1 Differentiable approach: A rough primer

Let an economy be parametrized by the endowment vector \( \omega \in \mathbb{R}_{++}^{LI} \) keeping preferences \( \{u^i\}_{i \in I} \) fixed. Furthermore, normalize \( p_L = 1 \) and eliminate the \( L \)-th component of the excess demand. Then

\[
z : \mathbb{R}_{++}^{L-1} \times \mathbb{R}_{++}^{LI} \to \mathbb{R}_{++}^{L-1}
\]

represents an aggregate excess demand for an exchange economy \( \omega = (\omega^i)_{i \in I} \in \mathbb{R}_{++}^{LI} \). Assume the economy is such that \( z(p, \omega) \) satisfies the following properties:

1. smoothness: \( z(p, \omega) \) is \( C^\infty \);
2. homogeneity of degree 0: \( z(\lambda p, \omega) = z(p, \omega) \), for any \( \lambda > 0 \);
3. Walras Law: \( pz(p, \omega) = 0 \), \( \forall p > 0 \);
4. Lower boundedness: \( \exists s \) such that \( z_l(p, \omega) > -s \), \( \forall l \in L \);
5. Boundary property:

\[
p^n \to p \neq 0, \text{ with } p_l = 0 \text{ for some } l, \Rightarrow \max \{ z_1(p^n, \omega), \ldots, z_L(p^n, \omega) \} \to \infty
\]

**Definition 1** A \( p \in \mathbb{R}_{++}^L \) such that \( z(p, \omega) = 0 \) is **regular** if \( D_p z(p, \omega) \) has rank \( L - 1 \).

**Definition 2** An economy \( \omega \in \mathbb{R}_{++}^{LI} \) is regular if \( D_p z(p, \omega) \) has rank \( L - 1 \) for any \( p \in \mathbb{R}_{++}^{L-1} \) such that \( z(p, \omega) = 0 \).

**Definition 3** An equilibrium price \( p \in \mathbb{R}_{++}^L \) is **locally unique** if \( \exists \) an open set \( P \) such that \( p \in P \) and for any \( p' \neq p \in P \), \( z(p', \omega) \neq 0 \).

**Proposition 4** A regular equilibrium price \( p \in \mathbb{R}_{++}^L \) is locally unique. Furthermore, the set of equilibrium prices of an economy \( \omega \in \mathbb{R}_{++}^{LI} \) is a smooth manifold (see Math Appendix) of dimension \( LI \).

**Proof.** Fix an arbitrary \( \omega \in \mathbb{R}_{++}^{LI} \). Since \( D_p z(p, \omega) \) has rank \( L - 1 \), by regularity of \( p \), the Inverse function theorem - Local (see Math Appendix) applied to the map \( z : \mathbb{R}_{++}^{L-1} \to \mathbb{R}_{++}^{L-1} \), directly implies local uniqueness of \( p \in \mathbb{R}_{++}^{L-1} \). Since \( D_p z(p, \omega) \) has rank \( L - 1 \), by regularity of \( p \), the Inverse function theorem - Global (see Math Appendix) applied \( z : \mathbb{R}_{++}^{L-1} \times \mathbb{R}_{++}^{LI} \to \mathbb{R}_{++}^{L-1} \), directly implies that the set \( p \in z^{-1}(0) \), as a function of \( \omega \in \mathbb{R}_{++}^{LI} \), is a smooth manifold of dimension \( LI \).
**Proposition 5** Any economy $\omega$ in a full measure Lebesgue subset of $\mathbb{R}^{LI}_{++}$ is regular.

**Proof.** The statement follows by the Transversality theorem (see Math Appendix), if $z \cap 0$. We now show that $z \cap 0$. Pick an arbitrary agent $i \in I$. It will be sufficient to show that, for any $(p, \omega) \in \mathbb{R}^{L-1}_{+} \times \mathbb{R}^{LI}_{++}$ such that $z(p, \omega) = 0$, we can find a perturbation $d\omega^i \in \mathbb{R}^I$ such that $dz = D_{z}z(p, \omega)d\omega^i = -d\omega^i$. But any perturbation $d\omega^i$ such that $pd\omega^i = 0$ leaves each agent $i \in I$ demand unchanged and hence it implies $D_{z}z(p, \omega)d\omega^i = -d\omega^i$. ■

**Definition 6** The index $i(p, \omega)$ of a price $p \in \mathbb{R}^{L-1}_{++}$ such that $z(p, \omega) = 0$ is defined as

$$i(p, \omega) = (-1)^{L-1} \text{sign} |D_{p}z(p, \omega)|.$$  

The index $i(\omega)$ of an economy $(\omega^i, \omega^i)_{i \in I}$ is defined as

$$i(\omega) = \sum_{p: z(p, \omega) = 0} i(p, \omega).$$

**Theorem 7** For any regular economy $\omega \in \mathbb{R}^{LI}_{++}$, $i(\omega) = 1$.

**Proof.** The theorem is a deep mathematical result whose proof is clearly beyond the scope of this class. Let it suffice to say that the proof relies crucially on the boundary property of excess demand. Adventurous reader might want to look at Mas Colell (1985), section 5,6, p. 201-15. ■

**Corollary 8** Any regular economy $\omega \in \mathbb{R}^{LI}_{++}$ has an odd number of equilibria. In particular, any regular economy $\omega \in \mathbb{R}^{LI}_{++}$ has at least one equilibrium.

