1. a. A function \( f \) of \( n \) variables \( x_1, \ldots, x_n \) is called a homogeneous function of degree \( s \) if

\[
 f(\lambda x_1, \ldots, \lambda x_n) = \lambda^s f(x_1, \ldots, x_n)
\]

where \( \lambda \) is an arbitrary constant. Prove that such a function satisfies

\[
 \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = s f(x_1, \ldots, x_n)
\]

b. Determine if the potentials \( V_{ee}, V_{eN} \) and \( V_{NN} \) are homogeneous functions and if they are, determine their degree \( s \).

2. Let \( |\psi\rangle \) be an eigenvector of a Hamiltonian \( H \) and let \( A \) be any Hermitian operator. Prove that

\[
 \langle \psi | [H, A] | \psi \rangle = 0
\]

3. Consider a molecule composed of \( N \) nuclei with positions \( R_1, \ldots, R_N \) and \( M \) electrons with position operators \( r_1, \ldots, r_M \). Let \( E_n(R) \) be any Born-Oppenheimer surface with \( R = R_1, \ldots, R_N \) and \( E_n(R) = \varepsilon_n(R) + V_{NN}(R) \). Here \( \varepsilon_n(R) \) is an eigenvalue of the electronic Hamiltonian \( H_e \) defined in class and \( V_{NN}(R) \) is the nuclear-nuclear Coulomb repulsion potential. Let \( |\psi_n(R)\rangle \) be the corresponding eigenvector. If \( T_e \) is the electronic kinetic energy and \( V = V_{ee} + V_{eN} + V_{NN} \), using the results of problems 1 and 2, prove that

\[
 \langle \psi_n(R) | T_e | \psi_n(R) \rangle = -E_n(R) - \sum_{I=1}^{N} R_I \cdot \frac{\partial E_n}{\partial R_I}
\]

and

\[
 \langle \psi_n(R) | V | \psi_n(R) \rangle = 2E_n(R) + \sum_{I=1}^{N} R_I \cdot \frac{\partial E_n}{\partial R_I}
\]

In particular, consider using the result of problem 2 with \( A \) chosen to be

\[
 A = \sum_{i=1}^{M} r_i \cdot p_i
\]

where \( p_1, \ldots, p_M \) are the momentum operators for the electrons.

4. Consider now a diatomic molecule with an equilibrium bond length \( R_0 \). Using the result of problem 3, show that the process of bringing two nuclei together from infinite separation to a separation of \( R_0 \) to form a stable diatomic results in a decrease in potential energy but an increase in the electron kinetic energy.

5. Consider an atom or molecule that has 2 electrons. Let \( \psi(x_1, x_2) \) be the wave electronic wave function. Suppose the spatial and spin components of \( \psi \) separate such that

\[
 \psi(x_1, x_2) = f(r_1, r_2) \chi(m_1, m_2)
\]

where \( m_1 \) and \( m_2 \) are the eigenvalues of \( S_z \) for each electron.
a. Suppose the electrons are in a singlet spin state with the total eigenvalue of $S_z$ equal to 0. What must be the relationship between $f(r_2, r_1)$ and $f(r_1, r_2)$? Is it possible for $f(r_1, r_2)$ to be a function of $|r_1 - r_2|$ alone?

b. Suppose the electrons are in a triplet spin state with the total eigenvalue of $S_z$ equal to 0. What must be the relationship between $f(r_2, r_1)$ and $f(r_1, r_2)$? Is it possible for $f(r_1, r_2)$ to be a function of $|r_1 - r_2|$ alone?