a simple matter to work out. Fix \( x \geq x_n \), and imagine any \( \theta \in \left[ x - \epsilon, x + \epsilon \right] \).

For such \( \theta \), everybody with a signal between \( \theta - \epsilon \) and \( x_n \) (such an interval may be empty, of course) will attack, by the recursive assumption. Because these are the only attackers (also by the recursive assumption), the government will yield iff

\[
\frac{1}{2\epsilon} \left[ x_n - (\theta - \epsilon) \right] \geq \alpha(\theta),
\]

or

\[
\theta + 2\epsilon\alpha(\theta) \leq x_n + \epsilon
\]

So we can define an implicit function \( h(x, \epsilon) \) such that the above inequality translates into

\[
\theta \leq h(x_n, \epsilon).
\]

Put another way, the implicit function \( h(x, \epsilon) \) solves the equation

\[
2
\]

It follows that if our person with signal \( x \) were to attack, her expected payoff would be given by

\[
\frac{1}{2\epsilon} \int_{x - \epsilon}^{h(x_n, \epsilon)} \left[ e - f(\theta) \right] d\theta - t.
\]

Now retrace the recursion starting all the way from \( n = 0 \): we have \( x_0 = \theta - \epsilon \).

Then (remembering that \( \alpha(\theta) = 0 \) for all \( \theta \leq \theta \)) it is easy to see that (3) reduces to

\[
\frac{1}{2\epsilon} \int_{x - \epsilon}^{\theta} \left[ e - f(\theta) \right] d\theta - t.
\]

For \( x \approx x_0 \), this is just

\[
\frac{1}{2\epsilon} \int_{x - 2\epsilon}^{\theta} \left[ e - f(\theta) \right] d\theta - t,
\]

which is certainly strictly positive. So \( x_1 \) is well-defined, and \( x_1 > x_0 \).

Now put \( x_1 \) in place of \( x_0 \), and repeat the process. Notice that \( h \) is increasing in \( x \), so if we replace \( x_0 \) by \( x_1 \) in (3), then, evaluated at \( x = x_1 \), the payoff must turn strictly positive.\(^2\) So the new \( x_2 \), which is the maximal signal for which people will sell under the belief that everyone less than \( x_1 \) sells, will be still higher than \( x_1 \). And so on: the recursion creates a strictly increasing sequence \( \{x_n\} \), which converges from below to \( x^* \), where \( x^* \) solves

\[
\frac{1}{2\epsilon} \int_{x^* - \epsilon}^{h(x^*, \epsilon)} \left[ e - f(\theta) \right] d\theta - t = 0.
\]

It is very easy to see that there is a unique solution to \( x^* \) defined in this way. In fact, something stronger can be established:

\(^2\)This is on the assumption that the sequence \( \{x_n\} \) stays bounded below \( \theta \). This will certainly be the case, see below, so it’s not really an assumption at all.
To prove this, consider any \( x^* \) in which a speculative attack is carried out by an individual if and only if \( x^* \), which is nonnegative as we've just argued. But this contradicts the Claim.

To learn a bit more about \( x^* \), use (2) to see that \( h(x, \epsilon) - x + \epsilon = 2\epsilon[1-a(h(x, \epsilon))] \), so that

\[
0 = \frac{1}{2\epsilon} \int_{x^* - \epsilon}^{h(x^*, \epsilon)} [e - f(\theta)]d\theta - t = [1 - a(h(x, \epsilon))]\epsilon - \frac{1}{2\epsilon} \int_{x^* - \epsilon}^{h(x^*, \epsilon)} f(\theta)d\theta - t,
\]

or

\[
e - \frac{1}{2\epsilon} \int_{x^* - \epsilon}^{h(x^*, \epsilon)} f(\theta)d\theta - t = a(h(x, \epsilon))\epsilon.
\]

A comparison of this equation with (1) categorically shows that \( x^* \) is bounded below \( \bar{\theta} \) for small \( \epsilon \).

So there is a unique solution to \( x^* \) and it is below \( \bar{\theta} \), which justifies the previous recursive analysis (see in particular, footnote 2). Notice also that our analysis shows that every equilibrium must involve attack for signals less than \( x^* \).

We covered the material up to this point in class. What is left is to show that the above strategy describes a unique equilibrium. This comes next.

To complete the proof, we must show that no signal above \( x^* \) can ever attack. Suppose, on the contrary, that in some equilibrium some signal above \( x^* \) finds it profitable to attack. Take the supremum of all signals under which it is weakly profitable to attack: call this \( x' \). Then at \( x' \) it is weakly profitable to attack. Suppose we now entertain a change in belief by supposing that everybody below \( x' \) attacks for sure; then this cannot change the weak profitability of attack at \( x' \). But the profit is

\[
\frac{1}{2\epsilon} \int_{x^* - \epsilon}^{h(x', \epsilon)} [e - f(\theta)]d\theta - t,
\]

which is nonnegative as we've just argued. But this contradicts the Claim.

So we have proved that there is a unique equilibrium to the “perturbed” game, in which a speculative attack is carried out by an individual if and only if \( x \leq x^* \). As \( \epsilon \to 0 \), this has an effect of refining the equilibrium correspondence dramatically. To describe this, calculate the threshold \( x^* \) as \( \epsilon \to 0 \). The easiest way to do this is the “sandwich” inequality:

\[
[e - f(h(x^*, \epsilon))][1-a(h(x^*, \epsilon))] \leq \frac{1}{2\epsilon} \int_{x^* - \epsilon}^{h(x^*, \epsilon)} [e - f(\theta)]d\theta \leq [e - f(h(x^*, \epsilon))][1-a(h(x^*, \epsilon))],
\]

which is obtained by noting that \( f(x^* - \epsilon) \leq f(\theta) \leq f(h(x^*, \epsilon)) \) for all \( \theta \in [x^* - \epsilon, h(x^*, \epsilon)] \). Both sides of the sandwich go to the same limit, because \( x^* \) and \( h(x^*, \epsilon) \)
— as well as the realization of the state — all go to a common limit, call it \( \theta^* \).

This limit solves the condition

\[
(e - f(\theta^*))[1 - a(\theta^*)] = t.
\]

It is obvious that there is a unique solution to (6).

Note: At this point be careful when reading Morris-Shin. There is an error in Theorem 2. See Heinemann (AER 2000) for a correction of this error which agrees with the calculations provided here.