**Proposition 9** Any economy $\omega \in \mathbb{R}^{LI}_{++}$ has at least one equilibrium price $p \in \mathbb{R}^{L-1}_{+}$.  

**Proof.** Sketch. Let $\omega \in \mathbb{R}^{LI}_{++}$ be an arbitrary regular economy. Pick the economy $\omega' \in \mathbb{R}^{LI}_{++}$ such that there exist a unique price $p \in \mathbb{R}^{L-1}_{++}$ such that $z(p) = 0$, and $D_{p}z(p)$ has rank $L - 1$. One such economy can always be constructed by choosing $\omega' \in \mathbb{R}^{LI}_{++}$ to be a Pareto optimal allocation (show that generic regularity holds in the subset of economies with Pareto optimal endowments). Let $t\omega + (1 - t)\omega'$, for $0 \leq t \leq 1$, represent a 1-dimensional subset of economies. Let $z(p, t)$ be the map $z : \mathbb{R}^{L-1}_{++} \times [0, 1]$ induced by $z(p, t) = z(p, t\omega + (1 - t)\omega')$ for given $(\omega, \omega')$. It follows from the Inverse function theorem - Global (see Math Appendix) that the set of equilibrium prices $p \in z^{-1}(0)$, as a function of $t \in [0, 1]$, is a smooth manifold of dimension 1. Since both $\omega$ and $\omega' \in \mathbb{R}^{LI}_{++}$ are regular, $i(\omega') = i(\omega') = 1$ by the Index theorem. By the Classification theorem (see Math Appendix), $z^{-1}(0)$ is homeomorphic to a countable set of lines and circles. Along a component of $z^{-1}(0)$ (a line or a circle), a change in index occurs when the manifold folds. Regularity of $z^{-1}(0)$ at the boundary, $t = 0$ and $t = 1$ implies that at least one component of $z^{-1}(0)$
is diffeomorphic to a line with boundary at \( t = 0 \) and \( t = 1 \). It looks confusing, but it’s easier with a figure.

Finally, we need to show that the assumption that \( \omega \in R^{LI}_{++} \) is regular is without loss of generality. In this respect, suppose \( \omega \in R^{LI}_{++} \) is critical (i.e., not regular). It is easy to show that the regularity of \( \omega' \in R^{LI}_{++} \) implies that there exists a regular economy \( \omega'' \in R^{LI}_{++} \) in an open ball of \( \omega \in R^{LI}_{++} \) such that

\[
\omega = t\omega' + (1-t)\omega'', \quad 0 < t < 1.
\]

But since \( \omega' \) and \( \omega'' \in R^{LI}_{++} \), are both regular, any economy \( \omega = t\omega' + (1-t)\omega'' \) has at least one equilibrium. ■

1.1 Mathematical Appendix

**Theorem 10** (Inverse function theorem - Local). Let \( f : R^m_{++} \rightarrow R^n_{++} \) be \( C^\infty \). If \( Df \) has rank \( n \), at some \( x \in R^n_{++} \), there exist an open set \( V \subseteq R^n_{++} \) and a function \( f^{-1} : V \rightarrow R^n_{++} \) such that \( f(x) \in V \) and \( f^{-1}(f(z)) = z \) in a neighborhood of \( x \).

**Definition 11** A subset \( X \subset R^n_{++} \) is a smooth manifold of dimension \( n \) if for any \( x \in X \) there exist a neighborhood \( U \subset R^n_{++} \) and a \( C^\infty \) function \( f : U \rightarrow R^n_{++} \) such that \( Df \) has rank \( n \) in the whole domain.

**Theorem 12** (Inverse function theorem - Global). Let \( f : R^m_{++} \rightarrow R^n_{++} \), \( m \geq n \), be \( C^\infty \). Suppose that \( D_x f \) has rank \( n \) for any \( x \in R^m_{++} \). Then \( f^{-1}(0) = \{ x \in R^m_{++} \mid f(x) = 0 \} \) is a smooth manifold of dimension \( m - n \).

**Definition 13** Let \( f : R^m_{++} \rightarrow R^n_{++} \), \( m \geq n \), be \( C^\infty \). \( f \) is transversal to \( 0 \), denoted \( f \pitchfork 0 \), if \( Df \) has rank \( n \) for any \( x \in R^m_{++} \) such that \( f(x) = 0 \).

**Theorem 14** (Transversality). Let \( f : R^m_{++} \rightarrow R^n_{++} \), \( m \geq n \), be \( C^\infty \) and transversal to \( 0 \), \( f \pitchfork 0 \). Decompose any vector \( x \in R^m_{++} \) as \( x = [x_1, x_2] \), with \( x_1 \in R^{m-n}_{++} \), \( x_2 \in R^n_{++} \). Then \( D_{x_2} f(x) \) has rank \( n \) for all \( x \) in a Lebesgue measure-1 subset of \( R^m_{++} \).

**Theorem 15** (Classification). Every smooth manifold of dimension \( 1 \) is homeomorphic to a disjoint union of countably many copies of \( R \) (the real line) and \( S \) (the circle).

1.2 References

The main reference is:


But, as Andreu told me once, "I do not hate students so much that I would give them this book to read." Similar comments hold, in my opinion, for

You are then left with:
