General equilibrium theory
Lecture notes

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November 9, 2009
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Preface

These notes constitute the material for the second half of the Micro I (first year) graduate course at NYU.

Other books: Mas Colell, Gale lecture notes.
Chapter 1

Introduction
Chapter 2

Demand theory: A quick review
Chapter 3

Arrow-Debreu economies

3.1 Competitive equilibrium

3.2 Existence

3.3 Welfare

3.4 Differentiable approach: A rough primer

Let an economy be parametrized by the endowment vector $\omega \in \mathbb{R}_{++}^{LI}$ keeping preferences $(u^i)_{i \in I}$ fixed. Furthermore, normalize $p_L = 1$ and eliminate the $L$-th component of the excess demand. Then

$$z : \mathbb{R}_{++}^{L-1} \times \mathbb{R}_{++}^{LI} \rightarrow \mathbb{R}^{L-1}$$

represents an aggregate excess demand for an exchange economy $\omega = (\omega^i)_{i \in I} \in \mathbb{R}_{++}^{LI}$. Assume the economy is such that $z(p, \omega)$ satisfies the following properties:

1. smoothness: $z(p, \omega)$ is $C^\infty$;
2. homogeneity of degree 0: $z(\lambda p, \omega) = z(p, \omega)$, for any $\lambda > 0$;
3. Walras Law: $pz(p, \omega) = 0$, $\forall p >> 0$;
4. Lower boundedness: $\exists s$ such that $z_l(p, \omega) > -s$, $\forall l \in L$;
5. Boundary property:

\[ p^n \to p \neq 0, \text{ with } p_l = 0 \text{ for some } l, \Rightarrow \max \{ z_1(p^n, \omega), \ldots, z_L(p^n, \omega) \} \to \infty \]

**Definition 1** A \( p \in \mathbb{R}^L_+ \) such that \( z(p, \omega) = 0 \) is **regular** if \( D_p z(p, \omega) \) has rank \( L - 1 \).

**Definition 2** An economy \( \omega \in \mathbb{R}^{LI}_+ \) is regular if \( D_p z(p, \omega) \) has rank \( L - 1 \) for any \( p \in \mathbb{R}^{L-1}_+ \) such that \( z(p, \omega) = 0 \).

**Definition 3** An equilibrium price \( p \in \mathbb{R}^{L-1}_+ \) is **locally unique** if \( \exists \) an open set \( P \) such that \( p \in P \) and for any \( p' \neq p, z(p', \omega) \neq 0 \).

**Proposition 4** A regular equilibrium price \( p \in \mathbb{R}^{L-1}_+ \) is locally unique. Furthermore, the set of equilibrium prices of an economy \( \omega \in \mathbb{R}^{LI}_+ \) is a smooth manifold (see Math Appendix) of dimension \( LI \).

**Proof.** Fix an arbitrary \( \omega \in \mathbb{R}^{LI}_+ \). Since \( D_p z(p, \omega) \) has rank \( L - 1 \), by regularity of \( p \), the Inverse function theorem - Local (see Math Appendix) applied to the map \( z : \mathbb{R}^{L+}_+ \to \mathbb{R}^{L-1} \), directly implies local uniqueness of \( p \in \mathbb{R}^{L-1}_+ \). Since \( D_p z(p, \omega) \) has rank \( L - 1 \), by regularity of \( p \), the Inverse function theorem - Global (see Math Appendix) applied \( z : \mathbb{R}^{L-1}_+ \times \mathbb{R}^{LI}_+ \to \mathbb{R}^{L-1} \), directly implies that the set \( p \in z^{-1}(0) \), as a function of \( \omega \in \mathbb{R}^{LI}_+ \), is a smooth manifold of dimension \( LI \).

**Proposition 5** Any economy \( \omega \) in a full measure Lebesgue subset of \( \mathbb{R}^{LI}_+ \) is regular.

**Proof.** The statement follows by the Transversality theorem (see Math Appendix), if \( z \pitchfork 0 \). We now show that \( z \pitchfork 0 \). Pick an arbitrary agent \( i \in I \). It will be sufficient to show that, for any \( (p, \omega) \in \mathbb{R}^{L-1}_+ \times \mathbb{R}^{LI}_+ \) such that \( z(p, \omega) = 0 \), we can find a perturbation \( d\omega^i \in \mathbb{R}^L \) such that \( dz = D_{\omega^i} z(p, \omega) d\omega^i = -d\omega^i \).

But any perturbation \( d\omega^i \) such that \( p d\omega^i = 0 \) leaves each agent \( i \in I \) demand unchanged and hence it implies \( D_{\omega^i} z(p, \omega) d\omega^i = -d\omega^i \). ■

**Definition 6** The **index** \( i(p, \omega) \) of a price \( p \in \mathbb{R}^{L-1}_+ \) such that \( z(p, \omega) = 0 \) is defined as

\[ i(p, \omega) = (-1)^{L-1} \text{sign} \left| D_p z(p, \omega) \right| . \]

The index \( i(\omega) \) of an economy \( (\omega^i, \omega^i)_{i \in I} \) is defined as

\[ i(\omega) = \sum_{p : z(p, \omega) = 0} i(p, \omega). \]
3.4 DIFFERENTIABLE APPROACH: A ROUGH PRIMER

Theorem 7 For any regular economy \( \omega \in \mathbb{R}^{L+} \), \( i(\omega) = 1 \).

Proof. The theorem is a deep mathematical result whose proof is clearly beyond the scope of this class. Let it suffice to say that the proof relies crucially on the boundary property of excess demand. Adventurous reader might want to look at Mas Colell (1985), section 5.6, p. 201-15.

Corollary 8 Any regular economy \( \omega \in \mathbb{R}^{L+} \) has an odd number of equilibria. In particular, any regular economy \( \omega \in \mathbb{R}^{L+} \) has at least one equilibrium.

Proposition 9 Any economy \( \omega \in \mathbb{R}^{L+} \) has at least one equilibrium price \( p \in \mathbb{R}^{L-1} \).

Proof. Sketch. Let \( \omega \in \mathbb{R}^{L+} \) be an arbitrary regular economy. Pick the economy \( \omega_0 \in \mathbb{R}^{L+} \) such that there exist a unique price \( p \in \mathbb{R}^{L-1} \) such that \( z(p) = 0 \), and \( D_p z(p) \) has rank \( L - 1 \). One such economy can always be constructed by choosing \( \omega' \in \mathbb{R}^{L+} \) to be a Pareto optimal allocation (show that generic regularity holds in the subset of economies with Pareto optimal endowments). Let \( t\omega + (1-t)\omega' \), for \( 0 \leq t \leq 1 \), represent a 1-dimensional subset of economies. Let \( z(p,t) \) be the map \( z : \mathbb{R}^{L-1} \times [0,1] \) induced by \( z(p,t) = z(p,t\omega + (1-t)\omega') \) for given \((\omega,\omega')\). It follows from the Inverse function theorem - Global (see Math Appendix) that the set of equilibrium prices \( p \in z^-(0) \), as a function of \( t \in [0,1] \), is a smooth manifold of dimension 1. Since both \( \omega \) and \( \omega' \in \mathbb{R}^{L+} \) are regular, \( i(\omega') = i(\omega) = 1 \) by the Index theorem. By the Classification theorem (see Math Appendix), \( z^-(0) \) is homeomorphic to a countable set of lines and circles. Along a component of \( z^-(0) \) (a line or a circle), a change in index occurs when the manifold folds. Regularity of \( z^-(0) \) at the boundary, \( t = 0 \) and \( t = 1 \) implies that at least one component of \( z^-(0) \) is diffeomorphic to a line with boundary at \( t = 0 \) and \( t = 1 \). It looks confusing, but it’s easier with a figure.

Finally, we need to show that the assumption that \( \omega \in \mathbb{R}^{L+} \) is regular is without loss of generality. In this respect, suppose \( \omega \in \mathbb{R}^{L+} \) is critical (i.e., not regular). It is easy to show that the regularity of \( \omega' \in \mathbb{R}^{L+} \) implies that there exists a regular economy \( \omega'' \in \mathbb{R}^{L+} \) in an open ball of \( \omega \in \mathbb{R}^{L+} \) such that
\[
\omega = t\omega' + (1-t)\omega'', \quad \text{for} \quad 0 < t < 1.
\]
But since \( \omega' \) and \( \omega'' \in \mathbb{R}^{L+} \) are both regular, any economy \( \omega = t\omega' + (1-t)\omega'' \) has at least one equilibrium. ■
3.4.1 Mathematical Appendix

**Theorem 10** (Inverse function theorem - Local). Let $f : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be $C^\infty$. If $Df$ has rank $n$, at some $x \in \mathbb{R}^n_+$, there exist an open set $V \subseteq \mathbb{R}^n_+$ and a function $f^{-1} : V \to \mathbb{R}^n_+$ such that $f(x) \in V$ and $f^{-1}(f(z)) = z$ in a neighborhood of $x$.

**Definition 11** A subset $X \subset \mathbb{R}^m_+$ is a smooth manifold of dimension $n$ if for any $x \in X$ there exist a neighborhood $U \subset X$ and a $C^\infty$ function $f : U \to \mathbb{R}^n_+$ such that $Df$ has rank $n$ in the whole domain.

**Theorem 12** (Inverse function theorem - Global). Let $f : \mathbb{R}^m_+ \to \mathbb{R}^n_+$, $m \geq n$, be $C^\infty$. Suppose that $D_x f$ has rank $n$ for any $x \in \mathbb{R}^m_+$. Then $f^{-1}(0) = \{ x \in \mathbb{R}^m_+ | f(x) = 0 \}$ is a smooth manifold of dimension $m - n$.

**Definition 13** Let $f : \mathbb{R}^m_+ \to \mathbb{R}^n_+$, $m \geq n$, be $C^\infty$. $f$ is transversal to 0, denoted $f \pitchfork 0$, if $Df$ has rank $n$ for any $x \in \mathbb{R}^m_+$ such that $f(x) = 0$.

**Theorem 14** (Transversality). Let $f : \mathbb{R}^m_+ \to \mathbb{R}^n_+$, $m \geq n$, be $C^\infty$ and transversal to 0, $f \pitchfork 0$. Decompose any vector $x \in \mathbb{R}^m_+$ as $x = [x_1 | x_2]$, with $x_1 \in \mathbb{R}^{m-n}_+, x_2 \in \mathbb{R}^n_+$. Then $D_{x_2} f(x)$ has rank $n$ for all $x$ in a Lebesgue measure-1 subset of $\mathbb{R}^m_+$.

**Theorem 15** (Classification). Every smooth manifold of dimension 1 is homeomorphic to a disjoint union of countably many copies of $\mathbb{R}$ (the real line) and $S$ (the circle).

3.4.2 References

The main reference is:


But, as Andreu told me once, "I do not hate students so much that I would give them this book to read." Similar comments hold, in my opinion, for
3.5 STRATEGIC FOUNDATIONS


You are then left with:


Production

3.5 Strategic foundations
Chapter 4

Two-period economies

In a two-period pure exchange economy we study financial market equilibria. In particular, we study the welfare properties of equilibria and their implications in terms of asset pricing.

In this context, as a foundation for macroeconomics and financial economics, we study sufficient conditions for aggregation, so that the standard analysis of one-good economies is without loss of generality, sufficient conditions for the representative agent theorem, so that the standard analysis of single agent economies is without loss of generality.

The No-arbitrage theorem and the Arrow theorem on the decentralization of equilibria of state and time contingent good economies via financial markets are introduced as useful means to characterize financial market equilibria.

4.0.1 Time and state contingent commodities

Consider an economy extending for 2 periods, \( t = 0, 1 \). Let \( i \in \{1, \ldots, I\} \) denote agents and \( l \in \{1, \ldots, L\} \) physical goods of the economy. In addition, the state of the world at time \( t = 1 \) is uncertain. Let \( \{1, \ldots, S\} \) denote the state space of the economy at \( t = 1 \). For notational convenience we typically identify \( t = 0 \) with \( s = 0 \), so that the index \( s \) runs from 0 to \( S \).

Define \( n = L(S + 1) \). The consumption space is denoted then by \( X \subseteq \mathbb{R}^n_+ \). Each agent is endowed with a vector \( \omega^i = (\omega^i_0, \omega^i_1, \ldots, \omega^i_S) \), where \( \omega^i_s \in \mathbb{R}^n_+ \), for any \( s = 0, \ldots, S \). Let \( u^i : X \rightarrow \mathbb{R} \) denote agent \( i \)'s utility function. We will assume:

**Assumption 1** \( \omega^i \in \mathbb{R}^n_+ \) for all \( i \)
Assumption 2 \( u^i \) is continuous, strongly monotonic, strictly quasiconcave and smooth, for all \( i \) (see Magill-Quinzii, p.50 for definitions and details). Furthermore, \( u^i \) has a Von Neumann-Morgernstern representation:

\[
 u^i(x^i) = u^i(x^i_0) + \sum_{s=1}^{S} \text{prob}_s u^i(x^i_s)
\]

Suppose now that at time 0, agents can buy contingent commodities. That is, contracts for the delivery of goods at time 1 contingently to the realization of uncertainty. Denote by \( x^i = (x^i_0, x^i_1, ..., x^i_S) \) the vector of all such contingent commodities purchased by agent \( i \) at time 0, where \( x^i_s \in R^L_+ \), for any \( s = 0, ..., S \). Also, let \( x = (x^1, ..., x^I) \).

Let \( \phi = (\phi_0, \phi_1, ..., \phi_S) \), where \( \phi_s \in R^L_+ \) for each \( s \), denote the price of state contingent commodities; that is, for a price \( \phi_{ls} \) agents trade at time 0 the delivery in state \( s \) of one unit of good \( l \).

Under the assumption that the markets for all contingent commodities are open at time 0, agent \( i \)'s budget constraint can be written as:

\[
 \phi_0(x^i_0 - \omega^i_0) + \sum_{s=0}^{S} \phi_s(x^i_s - \omega^i_s) = 0 \tag{4.1}
\]

Definition 16 An Arrow-Debreu equilibrium is a \((x^*, \phi^*)\) such that

1. \( x^{*i} \in \arg \max u^i(x^i) \) s.t. \( \phi_0(x^i_0 - \omega^i_0) + \sum_{s=0}^{S} \phi_s(x^i_s - \omega^i_s) = 0 \), and

2. \( \sum_{i=1}^{I} x^{*i} - \omega^i_s = 0 \), for any \( s = 0, 1, ..., S \)

Observe that the dynamic and uncertain nature of the economy (consumption occurs at different times \( t = 0,1 \) and states \( s \in S \)) does not manifests itself in the analysis: a consumption good \( l \) at a time \( t \) and state \( s \) is treated simply as a different commodity than the same consumption good \( l \) at a different time \( t' \) or at the same time \( t \) but different state \( s' \). This is the simple trick introduced in Debreu’s last chapter of the *Theory of Value*.

\(^1\)We write the budget constraint with equality. This is without loss of generality under monotonicity of preferences, an assumption we shall maintain.
It has the fundamental implication that the standard theory and results of static equilibrium economies can be applied without change to our dynamic environment. In particular, then, under the standard set of assumptions on preferences and endowments, an equilibrium exists and the First and Second Welfare Theorems hold.\(^2\)

**Definition 17** Let \((x^*, \phi^*)\) be an Arrow-Debreu equilibrium. We say that \(x^*\) is a Pareto optimal allocation if there does not exist an allocation \(y \in X^I\) such that

1. \(u(y^i) \geq u(x^{*i})\) for any \(i = 1, \ldots, I\) (strictly for at least one \(i\)), and
2. \(\sum_{i=1}^{I} y^i - \omega^i_s = 0\), for any \(s = 0, 1, \ldots, S\)

**Theorem 18** Any Arrow-Debreu equilibrium allocation \(x^*\) is Pareto Optimal.

**Proof.** By contradiction. Suppose there exist a \(y \in X\) such that 1) and 2) in the definition of Pareto optimal allocation are satisfied. Then, by 1) in the definition of Arrow-Debreu equilibrium, it must be that

\[
\phi_0^*(y^i_0 - \omega^i_0) + \sum_{s=0}^{S} \phi_s^*(y^i_s - \omega^i_s) \geq 0,
\]

for all \(i\); and

\[
\phi_0^*(y^i_0 - \omega^i_0) + \sum_{s=0}^{S} \phi_s^*(y^i_s - \omega^i_s) > 0,
\]

for at least one \(i\).

Summing over \(i\), then

\[
\phi_0^* \sum_{i=1}^{I} (y^i_0 - \omega^i_0) + \sum_{s=0}^{S} \phi_s^* \sum_{i=1}^{I} (y^i_s - \omega^i_s) > 0
\]

which contradicts requirement 2) in the definition of Pareto optimal allocation. ■

The proof exploits strict monotonicity of preferences. Where?

---

\(^2\)Having set definitions for 2-periods Arrow-Debreu economies, it should be apparent how a generalization to any finite \(T\)-periods economies is in fact effectively straightforward. Infinite horizon will be dealt with in successive notes.
4.0.2 Financial market economy

Consider the 2-period economy just introduced. Suppose now contingent commodities are not traded. Instead, agents can trade in spot markets and in \( j \in \{1, \ldots, J\} \) assets. An asset \( j \) is a promise to pay \( a^j_l \geq 0 \) units of good \( l = 1 \) in state \( s = 1, \ldots, S \).\(^3\) Let \( a_j = (a^1_j, \ldots, a^S_j) \). To summarize the payoffs of all the available assets, define the \( S \times J \) asset payoff matrix

\[
A = \begin{pmatrix}
a^1_1 & \cdots & a^1_J \\
\vdots & \ddots & \vdots \\
a^S_1 & \cdots & a^S_J
\end{pmatrix}.
\]

It will be convenient to define \( a_s \) to be the \( s \)-th row of the matrix. Note that it contains the payoff of each of the assets in state \( s \).

Let \( p = (p_0, p_1, \ldots, p_S) \), where \( p_s \in \mathbb{R}^L_+ \) for each \( s \), denote the \textit{spot price vector} for goods. That is, for a price \( p_{ts} \) agents trade one unit of good \( l \) in state \( s \). Recall the definition of prices for state contingent commodities in Arrow-Debreu economies, denoted \( \phi \). Note the difference. Let good \( l = 1 \) at each date and state represent the numeraire; that is, \( p_{ts} = 1 \), for all \( s = 0, \ldots, S \).

Let \( x_{si}^i \) denote the amount of good \( l \) that agent \( i \) consumes in state \( s \). Let \( q = (q_1, \ldots, q_J) \in \mathbb{R}^J_+ \), denote the prices for the assets.\(^4\) Note that the prices of assets are non-negative, as we normalized asset payoff to be non-negative.

Given prices \( (p, q) \) and the asset structure \( A \), any agent \( i \) picks a consumption vector \( x^i \in X \) and a portfolio \( z^i \in \mathbb{R}^J \) to

\[
\max u^i(x^i)
\]

s.t.

\[
p_0(x^i_0 - \omega^i_0) = -qz^i
\]

\[
p_s(x^i_s - \omega^i_s) = A_s z^i, \text{ for } s = 1, \ldots, S.
\]

\(^3\)The non-negativity restriction on asset payoffs is just for notational simplicity.

\(^4\)Quantities will be row vectors and prices will be column vectors, to avoid the annoying use of transposes.
Def. 19 A Financial markets equilibrium is a \((x^*, z^*, p^*, q^*)\) such that

1. \(x^i \in \arg\max u^i(x^i)\) s.t.
   \[ p_0(x_0^i - \omega_0^i) = -qz_i, \text{ and} \]
   \[ p_s(x_s^i - \omega_s^i) = a_s z_i, \text{ for } s = 1, \ldots, S; \text{ and furthermore} \]

2. \(\sum_{i=1}^I x^{i*} - \omega^{i*} = 0, \text{ for any } s = 0, 1, \ldots, S, \text{ and } \sum_{i=1}^I z^{i*} \)

Financial markets equilibrium is the equilibrium concept we shall care about. This is because i) Arrow-Debreu markets are perhaps too demanding a requirement, and especially because ii) we are interested in financial markets and asset prices \(q\) in particular. Arrow-Debreu equilibrium will be a useful concept insofar as it represents a benchmark (about which we have a wealth of available results) against which to measure Financial markets equilibrium.

Def. 20 Remark 21 The economy just introduced is characterized by asset markets in zero net supply, that is, no endowments of assets are allowed for. It is straightforward to extend the analysis to assets in positive net supply, e.g., stocks. In fact, part of each agent \(i\)’s endowment (to be specific: the projection of his/her endowment on the asset span, \(< A > = \{ \tau \in R^S : \tau = Az, z \in R^J \}\) can be represented as the outcome of an asset endowment, \(z^i_w\); that is, letting \(\omega^i = (\omega_{11}^i, \ldots, \omega_{1S}^i)\), we can write

\[ \omega^i = w^i + Az^i_w \]

and proceed straightforwardly by constructing the budget constraints and the equilibrium notion.

4.0.3 No Arbitrage

Before deriving the properties of asset prices in equilibrium, we shall invest some time in understanding the implications that can be derived from the milder condition of no-arbitrage. This is because the characterization of no-arbitrage prices will also be useful to characterize financial markets equilibria.

For notational convenience, define the \((S + 1) \times J\) matrix

\[ W = \begin{bmatrix} -q \\ A \end{bmatrix}. \]
**Definition 22**  \( W \) satisfies the No-arbitrage condition if there does not exist a \( z \in R^J \) such that \( Wz > 0 \).\(^5\)

The No-Arbitrage condition can be equivalently formulated in the following way. Define the span of \( W \) to be

\[
< W > = \{ \tau \in R^{S+1} : \tau = Wz, \; z \in R^J \}.
\]

This set contains all the feasible wealth transfers, given asset structure \( A \). Now, we can say that \( W \) satisfies the No-arbitrage condition if

\[
< W > \cap R^{S+1}_+ = \{0\}.
\]

Clearly, requiring that \( W = (-q, A) \) satisfies the No-arbitrage condition is weaker than requiring that \( q \) is an equilibrium price of the economy (with asset structure \( A \)). By strong monotonicity of preferences, No-arbitrage is equivalent to requiring the agent’s problem to be well defined. The next result is remarkable since it provides a foundation for asset pricing based only on No-arbitrage.

**Theorem 23** (No-Arbitrage theorem)

\[
< W > \cap R^{S+1}_+ = \{0\} \iff \exists \hat{\pi} \in R^{S+1}_+ \text{ such that } \hat{\pi}W = 0.
\]

First, observe that there is no uniqueness claim on the \( \hat{\pi} \), just existence. Next, notice how \( \hat{\pi}W = 0 \) implies \( \hat{\pi}\tau = 0 \) for all \( \tau \in < W > \). It then provides a pricing formula for assets:

\[
\hat{\pi}W = \begin{pmatrix}
... \\
-\hat{\pi}_0 q^j + \hat{\pi}_1 a^j_1 + ... + \hat{\pi}_s a^j_s \\
...
\end{pmatrix} = \begin{pmatrix}
0 \\
... 
\end{pmatrix}_{J \times 1}
\]

and, rearranging, we obtain for each asset \( j \),

\[
q^j = \pi_1 a^j_1 + ... + \pi_S a^j_S, \quad \text{for} \quad \pi_s = \frac{\hat{\pi}_s}{\hat{\pi}_0} \quad (4.2)
\]

---

\(^5\) \( Wz > 0 \) requires that all components of \( Wz \) are \( \geq 0 \) and at least one of them > 0.
Note how the positivity of all components of $\hat{\pi}$ was necessary to obtain (4.2).

**Proof.** Define the simplex in $R^{S+1}_+$ as $\Delta = \{ \tau \in R^{S+1}_+ : \sum_{s=0}^S \tau_s = 1 \}$. Note that by the No-arbitrage condition, $\langle W \rangle \cap \Delta$ is empty. The proof hinges crucially on the following separating result, a version of Farkas Lemma, which we shall take without proof.

**Lemma 24** Let $X$ be a finite dimensional vector space. Let $K$ be a non-empty, compact and convex subset of $X$. Let $M$ be a non-empty, closed and convex subset of $X$. Furthermore, let $K$ and $M$ be disjoint. Then, there exists $\hat{\pi} \in X \setminus \{0\}$ such that

$$\sup_{\tau \in M} \hat{\pi} \tau < \inf_{\tau \in K} \hat{\pi} \tau.$$  

Let $X = R^{S+1}_+$, $K = \Delta$ and $M = \langle W \rangle$. Observe that all the required properties hold and so the Lemma applies. As a result, there exists $\hat{\pi} \in X \setminus \{0\}$ such that

$$\sup_{\tau \in \langle W \rangle} \hat{\pi} \tau < \inf_{\tau \in \Delta} \hat{\pi} \tau. \tag{4.3}$$

It remains to show that $\hat{\pi} \in R^{S+1}_+$. Suppose, on the contrary, that there is some $s$ for which $\hat{\pi}_s \leq 0$. Then note that in (4.3), the RHS $\leq 0$. By (4.3), then, LHS $< 0$. But this contradicts the fact that $0 \in \langle W \rangle$.

We still have to show that $\hat{\pi}W = 0$, or in other words, that $\hat{\pi} \tau = 0$ for all $\tau \in \langle W \rangle$. Suppose, on the contrary that there exists $\tau \in \langle W \rangle$ such that $\hat{\pi} \tau \neq 0$. Since $\langle W \rangle$ is a subspace, there exists $\alpha \in R$ such that $\alpha \tau \in \langle W \rangle$ and $\hat{\pi} \alpha \tau$ is as large as we want. However, RHS is bounded above, which implies a contradiction.

$\Longleftrightarrow$ The existence of $\hat{\pi} \in R^{S+1}_+$ such that $\hat{\pi}W = 0$ implies that $\hat{\pi} \tau = 0$ for all $\tau \in \langle W \rangle$. By contradiction, suppose $\exists \tau^* \in \langle W \rangle$ and such that $\tau^* \in R^{S+1}_+ \setminus \{0\}$. Since $\hat{\pi}$ is strictly positive, $\hat{\pi} \tau^* > 0$, the desired contradiction.

A few final remarks to this section.

**Remark 25** An asset which pays one unit of numeraire in state $s$ and nothing in all other states (Arrow security), has price $\pi_s$ according to (4.2). Such asset is called Arrow security.
Remark 26 Is the vector \( \hat{\pi} \) obtained by the No-arbitrage theorem unique? Notice how (11) defines a system of \( J \) equations and \( S \) unknowns, represented by \( \pi \). Define the set of solutions to that system as
\[
R(q) = \{ \pi \in R^S_+ : q = \pi A \}.
\]
Suppose, the matrix \( A \) has rank \( J' \leq J \) (that it, \( A \) has \( J' \) linearly independent column vectors and \( J' \) is the effective dimension of the asset space). In general, then \( R(q) \) will have dimension \( S - J' \). It follows then that, in this case, the No-arbitrage theorem restricts \( \hat{\pi} \) to lie in a \( S - J' + 1 \) dimensional set. If we had \( S \) linearly independent assets, the solution set has dimension zero, and there is a unique \( \pi \) vector that solves (11). The case of \( S \) linearly independent assets is referred to as Complete markets.

Remark 27 Let preferences be Von Neumann-Morgenstern:
\[
u^i(x^i) = u^i(x^i_0) + \sum_{s=1}^{S} \text{prob}_s u^i(x^i_s)
\]
where \( \sum_{s=1}^{S} \text{prob}_s = 1 \). Let then \( m_s = \frac{\pi_s}{\text{prob}_s} \). Then
\[
q^j = E(m a_j)
\]
In this representation of asset prices the vector \( m \in R^S_+ \) is called Stochastic discount factor.

4.0.4 Equilibrium economies and the stochastic discount factor
In the previous section we showed the existence of a vector that provides the basis for pricing assets in a way that is compatible with equilibrium, albeit milder than that. In this section, we will strengthen our assumptions and study asset prices in a full-fledged economy. Among other things, this will allow us to provide some economic content to the vector \( \pi \).

Recall the definition of Financial market equilibrium. Let \( MRS^s_i(x^i) \) denote agent \( i \)’s marginal rate of substitution between consumption of the numeraire good 1 in state \( s \) and consumption of the numeraire good 1 at date 0:
Let $MRS^i(x^i) = \left( \ldots \ MRS^i_s(x^i) \ldots \right)$ denote the vector of marginal rates of substitution for agent $i$, an $S$ dimensional vector. Note that, under the assumption of strong monotonicity of preferences, $MRS^i(x^i) \in R^S_{++}$.

By taking the First Order Conditions (necessary and sufficient for a maximum under the assumption of strict quasi-concavity of preferences) with respect to $z^i_j$ of the individual problem for an arbitrary price vector $q$, we obtain that

$$q_j^i = \sum_{s=1}^S \text{prob}_s MRS^i_s(x^i)a^j_s = E\left( MRS^i(x^i) \cdot a^j \right),$$

for all $j = 1, \ldots, J$ and all $i = 1, \ldots, I$, where of course the allocation $x^i$ is the equilibrium allocation. At equilibrium, therefore, the marginal cost of one more unit of asset $j$, $q^i_j$, is equalized to the marginal valuation of that agent for the asset's payoff, $\sum_{s=1}^S \text{prob}_s MRS^i_s(x^i)a^j_s$.

Compare equation (4.4) to the previous equation (4.2). Clearly, at any equilibrium, condition (4.4) has to hold for each agent $i$. Therefore, in equilibrium, the vector of marginal rates of substitution of any arbitrary agent $i$ can be used to price assets; that is any of the agents' vector of marginal rates of substitution (normalized by probabilities) is a viable stochastic discount factor $m$.

In other words, any vector $\left( \ldots \frac{MRS^i(x^i)}{\text{prob}_s} \ldots \right)$ belongs to $R(q)$ and is hence a viable $\pi$ for the asset pricing equation (4.2). But recall that $R(q)$ is of dimension $S - J'$, where $J'$ is the effective dimension of the asset space. The higher the effective dimension of the asset space (sloppily said, the larger financial markets) the more aligned are agents’ marginal rates of substitution at equilibrium (sloppily said, the smaller are unexploited gains from trade at equilibrium). In the extreme case, when markets are complete (that is, when the rank of $A$ is $S$), the set $R(q)$ is in fact a singleton and hence the $MRS^i(x^i)$ are equalized across agents $i$ at equilibrium: $MRS^i(x^i) = MRS$, for any $i = 1, \ldots, I$.

**Problem 28** Write the Pareto problem for the economy and show that, at any Pareto optimal allocation, $x$, it is the case that $MRS^i(x^i) = MRS$, for
any \( i = 1, \ldots, I \). Furthermore, show that an allocation \( x \) which satisfies the feasibility conditions (market clearing) for goods and is such that \( MRS^i(x^i) = MRS \), for any \( i = 1, \ldots, I \), is Pareto optimal.

We conclude that, when markets are Complete, equilibrium allocations are Pareto optimal. That is, the First Welfare theorem holds for Financial market equilibria when markets are Complete.

**Problem 29 (Economies with bid-ask spreads)** Extend our economy by assuming that, given an exogenous vector \( \gamma \in \mathbb{R}_+^J \):

- the buying price of asset \( j \) is \( q_j + \gamma_j \)

while

- the selling price of asset \( j \) is \( q_j \)

for any \( j = 1, \ldots, J \) and exogenous. Write the budget constraint and the First Order Conditions for an agent \( i \)’s problem. Derive a generalized asset pricing relation (not an equation, is it?) that relates \( MRS^i(x^i) \) to asset prices.

### 4.0.5 Arrow theorem

The Arrow theorem is the fundamental decentralization result in financial economics. It states sufficient conditions for a form of equivalence between the Arrow-Debreu and the Financial market equilibrium concepts. It was essentially introduced by Arrow (1952). The proof of the theorem introduces a reformulation of the budget constraints of the Financial market economy which focuses on feasible wealth transfers across states directly, on the span of \( A \),

\[
<A> = \{ \tau \in \mathbb{R}^S : \tau = Az, \ z \in \mathbb{R}^J \}
\]

in particular. Such a reformulation is important not only in itself but as a lemma for welfare analysis in Financial market economies.

**Proposition 30** Let \((x^*, \phi^*)\) represent an Arrow-Debreu equilibrium. Suppose \( \text{rank}(A) = S \) (financial markets are Complete). Then \((x^*, z^*, p^*, q^*)\) is a Financial market equilibrium, where

\[
\phi^* = p^* \left( \frac{MRS^i_s(x^i_s)}{\text{prob}_s} \ldots \right) \quad \text{and} \quad q^* = \sum_{s=1}^{S} \text{prob}_s MRS^i_s(x^i_s) a_s
\]
Proof. Financial market equilibrium prices of assets $q^*$ satisfy No-arbitrage. There exists then a vector $\pi \in R_{++}^{S+1}$ such that $\pi W = 0$, or $q^* = \pi A$. The budget constraints in the financial market economy are

$$p_0 (x_0^i - \omega_0^i) + q^* z^i = 0, \quad p_s^* (x_s^i - \omega_s^i) = a_s z^i, \quad \text{for } s = 1, ..., S.$$ 

Substituting $q = \pi A$, expanding the first equation, and writing the constraints at time 1 in vector form, we obtain:

$$p_0^* (x_0^i - \omega_0^i) + \sum_{s=1}^{S} \pi_s p_s^* (x_s^i - \omega_s^i) = 0 \quad (4.5)$$

$$\begin{bmatrix}
p_0^* (x_0^i - \omega_0^i) \\
\cdot \\
p_s^* (x_s^i - \omega_s^i) \\
\cdot \\
\cdot 
\end{bmatrix} \in < A > \quad (4.6)$$

But if $\text{rank}(A) = S$, it follows that $< A > = R^S$, and the constraint

$$\begin{bmatrix}
p_0^* (x_0^i - \omega_0^i) \\
\cdot \\
p_s^* (x_s^i - \omega_s^i) \\
\cdot \\
\cdot 
\end{bmatrix} \in < A >$$

$A >$ is never binding. Each agent $i$’s problem is then subject only to

$$p_0^* (x_0^i - \omega_0^i) + \sum_{s=1}^{S} \pi_s p_s^* (x_s^i - \omega_s^i) = 0,$$

the budget constraint in the Arrow-Debreu economy with

$$\phi_s^* = \pi_s p_s^*, \quad \text{for any } s = 1, ..., S.$$ 

Furthermore, by No-arbitrage

$$q^* = \sum_{s=1}^{S} \text{prob}_s MRS_s^i(x^i) a_s.$$
Finally, using $\pi_s = \frac{MRS_i(x^s)}{\text{prob}_s}$, for any $s = 1, \ldots, S$, proves the result. (Recall that, with Complete markets $MRS_i(x^i) = MRS$, for any $i = 1, \ldots, I$.)

4.1 Constrained Pareto Optimality

Under Complete markets, the First Welfare Theorem holds for Financial market equilibrium. This is a direct implication of Arrow theorem.

Proposition 31 Let $(x^s, z^s, p^s, q^s)$ be a Financial market equilibrium of an economy with Complete markets (with rank$(A) = S$). Then $x^*$ is a Pareto optimal allocation.

However, under Incomplete markets Financial market equilibria are generically inefficient in a Pareto sense. That is, a planner could find an allocation that improves some agents without making any other agent worse off.

Theorem 32 At a Financial Market Equilibrium $(x^*, z^*, p^*, q^*)$ of an incomplete financial market economy, that is, of an economy with rank$(A) < S$, the allocation $x^*$ is generically\(^6\) not Pareto Optimal.

Proof. From the proof of Arrow theorem, we can write the budget constraints of the Financial market equilibrium as:

$$p_0^* (x_0^* - \omega_0^i) + \sum_{s=1}^{S} \pi_s p_s^* (x_s^* - \omega_s^i) = 0$$

$$\begin{bmatrix}
\vdots \\
p_s^* (x_s^* - \omega_s^i) \\
\vdots 
\end{bmatrix} \in <A>$$

\(^6\)We say that a statement holds \textit{generically} when it holds for a full Lebesgue-measure subset of the parameter set which characterizes the economy. In these notes we shall assume that the an economy is parametrized by the endowments for each agent, the asset payoff matrix, and a two-parameter parametrization of utility functions for each agent; see Magill-Shafer, ch. 30 in W. Hildenbrand and H. Sonnenschein (eds.), \textit{Handbook of Mathematical Economics}, Vol. IV, Elsevier, 1991.
for some \( \pi \in R^{S^+}_+ \). Pareto optimality of \( x^* \) requires that there does not exist an allocation \( y \) such that

1. \( u(y^i) \geq u(x'^i) \) for any \( i = 1, \ldots, I \) (strictly for at least one \( i \)), and
2. \( \sum_{i=1}^{I} y^i - \omega^i_s = 0 \), for any \( s = 0, 1, \ldots, S \)

Reproducing the proof of the First Welfare theorem, it is clear that, if such a \( y \) exists, it must be that

\[
\begin{bmatrix}
  \vdots \\
  \vdots \\
  \vdots
\end{bmatrix}
\]

\[
 p^*_s (y^i_s - \omega^i_s) \not< A \>
\]

for some \( i = 1, \ldots, I \); otherwise the allocation \( y \) would be budget feasible for all agent \( i \) at the equilibrium prices. Generic Pareto sub-optimality of \( x^* \) follows then directly from the following Lemma, which we leave without proof.\(^7\)

**Lemma 33** Let \( (x^*, z^*, p^*, q^*) \) represent a Financial Market Equilibrium of an economy with \( \text{rank}(A) < S \). For a generic set of economies, the constraints

\[
\begin{bmatrix}
  \vdots \\
  \vdots \\
  \vdots
\end{bmatrix}
\]

\[
 p^*_s (x^i_s - \omega^i_s) \in < A >
\]

are binding for some \( i = 1, \ldots, I \).

\(^7\)The proof can be found in Magill-Shafer, ch. 30 in W. Hildenbrand and H. Sonnenschein (eds.), *Handbook of Mathematical Economics*, Vol. IV, Elsevier, 1991. It requires mathematical techniques from differential topology which are not appropriate to be introduced in this course.

**Remark 34** The Lemma implies a slightly stronger result than generic Pareto sub-optimality of Financial market equilibrium for economies with incomplete markets. It implies in fact that a Pareto improving allocation can be found locally around the equilibrium, as a perturbation of the equilibrium.
Pareto optimality might however represent too strict a definition of social welfare of an economy with frictions which restrict the consumption set, as in the case of incomplete markets. In this case, markets are assumed incomplete exogenously. There is no reason in the fundamentals of the model why they should be, but they are. Under Pareto optimality, however, the social welfare notion does not face the same constraints. For this reason, we typically define a weaker notion of social welfare, Constrained Pareto optimality, by restricting the set of feasible allocations to satisfy the same set of constraints on the consumption set imposed on agents at equilibrium. In the case of incomplete markets, for instance, the feasible wealth vectors across states are restricted to lie in the span of the payoff matrix. That can be interpreted as the economy’s “financial technology” and it seems reasonable to impose the same technological restrictions on the planner’s reallocations. The formalization of an efficiency notion capturing this idea follows. Let $x_{t=1}^i = (x_{s}^i)_{s=1}^{S} \in R_{+}^{SL}$; and similarly $p_{t=1} = (p_{s})_{s=1}^{S} \in R^{SL}$

**Definition 35** (Diamond, 1968; Geanakoplos-Polemarchakis, 1986) Let $(x^{*}, z^{*}, p^{*}, q^{*})$ represent a Financial market equilibrium of an economy whose consumption set at time $t = 1$ is restricted by

$$x_{t=1}^i, \in B(p_{t=1}), \text{ for any } i = 1, \ldots , I$$

In this economy, the allocation $x^{*}$ is Constrained Pareto optimal if there does not exist a $(y, \theta)$ such that

1. $u(y^i) \geq u(x_i^{*})$ for any $i = 1, \ldots , I, \text{ strictly for at least one } i$

2. $\sum_{i=1}^{I} y^i_s - \omega^i_s = 0, \text{ for any } s = 0, 1, \ldots , S$

and

3. $y_{t=1}^i \in B(g_{t=1}^i(\omega, \theta)), \text{ for any } i = 1, \ldots , I$

where $g_{t=1}^i(\omega, \theta)$ is a vector of equilibrium prices for spot markets at $t = 1$ opened after each agent $i = 1, \ldots , I$ has received income transfer $A\theta^i$.

The constraint on the consumption set restricts only time 1 consumption allocations. More general constraints are possible but these formulation is consistent with the typical frictions we encounter in economics, e.g., on financial markets. It is important that the constraint on the consumption
4.1 CONSTRAINED PARETO OPTIMALITY

set depends in general on \( g^*_{t=1}(\omega, \theta) \), that is on equilibrium prices for spot markets opened at \( t = 1 \) after income transfers to agents. It implicit identifies income transfers (besides consumption allocations at time \( t = 0 \)) as the instrument available for Constrained Pareto optimality; that is, it implicitly constrains the planner implementing Constraint Pareto optimal allocations to interact with markets, specifically to open spot markets after transfers. On the other hand, the planner is able to anticipate the spot price equilibrium map, \( g^*_{t=1}(\omega, \theta) \); that is, to internalize the effects of different transfers on spot prices at equilibrium.

**Proposition 36** Let \( (x^*, z^*, p^*, q^*) \) represent a Financial market equilibrium of an economy with complete markets (rank(\( A \)) = \( S \)) and whose consumption set at time \( t = 1 \) is restricted by

\[
x^i_{t=1} \in B \subseteq \mathbb{R}^{SL}^+ \text{, for any } i = 1, ..., I
\]

In this economy, the allocation \( x^* \) is Constrained Pareto optimal.

Crucially, markets are Complete and \( B \) is independent of prices. The proof is then a straightforward extension of the First Welfare theorem combined with Arrow theorem. Constraint Pareto optimality of Financial market equilibrium allocations is guaranteed as long as the constraint set \( B \) is exogenous.

**Proposition 37** Let \( (x^*, z^*, p^*, q^*) \) represent a Financial market equilibrium of an economy with Incomplete markets (rank(\( A \)) < \( S \)). In this economy, the allocation \( x^* \) is not Constrained Pareto optimal.

**Proof.** By the decomposition of the budget constraints in the proof of Arrow theorem, this economy is equivalent to one with Complete markets whose consumption set at time \( t = 1 \) is restricted by

\[
x^i_{t=1} \in B(g^*_{t=1}(\omega, z)), \text{ for any } i = 1, ..., I, \text{ for any } i = 1, ..., I
\]

with the set \( B(g^*_{t=1}(\omega, z)) \) defined implicitly by

\[
\begin{bmatrix}
& . \\
& . \\
g^*_{s}(\omega, z) (x^i_s - \omega^i_s) \in < A >, \text{ for any } i = 1, ..., I
& . \\
& .
\end{bmatrix}
\]
Note first of all that, by construction, \( p_s^* \in g_s^*(\omega_s, z^*) \). Following the proof of Pareto sub-optimality of Financial market equilibrium allocations, it then follows that if a Pareto-improving \( y \) exists, it must be that
\[
p_s^* (y^*_s - \omega^*_s) \notin \langle A >, \text{ for some } i = 1, \ldots, I; \text{ while } g_s^*(\omega_s, \theta) (y^i_s - \omega^i_s) = A \theta^i, \text{ for all } i = 1, \ldots, I.
\]
Generic Constrained Pareto sub-optimality of \( x^* \) follows then directly from the following Lemma, which we leave without proof.\(^8\)

**Lemma 38** Let \( (x^*, z^*, p^*, q^*) \) represent a Financial Market Equilibrium of an economy with \( \text{rank}(A) < S \). For a generic set of economies, the constraints
\[
g_s^*(\omega_s, z^* + dz) (y^i_s - \omega^i_s) = A(z^i_s + dz^i), \text{ for some } dz \in R^I \setminus \{0\}
\]
such that \( \sum_{i=1}^I dz^i = 0 \), are weakly relaxed for all \( i = 1, \ldots, I \), strictly for at least one.\(^9\)

---

\(^8\)The proof is due to Geanakoplos-Polemarchakis (1986). It also requires differential topology techniques.

\(^9\)Once again, note that the Lemma implies that a Pareto improving allocation can be found locally around the equilibrium, as a *perturbation of the equilibrium*.
4.1 CONSTRAINED PARETO OPTIMALITY

There is a fundamental difference between incomplete market economies, which have typically not Constrained Optimal equilibrium allocations, and economies with constraints on the consumption set, which have, on the contrary, Constrained Optimal equilibrium allocations. It stands out by comparing the respective trading constraints

\[ g^*_s(\omega_s, \theta)(x^i_s - \omega^i_s) = a^i_s \theta^i, \text{ for all } i \text{ and } s, \quad \text{vs.} \quad x^i_{t=1} \in B, \text{ for all } i. \]

The trading constraint of the incomplete market economy is determined at equilibrium, while the constraint on the consumption set is exogenous. Another way to re-phrase the same point is the following. A planner choosing \((y, \theta)\) will take into account that at each \((y, \theta)\) is typically associated a different trading constraint \(g^*_s(\omega_s, \theta)(x^i_s - \omega^i_s) = a^i_s \theta^i\), for all \(i\) and \(s\); while any agent \(i\) will choose \((x^i, z^i)\) to satisfy \(p^*_s(x^i_s - \omega^i_s) = a^i_s z^i\), for all \(s\), taking as given the equilibrium prices \(p^*_s\).

**Remark 39** Consider an economy whose constraints on the consumption set depend on the equilibrium allocation:

\[ x^i_{t=1} \in B(x^i_{t=1}, z^i), \text{ for any } i = 1, ..., I. \]

This is essentially an externality in the consumption set. It is not hard to extend the analysis of this section to show that this formulation introduces inefficiencies and equilibrium allocations are Constraint Pareto sub-optimal.

**Corollary 40** Let \((x^*, z^*, p^*, q^*)\) represent a Financial market equilibrium of a 1-good economy \((L = 1)\) with Incomplete markets \((\text{rank}(A) < S))\). In this economy, the allocation \(x^*\) is Constrained Pareto optimal.

**Proof.** The constraint on the consumption set implied by incomplete markets, if \(L = 1\), can be written

\[ (x^i_s - \omega^i_s) = a^i_s z^i. \]

It is independent of prices, of the form \(x^i_{t=1} \in B\).

**Remark 41** Consider an alternative definition of Constrained Pareto opti-
mality, due to Grossman (1970), in which constraints 3 are substituted by

$$3'. \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ p_s^*(x_s^{i*} - \omega_s^i) \\ \cdot \\ \cdot \end{bmatrix} = Az^i, \text{ for any } i = 1, ..., I$$

where $p^*$ is the spot market Financial market equilibrium vector of prices. That is, the planner takes the equilibrium prices as given. It is immediate to prove that, with this definition of Constrained Pareto optimality, any Financial market equilibrium allocation $x^*$ of an economy with Incomplete markets is in fact Constrained Pareto optimal, independently of the financial markets available ($\text{rank}(A) \leq S$).

**Problem 42** Consider a Complete market economy ($\text{rank}(A) = S$) whose feasible set of asset portfolios is restricted by:

$$z^i \in Z \subseteq R^l, \text{ for any } i = 1, ..., I$$

A typical example is borrowing limits:

$$z^i \geq -b, \text{ for any } i = 1, ..., I$$

Are equilibrium allocations of such an economy Constrained Pareto optimal (also if $L > 1$)?

**Problem 43** Consider a 1-good ($L = 1$) Incomplete market economy ($\text{rank}(A) < S$) which lasts 3 periods. Define an Financial market equilibrium for this economy as well as Constrained Pareto optimality. Are Financial market equilibrium allocations of such an economy Constrained Pareto optimal?

### 4.2 Aggregation

Agent $i$’s optimization problem in the definition of Financial market equilibrium requires two types of simultaneous decisions. On the one hand, the
agent has to deal with the usual consumption decisions i.e., she has to decide how many units of each good to consume in each state. But she also has to make financial decisions aimed at transferring wealth from one state to the other. In general, both individual decisions are interrelated: the consumption and portfolio allocations of all agents $i$ and the equilibrium prices for goods and assets are all determined simultaneously from the system of equations formed by $(\mathbf{??})$ and $(\mathbf{??})$. The financial and the real sectors of the economy cannot be isolated. Under some special conditions, however, the consumption and portfolio decisions of agents can be separated. This is typically very useful when the analysis is centered on financial issue. In order to concentrate on asset pricing issues, most finance models deal in fact with 1-good economies, implicitly assuming that the individual financial decisions and the market clearing conditions in the assets markets determine the financial equilibrium, independently of the individual consumption decisions and market clearing in the goods markets; that is independently of the real equilibrium prices and allocations. In this section we shall identify the conditions under which this can be done without loss of generality. This is sometimes called "the problem of aggregation."

The idea is the following. If we want equilibrium prices on the spot markets to be independent of equilibrium on the financial markets, then the aggregate spot market demand for the $L$ goods in each state $s$ should must depend only on the incomes of the agents in this state (and not in other states) and should be independent of the distribution of income among agents in this state.

**Theorem 44 Budget Separation.** Suppose that each agent $i$'s preferences are separable across states, identical, homothetic within states, and von Neumann-Morgenstern; i.e. suppose that there exists an homothetic $u : R^L \rightarrow R$ such that

$$u^i(x^i) = u(x_0^i) + \sum_{s=1}^{S} \text{prob}_s u(x_s^i), \text{ for all } i = 1, \ldots, I.$$ 

Then equilibrium spot prices $p^*$ are independent of asset prices $q$ and of the income distribution; that is, constant in \( \left\{ \omega^i \in R^{L(S+1)}_+ \middle| \sum_{i=1}^{I} \omega^i \text{ given} \right\} \).

**Proof.** The consumer’s maximization problem in the definition of Financial market equilibrium can be decomposed into a sequence of spot commodity
allocation problems and an income allocation problem as follows. *The spot commodity allocation problems.* Given the current and anticipated spot prices $p = (p_0, p_1, \ldots, p_S)$ and an exogenously given stream of financial income $y^i = (y^i_0, y^i_1, \ldots, y^i_S) \in R_+^{S+1}$ in units of numeraire, agent $i$ has to pick a consumption vector $x^i \in R_{L(S+1)}$ to

$$\begin{align*}
\max & \quad u^i(x^i) \\
\text{s.t.} & \\
& \quad p_0 x^i_0 = y^i_0 \\
& \quad p_s x^i_s = y^i_s, \text{ for } s = 1, \ldots, S.
\end{align*}$$

Let the $L(S + 1)$ demand functions be given by $x^i_s(p, y^i)$, for $l = 1, \ldots, L$, $s = 0, 1, \ldots, S$. Define now the indirect utility function for income by

$$v^i(y^i; p) = u^i(x^i(p, y^i)).$$

*The Income allocation problem.* Given prices $(p, q)$, endowments $\omega^i$, and the asset structure $A$, agent $i$ has to pick a portfolio $z^i \in R^I$ and an income stream $y^i \in R_+^{S+1}$ to

$$\begin{align*}
\max & \quad v^i(y^i; p) \\
\text{s.t.} & \\
& \quad p_0 \omega^i_0 - q z^i = y^i_0 \\
& \quad p_s \omega^i_s + a_s z^i = y^i_s, \text{ for } s = 1, \ldots, S.
\end{align*}$$

By additive separability across states of the utility, we can break the consumption allocation problem into $S+1$ ‘spot market’ problems, each of which yields the demands $x^i_s(p_s, y^i_s)$ for each state. By homotheticity, for each $s = 0, 1, \ldots, S$, and by identical preferences across all agents,

$$x^i_s(p_s, y^i_s) = y^i_s x^i_s(p_s, 1);$$

and since preferences are identical across agents,

$$y^i_s x^i_s(p_s, 1) = y^i_s x_s(p_s, 1)$$
4.2 AGGREGATION

Adding over all agents and using the market clearing condition in spot markets \( s \), we obtain, at spot markets equilibrium,

\[
x_s(p_s^*, 1) \sum_{i=1}^{I} y_{is}^{i} - \sum_{i=1}^{I} \omega_{is}^{i} = 0.
\]

Again by homothetic utility,

\[
x_s(p_s^*, \sum_{i=1}^{I} y_{is}^{i}) - \sum_{i=1}^{I} \omega_{is}^{i} = 0. \tag{4.9}
\]

Recall from the consumption allocation problem that \( p_s x_s^{i} = y_{is}^{i} \), for \( s = 0, 1, ... S \). By adding over all agents, and using market clearing in the spot markets in state \( s \),

\[
\sum_{i=1}^{I} y_{is}^{i} = p_s \sum_{i=1}^{I} x_{is}^{i}, \text{ for } s = 0, 1, ... S \tag{4.10}
\]

\[
= p_s \sum_{i=1}^{I} \omega_{is}^{i}, \text{ for } s = 0, 1, ... S.
\]

By combining (4.9) and (4.10), we obtain

\[
x_s(p_s^*, \sum_{i=1}^{I} \omega_{is}^{i}) = \sum_{i=1}^{I} \omega_{is}^{i}. \tag{4.11}
\]

Note how we have passed from the aggregate demand of all agents in the economy to the demand of an agent owning the aggregate endowments. Observe also how equation (4.11) is a system of \( L \) equations with \( L \) unknowns that determines spot prices \( p_s^* \) for each state \( s \) independently of asset prices \( q \). Note also that equilibrium spot prices \( p_s^* \) defined by (4.11) only depend \( \omega^i \) through \( \sum_{i=1}^{I} \omega_{is}^{i} \).

**Remark 45** The Budget separation theorem can be interpreted as identifying conditions under which studying a single good economy is without loss of generality. To this end, consider the income allocation problem of agent \( i \),
given equilibrium spot prices $p^*$:

$$\max_{y^i \in \mathbb{R}^{S+1}} v^j(y^i; p^*)$$

s.t.

$$y^i_0 = p^*_0 \omega^i_0 - q z^i$$
$$y^i_s = p^*_s \omega^i_s + a_s z^i, \text{ for } s = 1, ... S$$

If preferences separable across states, identical, homothetic within states, and von Neumann-Morgenstern, it is straightforward to show that $v^j(y^i; p^*)$ is identical across agents $i$ and, seen as a function of $y^i$, it satisfies the assumptions we have imposed on $v^i$ as a function of $x^i$, in Assumption A.2. Let $w_0 = p^*_0 \omega^i_0$, $w_s = p^*_s \omega^i_s$, for any $s = 1, ..., S$; and disregard for notational simplicity the dependence of $v^j(y^i; p^*)$ on $p^*$. The income allocation problem becomes:

$$\max_{y^i \in \mathbb{R}^{S+1}} v(y^i)$$

s.t.

$$y^i_0 - w_0 = -q z^i$$
$$y^i_s - w_s = a_s z^i, \text{ for } s = 1, ... S$$

which is homeomorphic to any agent $i$’s optimization problem in the definition of Financial market equilibrium with $l = 1$. Note that $y^i_s$ gains the interpretation of agent $i$’s consumption expenditure in state $s$, while $w_s$ is interpreted as agent $i$’s income endowment in state $s$.

4.2.1 The Representative Agent Theorem

A representative agent is the following theoretical construct.

