Erratum

Valentina Corradi
Department of Economics
University of Exeter
Exeter, EX4 4PU
UK

Rajiv Sarin
Department of Economics
Texas A&M University
College Station, TX 77843
USA

September 16, 2001

There is an error in the fourth moment calculation in the proof of Proposition 2 of our paper, "Continuous approximations of stochastic evolutionary game dynamics," which appeared in *JET*, 94 (2000), 163-191. This has the consequence that Proposition 2 is incorrect. This was pointed out to us by William Sandholm.

The discrete model is defined in eq. (1) of the paper. It supposes that in any time period $n = 1, 2, \ldots$ N pairs are matched with replacement. If a type $j\,(k)$ pair is matched with a type $k\,(j)$ she switches her type, due to the mimicking process, to $k\,(j)$ with probability $q_{jk}\,(q_{kj})$. If an individual is matched with her own type she does not switch. In the continuous limit considered in Proposition 2, the probability with which all type j individuals in jk matches who can switch type is reduced as the length of the time period θ_N is shrunk. However, when switching due to mimicking occurs, it is of the same magnitude as in the discrete model. This has the consequence that there are rare but discrete jumps as the time interval shrinks and so the limit process does not have continuous paths as is required for it to be a stochastic differential equation (SDE). Such jumps are rules out by the condition on the fourth moment.

In this erratum we suggest another manner of taking the continuous limit which results in an approximating continuous process which is an SDE.[1] In particular, we suppose that in any period of length θ_N only a fraction of order θ_N of type j individuals in jk matchings can switch. Whereas in the paper the probability of switching was shrinking but the proportion of individuals who switched was fixed, we now suppose that the proportion of individuals who can switch is shrinking but keep the probability of switching fixed. In contrast to the limit we considered earlier, this limit reduces the size of the jump as we take the continuous limit. It also leads to the fourth moment conditions stated in Lemma 2 of the paper being satisfied.

[1] While many continuous approximations could be considered, some of which would result in identical continuous limits to those obtained in the published paper, we propose the following because it is intuitively the "closest" to the one presented in our paper.

Before stating the new limit more formally in the next paragraph, we contrast the proposed limit with the limit by which we obtained a ODE in the paper. We obtained an ODE limit (in Section 2 of the paper) when only a fraction $\alpha_N = K\theta_N$ of the individuals were matched. This had the consequence that when $N \rightarrow \infty$ both the matching and the mimicking noise vanish. In contrast, the limit we now propose, there are N matches in every period, however, of all the types matched with distinct others, only a fraction $a_N = K\theta_N$ is allowed to switch type. Therefore the mimicking noise vanishes, but the matching noise does not vanish for $\theta_N = N^{-1}$.

More formally, suppose that at time θ_N, $2\theta_N$, ... we have N matchings with replacement. A fraction $\alpha_N = K\theta_N$ of type j individuals in jk pairings switch their type to k, where $\theta_N = N^{-\delta}$, $\delta = 1$.[2] Using the same notation as in our paper, if $\left\|P^N(0) - \tilde{P}(0)\right\| \rightarrow 0$ as $N \rightarrow \infty$, then $P^N \rightarrow \tilde{P}$ in $D_S[0,\infty)$ where S is as defined in the paper in Proposition 2 and $\tilde{P}$ is a diffusion whose generator is

$$A(p) = \frac{1}{2}\sum_{j=1}^{J}\sum_{k=1}^{J} a_{jk}(p) + \sum_{j=1}^{J} b_j(p)$$

where,

$$b_j(p) = \frac{K}{2} p_j \sum_{k \neq j}^{J} p_k (q_{kj} - q_{jk})$$

and,

$$\begin{aligned} a_{jj}(p) &= p_j(1-p_j) \\ a_{jk}(p) &= -p_j p_k \quad \forall j \neq k \end{aligned}$$

In contrast to the result stated in Proposition 2, the drift term b_j is multiplied by K as the fraction switching is $K\theta_N$ whereas in the published paper we had a probability of switching equal to θ_N. The diffusion term differs because in the limit the second term on the RHS of (5) in the published paper is of order θ_N^2 and hence vanishes in the limit. The 4th moment in this case is of order θ_N^2 as required in Lemma 2. In particular, the fourth moment of the matching noise is of order θ_N^2 while the fourth moment of the mimicking noise is of order θ_N^4 as only a fraction $K\theta_N$ can switch. Discrete jumps in the sample paths are eliminated as the proportion of individuals allowed to switch decreases. The noise remaining in the limit process is purely due to the matching noise. To see this, note that the moments of the process are

$$E\left(\Delta P_j^N(n\theta_N)/p\right) = \frac{K\theta_N}{2} p_j \sum_{k \neq j}^{J} p_k (q_{kj} - q_{jk}) + O\left(\theta_N N^{-1}\right),$$

$$Var\left(\Delta P_j^N(n\theta_N)/p\right) = \frac{1}{N} p_j(1-p_j) + \frac{K\theta_N^2}{2} p_j^2 \sum_{k \neq j}^{J} p_k^2 \left(q_{kj}(1-q_{kj}) + q_{jk}(1-q_{jk})\right) + O\left(\theta_N N^{-1}\right),$$

[2] Note that $\alpha_N N$ need not always be an integer. See footnote 12 in the paper.

$$Cov\left(\Delta P_j^N\left(n\theta_N\right), \Delta P_k^N\left(n\theta_N\right)/p\right) = -\frac{1}{N}p_j p_k,$$

$$E\left(\Delta P_j^N\left(n\theta_N\right)^4/p\right) = O\left(\theta_N^2\right)$$

For $J = 2$ and denoting $\tilde{P} = \tilde{P}_1$, we get the following SDE:

$$d\tilde{P}(t) = \frac{K}{2}\tilde{P}(t)\left(1-\tilde{P}(t)\right)\left(q_{21}-q_{12}\right)dt + \tilde{P}(t)\left(1-\tilde{P}(t)\right)dW(t)$$

Note that, for reasons stated in the above paragraph, the switching noise due to the mimicking process vanishes in the limit and the noise that remains is due to the matching process. The approximating SDE is, hence, different from what was "derived" in the paper. This has a obvious implications on the asymptotic analyses of the SDE, which we discuss in the next paragraph. Another point worth mentioning is that the SDE limit arises only for $\delta = 1$. When $\delta \in (0, 1)$ we obtain an ODE. This ODE is the one obtained in Proposition 1 of the paper. The intuition why this arises is as stated in the paper: the slower the rate at which we take the continuous limit the less of the noise of the discrete dynamic that is preserved.

When $\delta \in (0, 1)$, the approximating equation is an ODE, and hence the asymptotics of Proposition 3 are those of the ODE which were provided in the paper. As regards Proposition 4, we can derive the following limits of the SDE above: In the case $q_{12} \neq q_{21}$ we get that $\tilde{P}(t) \to 1$ as $t \to \infty$ with probability $\frac{1-e^{-K(q_{21}-q_{12})\tilde{P}(0)}}{1-e^{-K(q_{21}-q_{12})}}$ and $\tilde{P}(t) \to 0$ as $t \to \infty$ with probability $1 - \frac{1-e^{-K(q_{21}-q_{12})\tilde{P}(0)}}{1-e^{-K(q_{21}-q_{12})}}$ and when $q_{12} = q_{21}$, $\tilde{P}(t) \to 1$ as $t \to \infty$ with probability $\tilde{P}(0)$. Hence, as mentioned in the paper, the asymptotics of the SDE resemble more closely those of the discrete model as compared with the asymptotics of the ODE.