Problem set #3
Due: March 1, 2005

1. Consider three spin-1/2 particles with spins $S_1$, $S_2$ and $S_3$. Define $S = S_1 + S_2 + S_3$ as the total spin. Find a set of eigenvectors for $S^2$ and $S_z$ in terms of the direct products of the spin eigenstates, $|1/2; m_i\rangle$, $i = 1, 2, 3$, of the individual particles.

2. Consider a system composed of two spin-1/2 particles fixed in space described by a Hamiltonian of the form

$$H = \omega_1 S_{1z} + \omega_2 S_{2z}$$

where $S_{1z}$ and $S_{2z}$ are the $z$-components of the spin operators $S_1$ and $S_2$ for the two particles, respectively.

   a. Suppose the initial state vector is a triplet state

   $$|\Psi(0)\rangle = \frac{1}{\sqrt{2}} (|1/2; -1/2\rangle + |1/2; 1/2\rangle)$$

   At time, $t$, $S^2 = (S_1 + S_2)^2$ is measured. What values can be found and with what probabilities?

   b. Suppose the initial state vector is the singlet state

   $$|\Psi(0)\rangle = \frac{1}{\sqrt{2}} (|1/2; -1/2\rangle - |1/2; 1/2\rangle)$$

   At time, $t$, $S^2 = (S_1 + S_2)^2$ is measured. What values can be found and with what probabilities?

   c. Can this Hamiltonian ever evolve a triplet state into a singlet state at any time $t$? Can you write down a Hamiltonian that can effect such an evolution?

3. Consider a single quantum mechanical particle in one dimension with position and momentum operators $X$ and $P$, respectively. The Hamiltonian of the system is

$$H = \frac{P^2}{2m} + V(X)$$

Recall that the time evolution of the system can be studied equally well using a stationary state picture (the Heisenberg picture) in which the operators $P$ and $X$ evolve in time according to Heisenberg’s equations of motion:

$$\frac{dX}{dt} = \frac{i}{\hbar} [X, H]$$
$$\frac{dP}{dt} = \frac{i}{\hbar} [P, H]$$

The general solution to Heisenberg’s equations will be

$$X(t; X, P) = e^{iHt/\hbar} X e^{-iHt/\hbar}$$
$$P(t; X, P) = e^{iHt/\hbar} P e^{-iHt/\hbar}$$

where $X$ and $P$ are the usual Schrödinger operators for position and momentum, respectively.
Consider a set of Hermitian operator-valued functions \( \{T_k(X, P)\} \) in terms of the Schrödinger operators, \( X \) and \( P \) that satisfy a Lie algebra:

\[
[T_k(X, P), T_l(X, P)] = \sum_m C_{klm} T_m(X, P)
\]

where \( C_{klm} \) are the structure constants of the group and are, therefore, assumed known. Such a set of functions forms an operator basis in terms of which any Hermitian operator-valued function \( F(X, P) \) can be expanded according to:

\[
F(X, P) = \sum_k f_k T_k(X, P)
\]

where \( f_k \) is a set of complex expansion coefficients.

a. Express the Hamiltonian and solutions to Heisenberg’s equations of motion as expansions with respect to the operator functions \( T_k(X, P) \). Which sets of expansion coefficients will be functions of time?

b. Starting from Heisenberg’s equations, derive equations of motion for the time-dependent coefficients in terms of the structure constants of the Lie group.

c. What are the initial conditions on your equations of motion?

d. Of the conditions:

\[
\begin{align*}
X^l(t; X, P) &= X(t; X, P) \\
P^l(t; X, P) &= P(t; X, P) \\
[X(t; X, P), P(t; X, P)] &= i\hbar I
\end{align*}
\]

which of these impose additional conditions on the equations of motion and what, if any, are these conditions?