Definition 46 Consider a Financial market equilibrium $(x^*, z^*, p^*, q^*)$ of an economy populated by $i = 1, ..., I$ agents with preferences $u^i : X \to \mathbb{R}$ and endowments $\omega^i$. A Representative agent for this economy is an agent with preferences $U^R : X \to \mathbb{R}$ and endowment $\omega^R$ such that the Financial market equilibrium of an associated economy with the Representative agent as the only agent has prices $(p^*, q^*)$. 
In this section we shall identify assumptions which guarantee that the Representative agent construct can be invoked without loss of generality. This assumptions are behind much of the empirical macro/finance literature.

**Theorem 47** *Representative agent.* Suppose there exists an homothetic $u : R^L \rightarrow R$ such that

$$u^i(x^i) = u(x^i_0) + \sum_{s=1}^{S} \text{prob}_s u(x^i_s), \text{ for all } i = 1, \ldots, I.$$ 

Let $p^*$ denote equilibrium spot prices. If $p^s \omega^i_s \in < A >$, then there exist a map $u^R : R^{S+1} \rightarrow R$ such that:

$$\omega^R = \sum_{i=1}^{I} \omega^i,$$

$$U^R(x) = u^R(y_0) + \sum_{s=1}^{S} \text{prob}_s u^R(y_s) \text{ where } y_s = p^s \sum_{i=1}^{I} x^i_s, \text{ s = 0, 1, \ldots, S}$$

constitutes a Representative agent.

Since the Representative agent is the only agent in the economy, her consumption allocation and portfolio at equilibrium, $(x^R, z^R)$, are:

$$x^* = \omega^R = \sum_{i=1}^{I} \omega^i$$

$$z^* = 0$$

If the Representative agent’s preferences can be constructed independently of the equilibrium of the original economy with $I$ agents, then equilibrium prices can be read out of the Representative agent’s marginal rates of substitution evaluated at $\sum_{i=1}^{I} \omega^i$. Since $\sum_{i=1}^{I} \omega^i$ is exogenously given, equilibrium prices are obtained without computing the consumption allocation and portfolio for all agents at equilibrium, $(x^*, z^*)$.

**Proof.** The proof is constructive. Under the assumptions on preferences in the statement, we need to show that, for all agents $i = 1, \ldots, I$, equilibrium asset prices $q^*$ are constant in $\left\{ \omega^i \in R^{I(S+1)}_+ \big| \sum_{i=1}^{I} \omega^i \text{ given} \right\}$. If preferences
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satisfy \( u^i(x^i) = u(x^i_0) + \sum_{s=1}^{S} \text{prob}_s u(x^i_s) \), for all \( i = 1, ..., I \), with an homothetic \( u \), by the Budget separation theorem, equilibrium spot prices \( p^* \) are independent of \( q^* \) and constant in \( \left\{ \omega^i \in R^{L(S+1)}_+ \left| \sum_{i=1}^{I} \omega^i \right. \text{given} \right\} \). Therefore, \( p^* \omega^i_s \in \langle A \rangle \) is an assumption on fundamentals, in particular on \( \omega^i \).

Furthermore, we can restrict our analysis to the single good economy derived in the previous remark, whose agent \( i \)'s optimization problem is:

\[
\begin{align*}
\max_{y^i \in R^{S+1}} & \quad v(y^i) \\
\text{s.t.} & \quad y^i_0 - w_0 = -qz^i \\
& \quad y^i_s - w_s = a_s z^i, \text{ for } s = 1, ..., S
\end{align*}
\]

Write the budget constraints

\[
\begin{bmatrix}
y^i_0 - w_0 \\
y^i_s - w_s
\end{bmatrix} \in \langle \begin{bmatrix} -q \\ A \end{bmatrix} \rangle.
\]

Under the homothetic representation of preferences \( u^i(x^i) \), we can show that \( v(y^i) \) is von Neumann-Morgernstern and

\[
v(y^i) = u^R(y^i_0) + \sum_{s=1}^{S} \text{prob}_s u^R(y^i_s)
\]

for some function \( u^R : R \to R \). By Arrow theorem, we can write budget constraints as

\[
y^i_0 - w_0 + \sum_{s=1}^{S} \pi_s (y^i_s - w_s) = 0
\]

\[
(y^i_s - w_s)_{s=1}^{S} \in \langle A \rangle
\]

But, \((w^i_s)_{s=1}^{S} \in \langle A \rangle \) implies that there exist a \( z^i_w \) such that \((w^i_s)_{s=1}^{S} = A z^i_w \).

Therefore, \((w^i_s)_{s=1}^{S} \in \langle A \rangle \) implies that \((y^i_s)_{s=1}^{S} = A (z^i + z^i_w) \). We can then
write each agent $i$’s optimization problem in terms of $(y_i^0 - w_i^0, z^i)$, and the value of agent $i$’s endowment is $w_i^0 + \sum_{s=1}^{S} \pi_s w_i^s = + \sum_{s=1}^{S} \pi_s a_s z_w^i = w_i^0 + q z_w^i$. By the fact that preferences are identical across agents and by homotheticity of $v$, then we can write

$$\left[ y_i^0 (q, w_i^0 + q z_w^i) - w_i^0 \right] = \left[ y_i^0 (q, w_i^0 + q z_w^i) - w_i^0 \right] = \left[ y_0 (q, 1) - w_i^0 \right]$$

At equilibrium then

$$\left[ y_0 (q^*, 1) - w_i^0 \right] = \left[ y_0 (q^*, \sum_{i=1}^{I} w_i^i + q^* \sum_{i=1}^{I} z_i^i) - \sum_{i=1}^{I} w_i^i \right] = \left[ z (q^*, \sum_{i=1}^{I} w_i^i + q^* \sum_{i=1}^{I} z_i^i) \right] = \left[ 0 0 \right]$$

and prices $q^*$ only depend on $\sum_{i=1}^{I} w_i^i$ and $\sum_{i=1}^{I} z_i^i$. But since $(w_i^s)_{s=1}^{S} = A z_i^i$, $\sum_{i=1}^{I} z_i^i$ is a linear translation of $\sum_{i=1}^{I} w_i^i$. Finally, let

$$U_R(x) = v(\sum_{i=1}^{I} y_i^i) \text{ where } y_i^i = p^* \sum_{i=1}^{I} x_s^i, \ s = 0, 1, ..., S$$

to end the proof. ■

The Representative agent theorem, as noted, allows us to obtain equilibrium prices without computing the consumption allocation and portfolio for all agents at equilibrium, $(x^*, z^*)$. Let $w = \sum_{i=1}^{I} w_i^i$. Under the assumptions of the Representative agent theorem,

$$q = \sum_{s=1}^{S} MRS_s(w) a_s, \text{ for } MRS_s(w) = \frac{\partial u_R(w)}{\partial w_s} \frac{\partial u_R(w_0)}{\partial w_0}$$

That is, asset prices can be computed from agents’ preferences $u_R: R \rightarrow R$ and from the aggregate endowment $w$. This is called the Lucas’ trick for pricing assets.

**Problem 48** Note that, under the Complete markets assumption, the span restriction on endowments, $p^*_s \omega_i^s, \in A >$, is trivially satisfied. Does the assumption $p^*_s \omega_i^s, \in A >$, for all agents $i$ imply Pareto optimal allocations in equilibrium.
CHAPTER 4 TWO-PERIOD ECONOMIES

Problem 49 Assume all agents have identical quadratic preferences. Derive individual demands for assets (without assuming \( p_s^* \omega_s^i \in < A > \)) and show that the Representative agent theorem is obtained.

Another interesting but misleading result is the "weak" representative agent theorem, due to Constantinides (1982).

Theorem 50 Suppose markets are complete \((\text{rank}(A) = S)\) and preferences \(u^i(x^i)\) are von Neumann-Morgernstern (but not necessarily identical nor homothetic). Let \((x^*, z^*, p^*, q^*)\) be a Financial markets equilibrium. Then,

\[
\omega^R = \sum_{i=1}^I \omega^i,
\]

\[
U^R(x) = \max_{(x^i)_{i=1}^I} \sum_{i=1}^I \theta^i u^i(x^i) \quad \text{s.t.} \quad \sum_{i=1}^I x^i = x, \text{ where } \theta^i = (\lambda_i)^{-1} \text{ and } \lambda_i = \frac{\partial u^i(x^i)}{\partial x^i_{10}}
\]

constitutes a Representative agent.

Clearly, then,

\[
q^* = \sum_{s=1}^S MRS^R_s(\omega^R_s)a_s,
\]

where \(MRS^R_s(x) = \frac{\partial U^R_s(x)}{\partial x^i_{10}}\).

Proof. Consider a Financial market equilibrium \((x^*, z^*, p^*, q^*)\). By complete markets, the First welfare theorem holds and \(x^*\) is a Pareto optimal allocation. Therefore, there exist some weights that make \(x^*\) the solution to the planner’s problem. It turns out that the required weights are given by

\[
\theta^i = \left( \frac{\partial u^i(x^i)}{\partial x^i_{10}} \right)^{-1}.
\]

This is left to the reader to check; it's part of the celebrated Negishi theorem.

This result is certainly very general, as it does not impose identical homothetic preferences, however, it is not as useful as the “real” Representative agent theorem to find equilibrium asset prices. The reason is that to define the specific weights for the planner’s objective function, \((\theta^i)_{i=1}^I\), we need to know what the equilibrium allocation, \(x^*\), which in turn depends on the whole distribution of endowments over the agents in the economy.
4.3 Asset Pricing

Relying on the aggregation theorem in the previous section, in this section we will abstract from the consumption allocation problems and concentrate on one-good economies. This allows us to simplify the equilibrium definition as follows.

4.3.1 Some classic representation of asset pricing

Often in finance, especially in empirical finance, we study asset pricing representation which express asset returns in terms of risk factors. Factors are to be interpreted as those component of the risks that agents do require a higher return to hold.

How do we go from our basic asset pricing equation

\[ q = E(mA) \]

to factors?

Single factor beta representation

Consider the basic asset pricing equation for asset \( j \),

\[ q_j = E(ma_j) \]

Let the return on asset \( j \), \( R_j \), be defined as \( R_j = \frac{A_j}{q_j} \). Then the asset pricing equation becomes

\[ 1 = E(mR_j) \]

This equation applied to the risk free rate, \( R^f \), becomes \( R^f = \frac{1}{Em} \). Using the fact that for two random variables \( x \) and \( y \), \( E(xy) = ExEy + cov(x, y) \), we can rewrite the asset pricing equation as:

\[ ER_j = \frac{1}{Em} - \frac{cov(m, R_j)}{Em} = R^f - \frac{cov(m, R_j)}{Em} \]

or, expressed in terms of excess return:

\[ ER_j - R^f = -\frac{cov(m, R_j)}{Em} \]
Finally, letting
\[ \beta_j = -\frac{\text{cov}(m, R_j)}{\text{var}(m)} \]
and
\[ \lambda_\pi = \frac{\text{var}(m)}{Em} \]
we have the beta representation of asset prices:
\[ ER^j = R^f + \beta_j \lambda_m \]  
(4.12)

We interpret \( \beta_j \) as the "quantity" of risk in asset \( j \) and \( \lambda_m \) (which is the same for all assets \( j \)) as the "price" of risk. Then the expected return of an asset \( j \) is equal to the risk free rate plus the correction for risk, \( \beta_j \lambda_m \). Furthermore, we can read (4.12) as a single factor representation for asset prices, where the factor is \( m \), that is, if the representative agent theorem holds, her intertemporal marginal rate of substitution.

