Intro to Economic analysis

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1 Finance

• Consider an economy with two states of nature \( \{s_1, s_2\} \) and with two primary assets, \( \{1, 2\} \). The first asset is a stock with price \( q_1 \) and random uncertain future payoff \( (x_{s_1}^1, x_{s_2}^1) \), \( x_{s_1}^1 \neq x_{s_2}^1 \). There is also a bond (an asset with constant certain future payoff) with price \( q_2 \) and payoff \( (x^2, x^2) \).

1.1 No-Arbitrage

• Consider the problem of pricing a derivative asset, asset \( d \), with returns \( (x_d^{s_1}, x_d^{s_2}) \). Denote the price of the derivative asset as \( q_d \).

• Construct a portfolio with the two primary assets which replicates the payoff vector of the new asset. This requires solving the following:

\[
\begin{align*}
    x_d^{s_1} &= \theta_1 x_{s_1}^1 + \theta_2 x^2 \\
    x_d^{s_2} &= \theta_1 x_{s_2}^1 + \theta_2 x^2
\end{align*}
\]

for \( (\theta_1^*, \theta_2^*) \). The interpretation is that \( \theta_1 \) and \( \theta_2 \) are, respectively the quantity of asset 1 (resp. 2) that an agent has to buy (if > 0) or sell (if < 0) to construct a portfolio which has the same payoff as the payoff of the derivative asset \( d \), \( (x_d^{s_1}, x_d^{s_2}) \).

• Definition 1 Consider the economy in which each agent trades \( (\theta_1, \theta_2, \theta_d) \) units of assets \( (1, 2, d) \) at prices \( (q_1, q_2, q_d) \). We say that \( (q_1, q_2, q_d) \) is a No-arbitrage price system if there does not exist a portfolio \( (\theta_1, \theta_2, \theta_d) \) such that

\[
\begin{align*}
    \theta_1 x_{s_1}^1 + \theta_2 x_{s_1}^2 + \theta_d x_d^{s_1} &= 0 \quad (1) \\
    \theta_1 x_{s_2}^1 + \theta_2 x_{s_2}^2 + \theta_d x_d^{s_2} &= 0 \quad (2)
\end{align*}
\]
\[ \theta_1 q_1 + \theta_2 q_2 + \theta_d q_d < 0 \]  

Note that, such a portfolio \((\theta_1, \theta_2, \theta_d)\) would guarantee an agent a 0 payoff in the future, at a negative cost for the agent (that is, at a positive payout to the agents in the present). A portfolio \(2(\theta_1, \theta_2, \theta_d)\) would then guarantee still 0 payoff in the future and a twice a positive payout in the present. Keep proceeding this way, with \(3(\theta_1, \theta_2, \theta_d), 4(\theta_1, \theta_2, \theta_d),\) and so on; to show that the agent can receive an arbitrary large payout in the present (that is, an arbitrage).

- **Proposition 1** Any No-arbitrage price system satisfies

  \[ q_d = \theta_1 q_1 + \theta_2 q_2. \]

**Proof.** If \(q_d > \theta_1^* q_1 + \theta_2^* q_2\), then by selling one unit of asset \(d\) (that is, \(\theta_d = -1\)) and trading \(\theta_1^*\) units of asset 1 and \(\theta_2^*\) units of asset 2 we construct a portfolio which satisfies (1-3).

If instead \(q_d < \theta_1^* q_1 + \theta_2^* q_2\), then by buying one unit of asset \(d\) (that is, \(\theta_d = 1\)) and trading \(-\theta_1^*\) units of asset 1 and \(-\theta_2^*\) units of asset 2 we construct a portfolio which satisfies (1-3).

If \(q_d = \theta_1^* q_1 + \theta_2^* q_2\), we cannot construct a portfolio which satisfies (1-3). 

- In the context of this economy we say that the derivative \(d\) is priced by No-arbitrage. Complex forms of such pricing are an important Wall Street activity.

- Suppose that the initial prices you are given are the prices of assets 1 and 2 whose payoff is, respectively, 1 unit in state \(s_1\) and none in state \(s_2\), and 1 unit in state \(s_2\) and none in state \(s_1\). (We call these assets Arrow securities). Then you can price any sort of other assets in our economy (with two states of the world \(s_1, s_2\)) by No-arbitrage, using the price of these.

- Are the prices of Arrow securities positive?

  The affirmative answer follows from considering the maximization problem of an agent who constructs his portfolio to maximize utility.
Let $E(u(c_{t+1})) = pu(c_{s_1}) + (1 - p)u(c_{s_2})$. Then the agent’s problem is:

$$\max u(c_t) + \beta E(u(c_{t+1}))$$

subject to

$$c_t = y_t - q_1\theta_1 - q_2\theta_2 - \cdots$$
$$c_{s_1} = y_{s_1} + x_{s_1}^1\theta_1 + x_{s_1}^2\theta_2 + \cdots$$
$$c_{s_2} = y_{s_2} + x_{s_2}^1\theta_1 + x_{s_2}^2\theta_2 + \cdots$$

Solve by substituting, and deriving with respect to $\theta_1$, $\theta_2$, $\cdots$. The first order conditions with respect to $\theta_1$, for instance are:

$$-u'(c_t)q_1 + \beta\left( pu'(c_{s_1})x_{s_1}^1 + (1 - p)u'(c_{s_2})x_{s_2}^1 \right)$$

Since we assumed that asset 1 is the Arrow security paying 1 unit in state $s_1$ only, the first order condition becomes:

$$-u'(c_t)q_1 + \beta pu'(c_{s_1})x_{s_1}^1$$

and hence

$$q_1 = \frac{\beta pu'(c_{s_1})}{u'(c_t)} > 0$$

2  CAPM-Capital Asset Pricing Model

- Consider an agent living two periods, $t$ and $t+1$. His utility in terms of consumption $c_t$ and $c_{t+1}$ is:

$$u(c_t) + \beta u(c_{t+1})$$

where $\beta < 1$ is the discount rate, and $c_{t+1}$ is a random variable.

The agent faces a wealth process $y_t$, $y_{t+1}$. He can trade $J$ assets; asset $j = 1, \ldots, J$ has payoff $x_{t+1}^j$ at $t+1$.
• The agent’s problem is the following:\footnote{This is the same problem of the last section. The only few differences are in the more compact notation.}

\[
\max_{\theta^j, j=1,...,J} u(c_t) + \beta E u(c_{t+1})
\]

subject to:

\[
c_t = y_t - \sum_{j=1}^{J} q_t^j \theta^j
\]

\[
c_{t+1} = y_{t+1} + \sum_{j=1}^{J} x_{t+1}^j \theta^j
\]

where \(q_t^j\) is the price of asset \(j\) at time \(t\), and \(\theta^j\) is the amount of asset \(j\) in the agent portfolio, the choice variable.

• The first order conditions for the maximization problem imply the fundamental asset pricing equation:

\[
q_t^j = E \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1}^j \right)
\]

If the asset is long lived, e.g., a stock, its payoff at \(t + 1\) is the sum of its resale price, \(q_{t+1}^j\) and its cash flow at time \(t + 1\), e.g., its dividend; we write \(x_{t+1}^j = q_{t+1}^j + d_{t+1}^j\).

If preferences are logarithmic:

\[
q_t^j = E \left( \frac{c_t}{c_{t+1}} x_{t+1}^j \right)
\]

Let \(R_{t+1}^j = \frac{d_{t+1}^j + q_{t+1}^j}{q_t^j}\) denote the return of asset \(j\). Then the pricing equation implies:

\[
1 = E \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} R_{t+1}^j \right)
\]

• Let’s examine the asset pricing equation (5).
If there is no uncertainty, that is if the agent is able to perfectly insure, so that $c_t = c_{t+1}$, or if he is risk neutral, so that $u'(c)$ is a constant,

$$q^j_t = \beta E_t x^j_{t+1}$$

and the price of asset $j$ is the net present value of its expected payoff at $t + 1$ discounted at the pure discount rate $\beta$. An additional (very) special case is important:

If you assume that $\beta = 1$ (no discounting) and $d^j_{t+1} = 0$ (no dividends), then

$$q^j_t = E q^j_{t+1}$$

that is,

$$q^j_{t+1} = q^j_t + \text{random shock with zero mean}$$

This is the Random Walk of A Random Walk Down Wall Street, the famous book by Malkiel. Note all the assumptions we needed: i) risk neutrality, ii) no discounting, iii) no dividends.

If instead the agent is risk averse and his consumption is random, then by (5), the discount factor is also random: $\beta u'(c_{t+1}) u'(c_t)$. In this case, the discount contains a risk correction. To see this, note that (5) can be written:

$$q^j_t = E \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} \right) E (x^j_{t+1}) + cov \left( \beta \frac{u'(c_{t+1})}{u'(c_t)}, x^j_{t+1} \right)$$

(6)

and the covariance of the asset payoff with the marginal rate of substitution of the agent is the relevant component of the risk of the asset which enters in the price. Recall that the covariance between two random variable is a measure of how these random variable move together, that is, for instance one tends to be high when the other is high; see the section of the notes on "Time and Uncertainty."

Because of the concavity of the utility function $u(c)$ (that is, $u''(c) < 0$), equation (6) implies that assets whose payoff is negatively correlated with the agent’s future consumption are more valued by the agent and hence have a higher price (and a lower return).
Consider log preferences as an example; equation (6) in this case can be written as:

\[ q^j_t = E\left(\beta \frac{c_t}{c_{t+1}}\right) E \left( x^j_{t+1} \right) + \text{cov} \left( \beta \frac{c_t}{c_{t+1}}, x^j_{t+1} \right) \]

Why are assets whose payoff is negatively correlated with the agent’s future consumption more valued by the agents, in intuitive terms? Because they allow the agent to insure, that is to reduce the risk (roughly, variance) of his consumption process. It follows that their price is higher (better asset, hence more expensive), and equivalently that their return is smaller (better asset, willing to accept smaller return).

Remember:

Agents care about the variance of their consumption and hence about the covariance of the asset’s payoff with his consumption.

You cannot overestimate the importance of equation (6): asset prices and returns are driven by their expected value and by their covariance with the consumption process of their buyers and sellers.

- Often the pricing equation is written as an equation for excess returns.

The return of asset \( j \), is written \( R^j_{t+1} = \frac{x^j_{t+1}}{q^j_t} \); its excess return is the difference of its return and a risk free rate of return. Let’s construct the risk free rate first. Write the pricing equation (5) as:

\[ 1 = E \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} R^j_{t+1} \right) \]  

(7)

Consider a risk free return \( R^f_{t+1} \), that is, a return which is is known at \( t \). In this case equation (7) can be written:

\[ R^f_{t+1} = \frac{1}{E \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} \right)} \]  

(8)

We can now rewrite (6) as:
$$E \left( R^j_{t+1} - R^f_{t+1} \right) = \left( \frac{\text{cov} \left( \beta \frac{u'(c_{t+1})}{u'(c_t)}, R^j_{t+1} \right)}{\text{var} \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} \right)} \right) \left( - \frac{\text{var} \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} \right)}{\text{E} \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} \right)} \right)$$

(9)

Then $\beta^j = \left( \frac{\text{cov} \left( \beta \frac{u'(c_{t+1})}{u'(c_t)}, R^j_{t+1} \right)}{\text{var} \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} \right)} \right)$ is a measure of the riskiness of return $j$, its beta (not to be confused with the pure discount factor); while $\left( - \frac{\text{var} \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} \right)}{\text{E} \left( \beta \frac{u'(c_{t+1})}{u'(c_t)} \right)} \right)$ is a measure of the price of risk. Naturally, the price of risk is 0 if the variance of consumption is 0.

A special case is important:

Suppose there is no risk in the economy, $c_{t+1} = c_t$, or the agent is risk neutral, then the excess return equation, (9), is:

$$E \left( R^j_{t+1} - R^f_{t+1} \right) = 0,$$

that is,

$$R^j_{t+1} = R^f_{t+1} + \text{random shock with zero mean}$$

This is the returns are unpredictable result. As the prices are a random walk result it is only valid under risk neutrality (or, no risk in the economy).