1 Asset Pricing: Replicating portfolios

Consider an economy with two states of nature \{s_1, s_2\} and with two primary assets, \{1, 2\}. The first asset is a stock with price \(q_1\) and returns \((x_1^{s_1}, x_1^{s_2})\), \(x_1^{s_1} \neq x_1^{s_2}\). There is also a bond with price \(q_2\) and returns \((x_2, x_2)\). Taking this data as parameters, consider the problem of pricing a derivative, asset \(d\), with returns \((x_d^{s_1}, x_d^{s_2})\). Denote the price of the derivative asset as \(q_d\). Construct a portfolio with the two primary assets which replicates the payoff vector of the new asset. This requires solving the following system

\[
\begin{pmatrix}
x_2^2 \\
x_2^1 
\end{pmatrix}
= \xi_1 \begin{pmatrix}
x_1^1 \\
x_1^2 
\end{pmatrix} + \xi_2 \begin{pmatrix}
x_2^1 \\
x_2^2 
\end{pmatrix}.
\]

Note that the solution to this system exists and is unique. Now, by the “Law of One price”, the replicating portfolio and the derivative asset must have the same price since they have the same returns vector. Therefore, \(q_d = \xi_1 q_1 + \xi_2 q_2\). Note how this approach relied totally on the non existence of arbitrage opportunities that lies behind the “Law of One price”.

But how can you price an asset from fundamentals? that is, without knowing the prices of the components of a replicating portfolio?

Most importantly: the price of an asset depends on its expected returns and on its risk, but what is the relevant component of risk?

2 Asset Pricing: Consumption Capital Asset Pricing Model (Consumption CAPM)

Consider an agent living two periods, \(t\) and \(t+1\). His utility in terms of consumption \(c_t\) and \(c_{t+1}\) is:

\[u(c_t) + \beta u(c_{t+1})\]

where \(\beta < 1\) is the discount rate. We assume \(u(c)\) is smooth, strictly monotonic, and strongly concave, and the agent is an expected utility maximizer.

The agent faces a wealth process \(w_t, w_{t+1}\). He can trade \(J\) assets; asset \(j = 1, \ldots, J\) has payoff \(x_{t+1}^j\) at \(t+1\).

The agent’s problem is the following:

\[
\max_{\xi_j, j=1,\ldots,J} u(c_t) + \beta E_t u(c_{t+1})
\]
subject to:

\[ c_t = w_t - \sum_{j=1}^{J} p_t^j \xi_j \]

\[ c_{t+1} = w_{t+1} + \sum_{j=1}^{J} x_{t+1}^j \xi_j \]

where \( p_t^j \) is the price of asset \( j \), and \( \xi_j \) is the amount of asset \( j \) in the agent portfolio, the choice variable. The first order conditions for the maximization problem imply the fundamental asset pricing equation:

\[ p_t^j = E_t \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1}^j \right) \] (2)

If the asset is long lived, e.g., a stock, its payoff at \( t + 1 \) is the sum of its resale price, \( p_{t+1}^j \) and its cash flow at time \( t + 1 \), e.g., its dividend; we write \( x_{t+1}^j = p_{t+1}^j + d_{t+1}^j \). In this case, by repeatedly substituting updated versions of (2) into itself, that is by solving (2) as a stochastic difference equation, we can write the pricing equation in its present value form:

\[ p_t^j = E_t \left( \sum_{\tau=1}^{\infty} \beta^\tau \frac{u'(c_{t+\tau})}{u'(c_t)} d_{t+\tau}^j \right) \] (3)

### 2.1 Time and Risk Correction

Let’s examine the asset pricing equation (2). If there is no uncertainty, that is if the agent is able to perfectly insure, so that \( c_t = c_{t+1} \), or if he is risk neutral, so that \( u'(c) \) is a constant,

\[ p_t^j = \beta E_t x_{t+1}^j \]

and the price of asset \( j \) is the net present value of its expected payoff at \( t + 1 \) discounted at the pure discount rate \( \beta \).

If instead the agent is risk averse and his consumption is a stochastic process, then by (2), the discount factor is stochastic: \( \beta \frac{u'(c_{t+\tau})}{u'(c_t)} \). In this case, the discount contains a risk correction. To see this, note that (2) can be written:

\[ p_t^j = E_t \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} \right) E_t \left( x_{t+1}^j \right) + \text{cov}_t \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} , x_{t+1}^j \right) \] (4)

and the covariance of the asset payoff with the marginal rate of substitution of the agent is the relevant component of the risk of the asset which enters in the price.

Because of the concavity of the utility function \( u(c) \) (that is, \( u''(c) < 0 \)), equation (4) implies that assets whose payoff is negatively correlated with the

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1Some important technical issues regarding the possibility of “bubbles” in the solution of the difference equation are overlooked here.
agent’s consumption are more valued by the agent and hence have a higher price. Why are they more valued, in intuitive terms? Because they allow the agent to insure, that is to reduce the risk (roughly, variance) of his consumption process. Remember: Agents care about the variance of their consumption and hence about the covariance of the asset’s payoff with his consumption.

Note that the consumption which enters the stochastic discount factor is each single agent’s consumption in the economy. If financial markets are developed enough (they are complete, in the economist’s parlance), and utility functions well behaved and identical, then \( \beta u’(c_{t+1}) \) is equalized for any agent in equilibrium and we can think without loss of generality of \( c_{t+1} \) as of the economy’s consumption.\(^2\)

In present value form, equation (4) becomes:

\[
p_j^t = \sum_{\tau=1}^{\infty} E_t \left( \beta \frac{u'(c_{t+\tau})}{u'(c_t)} \right) E_t \left( d_{t+\tau}^j \right) + \sum_{\tau=1}^{\infty} \text{cov}_t \left( \beta \frac{u'(c_{t+\tau})}{u'(c_t)}, d_{t+\tau}^j \right)
\]

(5)

2.2 Beta Representation

Often the pricing equation is written as an equation for excess returns. The return of asset \( j \), is written \( R_{t+1}^j = \frac{x_{t+1}^j}{p_t} \); its excess return is the difference of its return and a risk free rate of return. Let’s construct the risk free rate first. Write the pricing equation (2) as:

\[
1 = E_t \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} R_{t+1}^j \right)
\]

(6)

Consider a risk free return \( R_{t+1}^f \), that is, a return which is is known at \( t \). In this case equation (6) can be written:

\[
R_{t+1}^f = \frac{1}{E_t \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} \right)}
\]

(7)

We can now rewrite (4) as:

\[
E_t \left( R_{t+1}^j - R_{t+1}^f \right) = \frac{\text{cov}_t \left( \beta \frac{u'(c_{t+1})}{u'(c_t)}, R_{t+1}^j \right)}{\text{var}_t \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} \right)} \cdot \left( 1 - \frac{\text{var}_t \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} \right)}{E_t \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} \right)} \right)
\]

(8)

Then \( \beta_t^j = \left( \frac{\text{cov}_t \left( \beta \frac{u'(c_{t+1})}{u'(c_t)}, R_{t+1}^j \right)}{\text{var}_t \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} \right)} \right) \) is a measure of the riskiness of return \( j \), its conditional beta (not to be confused with the pure discount factor); while

\(^2\)This is a fundamental result in economic theory. If you want to know more about it, look at J. Cochrane, *Asset Pricing*, Princeton University Press, 2001; this is not part of the course though.
\[
\left(-\frac{\text{var}_t \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} \right)}{E_t \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} \right)} \right)
\]
is a measure of the price of risk. Naturally, the price of risk is 0 if the variance of consumption is 0; and in this case, \( R^m_{t+1} = R^f_{t+1} \).

Yet another related representation of the same equation. Consider asset \( m \), the market, whose return is

\[
R^m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}
\]

Then, using (8), and noting that \( \beta^m_t = 1 \) by definition, we get:

\[
E_t \left( R^m_{t+1} - R^f_{t+1} \right) = \left(-\frac{\text{var}_t \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} \right)}{E_t \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} \right)} \right)
\]

and hence

\[
E_t \left( R^m_{t+1} - R^f_{t+1} \right) = \beta^m_t E_t \left( R^m_{t+1} - R^f_{t+1} \right) \tag{9}
\]

We think of \( \beta^m_t \left( R^m_{t+1} - R^f_{t+1} \right) \) as of the risk premium associated to asset \( j \).

Naturally, \( E_t R^m_{t+1} = R^f_{t+1} \) when \( \beta^m_t = 0 \); while \( E_t R^f_{t+1} = E_t R^m_{t+1} \) when \( \text{cov}_t \left( \beta \frac{u'(c_{t+1})}{u'(c_t)}, R^f_{t+1} \right) = \text{var}_t \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} \right) \) and \( \beta^m_t = 1 \). Moreover, note that if \( \text{cov}_t \left( \beta \frac{u'(c_{t+1})}{u'(c_t)}, R^f_{t+1} \right) < 0 \), \( R^m_t < R^f_t \) (the risk premium is negative).

### 2.3 Equivalent Risk Neutral Representation [extra material]

Consider the random variable \( x^j_{t+1} \): 

\[
E_t \left( x^j_{t+1} \right)
\]
is its expected value with respect to our basic probability distribution. Let \( m_{t+1} \) be a non-negative random variable, and let \( E^*_t \left( x^j_{t+1} \right) = E_t \left( \frac{m_{t+1}}{E_t \left( m_{t+1} x^j_{t+1} \right)} \right) \). Notice that we have implicitly defined a different probability distribution with respect to the basic one.\(^3\)

Recalling that \( R^f_{t+1} = \frac{1}{E_t \left( \frac{u'(c_{t+1})}{u'(c_t)} \right)} \), and letting \( m_{t+1} = \left( \frac{u'(c_{t+1})}{u'(c_t)} \right) \) we now have:

\[
p^j_t = \frac{E^*_t \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} \right)}{R^f_{t+1}} \tag{10}
\]

Notice that (10) is the pricing equation of an economy with risk neutral agents with the different probability distribution over uncertainty we constructed; we call therefore this probability distribution "risk neutral". The risk neutral probability distribution corrects for risk; in fact, with respect to the basic probability distribution, the risk neutral probability distribution weights more the states with lower consumption. (This is to say: a risk averse agent is like a risk neutral agent who believes that the "bad" states of uncertainty are more probable.)

\(^3\)That is, \( E^*_t(x) \geq 0 \) if \( x \geq 0 \) and \( E^*_t(x) = x \) if \( x \) is constant.
3 Properties of Efficient Market Hypothesis

The analysis of asset pricing and valuation in the previous section is often referred to as the Efficient Market Hypothesis. It has several implications that have been tested over the years. For a critical review, see A. Schleifer, Inefficient Markets, Oxford University Press, 2000.

3.1 Idiosyncratic Risk Does Not Affect Prices

An asset $j$ is one of idiosyncratic risk if its payoff is independent of the economy’s aggregate consumption. You can check immediately that idiosyncratic risk does not affect prices by looking at (4). Note that this is true no matter how large is $\text{var}_t(x_{t+1})$.

3.2 Prices Adjust Immediately to All Available Information

In the fundamental pricing equation (2), the information included in the price at time $t$ is the whole information available at time $t$. This you see by noting that the conditional expectation is taken with respect to the time $t$ information. See Figure 1 for validation.

3.3 Risk Adjusted Returns Are Not Predictable

Suppose agents are risk neutral ($u(c)$ is linear) and do not discount the future ($\beta$ close to 1). Consider a stock $j$ which pays no dividends, $x_{t+1}^j = p_{t+1}^j$; then, using (2):

$$p_t^j = E_t p_{t+1}^j$$ (11)

and its price is a random walk (or, more precisely, a martingale). This is the typical non-predictability example. It relies on extreme implausible assumptions.

In general prices are predictable, as are returns and excess returns. For instance, from (8), one sees that even if the price of risk

$$-\text{var}_t \left( \beta \frac{x^c_{t+1}}{\text{var}(x_{t+1})} \right)$$

is constant over time, the predictability of excess returns follows from the predictability of the conditional beta.