**Multi-factor beta representations**

A multi-factor beta representation for asset returns has the following form:
\[ ER^j = R^f + \sum_{f=1}^{F} \beta_{jf} \lambda_{mf} \]  
(4.13)

where \( (m_f)_{f=1}^{F} \) are orthogonal random variables which take the interpretation of risk factors and
\[ \beta_{jf} = -\frac{\text{cov}(m_f, R_j)}{\text{var}(m_f)} \]
is the beta of factor \( f \), the loading of the return on the factor \( f \).

**Proposition 51** A single factor beta representation
\[ ER_j = R^f + \beta_j \lambda_m \]
is equivalent to a multi-factor beta representation
\[ ER_j = R^f + \sum_{j=1}^{F} \beta_{jf} \lambda_{mf} \quad \text{with} \quad m = \sum_{f=1}^{F} b_fm_f \]
4.3 ASSET PRICING

In other words, a multi-factor beta representation for asset returns is consistent with our basic asset pricing equation when associated to a linear statistical model for the stochastic discount factor $m_f$, in the form of $m = \sum_{f=1}^{F} b_f m_f$.

**Proof.** Write $1 = E(m R_j)$ as $R_j = R_f - \frac{\text{cov}(m, R_j)}{Em}$ and then to substitute $m = \sum_{f=1}^{F} b_f m_f$ and the definitions of $\beta_{j_f}$, to have

$$\lambda_{m_f} = \frac{\text{var}(m_f)b_f}{Em_f}$$

\[\blacksquare\]

**The CAPM**

The CAPM is nothing else than a single factor beta representation of the following form:

$$ER^i = R_f + \beta_{j_f} \lambda_{m_f}$$

where

$$m_f = a + bR^w$$

the return on the market portfolio, the aggregate portfolio held by the investors in the economy.

It can be easily derived from an equilibrium model under special assumptions.

For example, assume preferences are quadratic:

$$u(x^i_0, x^i_1) = -\frac{1}{2}(x^i - x^#)^2 - \frac{1}{2} \beta \sum_{s=1}^{S} \text{prob}_s(x^i_s - x^#)^2$$

Moreover, assume agents have no endowments at time $t = 1$. Let $\sum_{i=1}^{I} x^i_s = x_s$, $s = 0, 1, ..., S$; and $\sum_{i=1}^{I} w^i_0 = w_0$. Then budget constraints include

$$x_s = R^w_s (w_0 - x_0)$$

Then,

$$m_s = \beta \frac{x^s - x^#}{x_0 - x^#} = \frac{\beta (w_0 - x_0)}{(x_0 - x^#) R^w_s} - \frac{\beta x^#}{x_0 - x^#}$$

which is the CAPM for $a = -\frac{\beta x^#}{x_0 - x^#}$ and $b = \frac{\beta (w_0 - x_0)}{(x_0 - x^#)}$. 
Note however that \( a = \frac{\beta x^\#}{x_0 - x^\#} \) and \( b = \frac{\beta(x_0 - x^\#)}{(x_0 - x^\#)} \) are not constant, as they do depend on equilibrium allocations. This will be important when we study conditional asset market representations, as it implies that the CAPM is intrinsically a conditional model of asset prices.

### Bounds on stochastic discount factors

Write the beta representation of asset returns as:

\[
ER_j - R^f = \frac{cov(m, R_j)}{Em} = \frac{\rho(m, R_j)\sigma(m)\sigma(R_j)}{Em}
\]

where \( 0 \leq \rho(m, R_j) \leq 1 \) denotes the correlation coefficient and \( \sigma(.) \), the standard deviation. Then

\[
\left| \frac{ER_j - R^f}{\sigma(R_j)} \right| \leq \frac{\sigma(m)}{Em}
\]

The left-hand-side is the *Sharpe-ratio* of asset \( j \).

The relationship implies a lower bound on the standard deviation of any stochastic discount factor \( m \) which prices asset \( j \). Hansen-Jagannathan are responsible for having derived bounds like these and shown that, when the stochastic discount factor is assumed to be the intertemporal marginal rate of substitution of the representative agent (with CES preferences), the data does not display enough variation in \( m \) to satisfy the relationship.

A related bound is derived by noticing that no-arbitrage implies the existence of a *unique* stochastic discount factor in the space of asset payoffs, denoted \( m_p \), with the property that any other stochastic discount factor \( m \) satisfies:

\[
m = m_p + \epsilon
\]

where \( \epsilon \) is orthogonal to \( m_p \).

The following corollary of the No-arbitrage theorem leads us to this result.

**Corollary 52** Let \((A, q)\) satisfy No-arbitrage. Then, there exists a unique \( \tau^* \in <A> \) such that \( q = A\tau^* \).

**Proof.** By the No-arbitrage theorem, there exists \( \pi \in R^S_{++} \) such that \( q = \pi A \). We need to distinguish notationally a matrix \( M \) from its transpose, \( M^T \). We write then the asset prices equation as \( q^T = A^T \pi^T \). Consider \( \pi_p \):

\[
\pi_p^T = A(A^T A)^{-1} q.
\]
Clearly, $q^T = A^T \pi_p^T$, that is, $\pi_p^T$ satisfies the asset pricing equation. Furthermore, such $\pi_p^T$ belongs to $< A >$, since $\pi_p^T = Az_p$ for $z_p = (A^T A)^{-1} q$. Prove uniqueness.

We can now exploit this uniqueness result to yield a characterization of the “multiplicity” of stochastic discount factors when markets are incomplete, and consequently a bound on $\sigma(m)$. In particular, we show that, for a given $(q, A)$ pair a vector $m$ is a stochastic discount factor if and only if it can be decomposed as a projection on $< A >$ and a vector-specific component orthogonal to $< A >$. Moreover, the previous corollary states that such a projection is unique.

Let $m \in R^{S+}_+$ be any stochastic discount factor, that is, for any $s = 1, \ldots, S$, $m_s = \frac{q_s}{\text{prob}}$ and $q_j = E(mA_j)$, for $j = 1, \ldots, J$. Consider the orthogonal projection of $m$ onto $< A >$, and denote it by $m_p$. We can then write any stochastic discount factors $m$ as $m = m_p + \varepsilon$, where $\varepsilon$ is orthogonal to any vector in $< A >$, in particular to any $A_j$. Observe in fact that $m_p + \varepsilon$ is also a stochastic discount factors since $q_j = E((m_p + \varepsilon)a_j) = E(m_pa_j) + E(\varepsilon a_j) = E(m_pa_j)$, by definition of $\varepsilon$. Now, observe that $q_j = E(m_p a_j)$ and that we just proved the uniqueness of the stochastic discount factors lying in $< A >$.

In words, even though there is a multiplicity of stochastic discount factors, they all share the same projection on $< A >$. Moreover, if we make the economic interpretation that the components of the stochastic discount factors vector are marginal rates of substitution of agents in the economy, we can interpret $m_p$ to be the economy’s aggregate risk and each agents $\varepsilon$ to be the individual’s unhedgeable risk.

It is clear then that

$$\sigma(m) \geq \sigma(m_p)$$

the bound on $\sigma(m)$ we set out to find.
Chapter 5

Asymmetric information

Do competitive insurance markets function orderly in the presence of moral hazard and adverse selection? What are the properties of allocations attainable as competitive equilibria of such economies? And in particular, are competitive equilibria incentive efficient?

The fundamental contribution on competitive markets for insurance contracts is Prescott and Townsend (1984). They analyze Walrasian equilibria of economies with moral hazard and with adverse selection when exclusive contracts are enforceable. While for moral hazard economies they prove existence and constrained versions of the first and second theorems of welfare economics, their method does not succeed in the case of adverse selection economies. They conclude (p. 44) that “there do seem to be fundamental problems for the operation of competitive markets for economies or situations which suffer from adverse selection.1”

More generally, the analysis of competitive equilibria of economies with

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1The standard strategic analysis of competition in insurance economies, due to Rothschild-Stiglitz (1976), considers the Nash equilibria of a game in which insurance companies simultaneously choose the contracts they issue, and the competitive aspect of the market is captured by allowing the free entry of insurance companies. Such equilibrium concept does not perform too well: equilibria in pure strategies do not exist for robust examples (Rothschild-Stiglitz (1976)), while equilibria in mixed strategies exist (Dasgupta-Maskin (1986)) but, in this set-up, are of difficult interpretation. Even when equilibria in pure strategies do exist, it is not clear that the way the game is modelled is appropriate for such markets, since it does not allow for dynamic reactions to new contract offers (Wilson (1977) and Riley (1979); see also Maskin-Tirole (1992)). Moreover, once sequences of moves are allowed, equilibria are not robust to ‘minor’ perturbations of the extensive form of the game (Hellwig (1987)).
asymmetric information has recently received renewed attention. For such economies the interaction between the private information dimension (e.g., the unobservable action in the moral hazard case, the unobservable type in the adverse selection case) and the observability of agents’ trades plays a crucial role, since trades have typically informational content over the agents’ private information. In particular, to decentralize incentive efficient Pareto optimal allocations the availability of fully exclusive contracts, i.e., of contracts whose terms (price and payoff) depend on the transactions in all other markets of the agent trading the contract, is generally required. The implementation of these contracts imposes typically the very strong informational requirement that all trades of an agent need to be observed. Full observability of trades is in fact the economic environment which Prescott and Townsend study. It is then of interest to analyze also situations where contracts traded are necessarily non-exclusive, because perfect monitoring of trades is not available. The case of complete anonymity of trades, where no transaction of the agents is observable, constitutes an important benchmark in this respect.

In summary, an important dichotomy arises in the study of economies with asymmetric information: economies in which each agent’s trades are observable behave very differently from economies in which trades are not observable (often these economies are referred to, respectively, as economies with exclusive and non-exclusive contractual relationships).

Many different approaches have been taken in the literature to analyze equilibria of economies with asymmetric information. Even restricting to Walrasian equilibria, many different definitions are available (a situation that Douglas Gale has referred to as “Balkanization”). In these notes we attempt an analysis of such concepts. To facilitate comparisons we apply all such concepts to the same simple moral hazard and adverse selection economy.

We will show that the equilibrium concepts developed for moral hazard economies have an analogous applications to adverse selection economies, and viceversa. We will also show that, in economies with observable trades, for which different concepts have been developed, all such concepts generate coincident predictions in terms of equilibrium allocations and prices. In other words, different equilibrium concepts give rise to different equilibrium predictions only when they capture different assumptions about the observability of trades. All this for both moral hazard and adverse selection economies, in our simple example economies.

Our classification of equilibrium concepts follows. In the framework of a Walrasian competitive equilibrium model, alternative assumptions on the ob-
servability of agents’ trades may be captured in a reduced form by alternative assumptions on the possible non-linearities of equilibrium prices.