Risk adjusted returns are returns minus the risk premium. The fact that they are unpredictable is an immediate consequence of equation (9). Another way to say the same thing is noticing that the price of a stock which pays no dividends is always a martingale in the risk neutral probability measure (which adjusts for risk), see equation (10), but not for the basic probability measure (which drives the uncertainty in the economy).

The classical examples of predictability (see e.g., Figure 2) are hardly a compelling critique of the Efficient Market Hypothesis.
3.4 Arbitrage Opportunities Are Not Present in Financial Markets

An arbitrage opportunity exist when a portfolio exists whose price is either 0 or negative and its payoff is always non-negative and strictly positive with positive probability; see example 1 for a financial market in which an arbitrage exists.

Because our discount factor, \( \beta u'(c_{t+1}) \) is strictly positive (utility is strictly monotonic, and hence \( u'(c) > 0 \)), any asset with a payoff which is non-negative and strictly positive with some probability has a strictly positive price, by (2). See Figure 3 for what seems like an arbitrage opportunity in the market.

4 The CAPM in Practice

How can we compute the price we expect an asset \( j \) to be traded at in the market? Is an asset (or a firm) cheap or expensive? Should we buy or sell? [the last two questions are meaningless in an Efficient Market; extra care should be used in interpreting them] Which data do we need to perform such valuations?

Look at equation (9). We need a risk free rate \( R^f_{t+1} \), a market rate of return \( R^m_{t+1} \), and a beta of the asset or firm \( j \), \( \beta^j_t \).

4.1 The Risk Free Rate

The risk free rate is usually directly available in the market; it is the rate of return of a 1 period T-bill (as risk free as possible a bond). The real issue is:

What is a period? risk free rates for many different maturities are traded in financial markets.

Must match maturities of the risk free rate with maturities of the payoffs of the asset. To see this, note that (7) can be written:

\[
\left( R^f_{t+1} \right)^{-1} = E_t \left( \beta u'(c_{t+1}) \frac{u'(c_t)}{u'(c_t)} \right) \tag{12}
\]

and \( \left( R^f_{t+1} \right)^{-1} \) is the value of a one period ahead zero-coupon risk free bond.

The value of a two period ahead zero-coupon risk free bond is

\[
\left( R^f_{t+2} \right)^{-1} = E_t \left( \beta^2 u'(c_{t+2}) \frac{u'(c_t)}{u'(c_t)} \right) \tag{13}
\]

and so on. Looking at the formulation in (5) it is now clear that the risk free rate to be used in evaluating a stream of payoffs needs to be matched with the maturity of the payoff.

What is the mistake one makes if mismatching maturities?

From (12) and (13) one gets an expression for the term structure of interest rates:
where \((R_{t+1}^f)_{t+2}^{-1}\) is the expected value at time \(t\) of a one period ahead zero-coupon risk free bond traded in the market at time \(t+1\). The mistake of mismatching is one of overlooking risk, as tomorrow’s risk free rate is not risk free from the point of view of today.

4.2 The Market Rate of Return and the betas

If we knew \(u(c)\) and we could measure precisely \(c_{t+1}, c_t\) (aggregate consumption), we would have no problem; and we could compute

\[
R_{t+1}^m = \beta u'(c_{t+1}) u'(c_t)
\]

and

\[
\beta_j = \frac{\text{cov}_t \left( \beta \frac{u'(c_{t+1})}{u'(c_t)}, R_{t+1}^j \right)}{\text{var}_t \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} \right)}
\]

In fact we do not know what \(u(c)\) and aggregate consumption data is quite bad. Therefore the standard procedure is to estimate a multi-factor beta equation of the form:

\[
R_{t+1}^j = R_t^j + \beta_{j1}^f f_1^t + \beta_{j2}^f f_2^t + \ldots + \epsilon_{jt}
\]

for any asset \(j\); where \(f^i\) are factors, i.e., proxies for the intertemporal marginal rate of substitution, like several indices of stock returns, GNP, inflation, and so on. (Note that only unconditional betas can be estimated; hence under the assumption that betas are constant over time).