Complete anonymity of trades (full non-exclusivity) corresponds to restricting price schedules to be a linear function of trades. The intermediate case in which only short and long trading positions can be distinguished, will turn out to be central in our analysis: a minimal form of non-linearity, e.g., the possibility of having a different price for buyers and sellers (a bid-ask spread), is in fact necessary and sufficient for competitive equilibria to exist; see Dubey and Geanakoplos (2004), Bisin and Gottardi (2000), Bisin, Geanakoplos, Gottardi, Minelli and Polemarchakis (2001).

At the other extreme, complete observability of trades (exclusivity) is captured by allowing price schedules to be general non-linear function of agents’ trades. We distinguish two main approaches to the analysis of such economies:

Prices are arbitrarily non-linear maps; as a consequence minimal restrictions are imposed by the equilibrium notion, and hence the plethora of resulting equilibria is refined by a formal concept in the spirit of sequential equilibria; see Gale (1993), Dubey and Geanakoplos (2004).

The specification of agents’ budget sets restricts admissible trades to lie in the set of incentive compatible trades; in other words, non-incentive compatible trades are just non available for trade, or, say, are traded at infinite price; Prescott and Townsend (1984); see also Bisin and Gottardi (2004) for the adverse selection case.

The literature on the strategic analysis of economies of asymmetric information presents us with many equilibrium concepts and strategic game forms, and few robust predictions about equilibrium allocations. We survey in this note instead Walrasian equilibrium concepts, with the objective of identifying robust predictions in terms of equilibrium allocations.

5.1 The Economy: Moral Hazard and Adverse Selection

We study two simple economies, workhorses of economics of uncertainty. The first economy is characterized by moral hazard in the form of hidden action;
CHAPTER 5 ASYMMETRIC INFORMATION

see Grossman and Hart (1983). The second economy is characterized by adverse selection in the form of unobservable risk types; see Rothschild and Stiglitz (1977), Wilson (19??). We will introduce such economies in a as much unified way as possible.

There is measure 1 of agents who live two periods, \( t = 0, 1 \), and consume, only in period 1, a single consumption good. Uncertainty is purely idiosyncratic, and is described by the collection of random variables \( \tilde{s}^\tau \), where \( \tau \) indexes the names of the agent and lies in a countable space. It is assumed that \( \tilde{s}^\tau \) are identically and independently distributed, with support \( S = \{ H, L \} \); the realization of all \( \tilde{s}^\tau \) variables is commonly observable.\(^2\) Uncertainty enters the economy via the agents’ endowments. The (date 1) endowment of an agent \( \tilde{w}^\tau = w(\tilde{s}^\tau) \); let \( w^H \equiv w(H) \), \( w^L \equiv w(L) \) be the agent’s endowment in, respectively, the idiosyncratic state \( H \) and state \( L \).

The probability distribution of the period 1 endowment that each agent faces depends from the value taken by a variable \( e \in \{ h, l \} \). The interpretation of \( e \) is what distinguishes moral hazard from adverse selection economies.

In moral hazard economies, \( e \) is an unobservable level of effort which is chosen by the agent. In adverse selection economies \( e \) describes the exogenously given risk type of an agent, and its realization is only privately observable. Let \( \xi^h \) (resp. \( \xi^l = 1 - \xi^l \) ) be the probability that an agent is of type \( e = h \) (resp. \( e = l \) ); by the Law of Large Numbers, \( \xi^h \) is then also the fraction of agents in the population which are of type \( h \). Importantly, in adverse selection economies, all markets open after agents observe the probability distribution of their endowments.

Let \( \pi^e_s \) be the probability of the realization \( s \) given \( e \in \{ h, l \} \) (obviously \( \pi^e_H = 1 - \pi^e_L \), for any \( e \)). By the Law of Large Numbers, \( \pi^e_s \) is also the fraction of agents with \( e \) for which state \( s \) is realized. Agents’ preferences are represented by a (Von Neumann - Morgenstern) utility function of the following form:

\[
\pi^e_su(c^H) + (1 - \pi^e_s)u(c^L) - v(e)
\]

where \( (c^H, c^L) \) denotes consumption respectively in state \( H \) and \( L \); let \( c \equiv (c^H, c^L) \), \( w \equiv (w^H, w^L) \).

In moral hazard economies, \( v(e) \) denotes the disutility of effort \( e \), and we

\(^2\) Measurability issues arise in probability spaces with a continuum of independent random variables. We adopt the usual abuse of the Law of Large Numbers.
5.2 THE SYMMETRIC INFORMATION BENCHMARK

assume that
\[ \pi_H^h > \pi_H^l, \quad v(h) > v(l), \quad w^H > w^L > 0 \]

so that h is the ‘high’ effort and H is the ‘good’ state.

In adverse selection economies, v(e) is just a utility constant and hence is disregarded from the analysis without loss of generality.

Assumption 1 Preferences are strictly monotonic, strictly concave, twice continuously differentiable, and \( \lim_{x \to 0} u'(x) = \infty \).

Let \( \Omega \) be the set of parameter values \((v(h), v(l), \pi_H^h, \pi_H^l, w^H, w^L)\) of the economy which satisfy the above assumptions.

5.2 The Symmetric Information Benchmark

We consider now the benchmark case of symmetric information, in which e, be it endogenous effort or risk type, is commonly observed. Even though, tautologically, no issues of moral hazard nor adverse selection arise with symmetric information, we nonetheless refer to the economy in which e is chosen by each agent (resp. e is exogenous) as a ‘moral hazard’ economy (resp. an ‘adverse selection’ economy).

Definition 53 An allocation \((c, e) \in \mathcal{P}_+ \times \{h, l\}\) of consumption and effort is optimal in the moral hazard economy under symmetric information if it solves:

\[
\max_{c, e} \sum_s \pi_s^e u(c_s) - v(e) \tag{5.1}
\]

s.t.

\[
\sum_s \pi_s^e (c_s - w_s) = 0
\]

Definition 54 An allocation \((c^h, c^l) \in \mathcal{P}_+^l\) of consumption is optimal in the adverse selection economy under symmetric information if it solves:

\[
\max_{c^h, c^l} \sum_e \kappa^e \sum_s \pi_s^e u(c_s^e) \tag{5.2}
\]
s.t. 
\[ \sum_e \xi^e \sum_s \pi^e_s (c^e_s - w_s) = 0 \]

for some \((\kappa^h, \kappa^l) \gg 0\) such that \(\kappa^h = 1 - \kappa^l\).

Let \(q^e_s\) denote the (linear) price of consumption in state \(s\) for agents \(e\). By allowing the prices of the securities whose payoff is contingent on the idiosyncratic uncertainty to depend on \(e\), we effectively allow agents to trade in a complete set of markets.

In addition to consumers we introduce firms. Firms ‘pool’ payments in different states of the world. The Law of Large Numbers provides, in the economy under consideration, a mechanism - or a technology - for transforming aggregates of the commodity contingent on different individual states. Thus firms are characterized by the following constant returns to scale technology:

\[ Y = \{y \in \mathbb{R}^4 : \sum_e \sum_s \pi^e_s y^e_s \leq 0\} \]

where \(y = [y^e_s]_{e \in S}\).

The firms’ problem is then the choice of a vector \(y\) of the commodity contingent on the agents’ individual states, lying in the set \(Y\) (i.e., a collection of trades, or contracts to offer; contracts of the same type are then pooled and transformed according to the Law of Large Numbers) so as to maximize profits:

\[ \max_{y \in Y} \sum_e \sum_s q^e_s y^e_s \]  \hspace{1cm} \text{(Pf)}

taking prices \(q\) as given.

**Definition 55** A Walrasian equilibrium with symmetric information in the moral hazard economy is given by prices \(q^e \in \Delta^2\), for all \(e\), a consumption allocation and effort choice \((c, e) \in \mathbb{R}^2_+ \times \{h, l\}\), a production vector \(y \in \mathbb{R}^4\), such that:

(i) \((c, e)\) solves the agent’s optimization problem

\[ \max_{c, e} \sum_s \pi^e_s u(c_s) - v(e) \]  \hspace{1cm} \text{(5.3)}

s.t.

\[ \sum_s q^e_s (c_s - w_s) = 0 \]
(ii) $y$ solves the firms’ profit maximization problem ($P^f$), at prices $q$;
(iii) markets clear:

$$ (c_s - w_s) \leq y^c_s, \ \forall s, e $$

(5.4)

**Definition 56** A Walrasian equilibrium with symmetric information in the adverse selection economy is given by prices $q^e \in \Delta^2$ and a consumption allocation $c^e \in \mathbb{R}^2_+$, for all $e$, such that:

(i) $c^e$ solves the optimization problem of agents of type $e$:

$$ \max_{c^e} \sum_s \pi^e_s u(c^e_s) $$

(5.5)

s.t.

$$ \sum_s q^e_s (c^e_s - w_s) = 0 $$

(ii) $y$ solves the firms’ profit maximization problem ($P^f$), at prices $q$;
(iii) markets clear:

$$ (c^e_s - w_s) \leq y^c_s, \ \forall s, e $$

(5.6)

The First and Second Welfare theorems hold straightforwardly for both the moral hazard and the adverse selection economies under symmetric information.

Any Walrasian equilibrium with symmetric information in the moral hazard economy is optimal.

Any optimal allocation of a moral hazard economy with symmetric information can be decentralized as a Walrasian equilibrium with symmetric information with transfers.

Any Walrasian equilibrium with symmetric information in the adverse selection economy is optimal.

Any optimal allocation of an adverse selection economy with symmetric information can be decentralized as a Walrasian equilibrium with symmetric information with transfers.
5.3 Incentive Constrained Pareto Optimality

**Definition 57** An allocation \((c, e) \in \mathbb{R}^2_+ \times \{h, l\}\) of consumption and effort is incentive constrained optimal in the moral hazard economy if it solves:

\[
\max_{c, e} \sum_s \pi_s^e u(c_s) - v(e) \tag{5.7}
\]

s.t.

\[
\sum_s \pi_s^e (c_s - w_s) = 0
\]

\[
\sum_s \pi_s^e u(c_s) - v(e) \geq \sum_s \pi_s^{e'} u(c_s) - v(e'), \ \forall e, e'
\]

**Definition 58** An allocation \((c^h, c^l) \in \mathbb{R}^4_+\) of consumption is incentive constrained optimal in the adverse selection economy if it solves:

\[
\max_{c^h, c^l} \sum_{e} \kappa^e \sum_s \pi_s^e u(c_s^e) \tag{5.8}
\]

s.t.

\[
\sum_e \xi^e \sum_s \pi_s^e (c_s - w_s) = 0
\]

\[
\sum_s \pi_s^h u(c_s^h) \geq \sum_s \pi_s^h u(c_s^l)
\]

\[
\sum_s \pi_s^l u(c_s^l) \geq \sum_s \pi_s^h u(c_s^h)
\]

for some \((\kappa^h, \kappa^l) \in \mathbb{R}^2_+\) such that \(\kappa^h = 1 - \kappa^l\).

5.4 Walrasian Equilibria: Fully Observable Trades

5.5 Walrasian Equilibria: Non-Observable Trades

5.6 References

5.6 REFERENCES


Chapter 6

Infinite Horizon Economies

6.1 Bubbles

6.2