Ethnic Diversity and Democratic Stability

Paper presented at Politics Department Seminar Series,
New York University
October 27 2003

.

Kanchan Chandra
Department of Political Science, MIT
kchandra@mit.edu

Cilanne Boulet
Department of Mathematics, MIT
cilanne@math.mit.edu

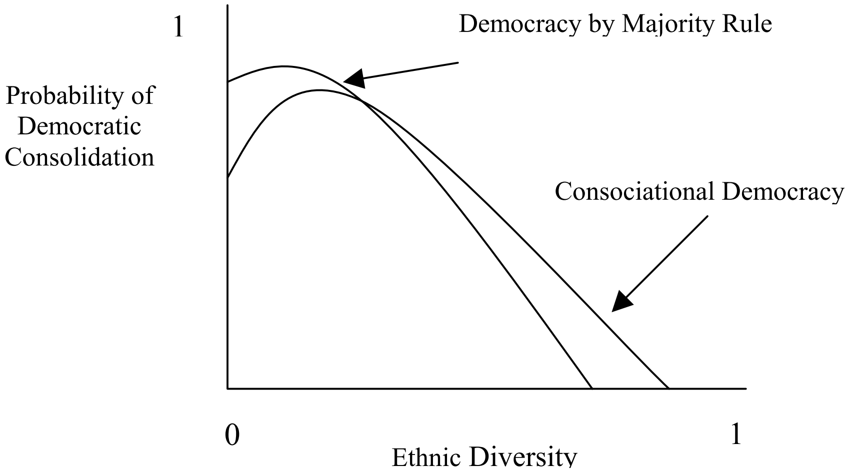
Comments Welcome

ETHNIC DIVERSITY AND DEMOCRATIC STABILITY
Kanchan Chandra and Cilanne Boulet

I Introduction

Ethnic diversity has a bad name in empirical democratic theory. Practically every major work on the determinants of democratic transition and consolidation describes ethnic diversity as a threat: The disagreement is only over the *degree* of threat relative to other variables and in different contexts (Mill [1961] 1991; Rustow 1970; Dahl 1971; Horowitz 1985; Lijphart 1977; Przeworski 1991). Figure 1, adapted from the work of Arend Lijphart, who is ironically among the most optimistic of political scientists on this subject, summarizes the presumed negative relationship between ethnic diversity and the probability of democratic consolidation. (Lijphart 1977, 237).

Figure 1



Perhaps the most widely cited of the mechanisms by which ethnic diversity is believed to threaten democracy is the prospect of permanent exclusion. Democracy, according to this argument, requires fluid political majorities and minorities in order to function. The politicization of ethnic divisions, the argument runs, is prone to produce a politics of fixed majorities and minorities based on ethnic demography (Horowitz 1985, p. 84). Consequently, ethnic divisions are more likely to be threatening for democracy than other types of divisions.

This paper accepts at face value the proposition that a stable democracy requires fluid majorities and minorities. But it challenges the claim that the politicization of ethnic divisions produces a politics of permanent majorities and minorities. This claim is based on the assumption that individuals have fixed ethnic identities that exist prior to, and remain unchanged by, political competition. Commonly associated with primordialist approaches to ethnic identity, this assumption has been discredited by three decades of research on the constructed origins of ethnic groups, which suggest that ethnic identities change in part through the political process. This paper develops a conceptual vocabulary in which to express constructivist insights with greater precision, and then uses this vocabulary to construct an alternative model of the relationship between ethnic diversity and democratic stability. Throughout, democracy is minimally defined to mean simply a political system in which the leadership is chosen through competitive elections. A stable democracy, then, is simply a democracy in which all parties acquiesce to the competitive system, while an unstable democracy is one in which the competitive system is under threat of subversion.

Our model is built on a distinction between an identity *category*, defined as a social group in which an individual is eligible for membership, and the *attribute(s)* that qualify an individual for membership in that category. The central insight of the model is that the existence of fluid majorities and minorities depends, not upon the ***nature*** of identity categories that are politically mobilized (i.e. whether they are ethnic or non-ethnic) but on the ***distribution*** of attributes that define membership in these categories in the population. Under majority rule, some distributions of attributes will produce only a single possible minimum winning identity category. Such distributions are likely to produce permanent results, repeatedly returning members of the same category to power. Regardless of whether they consist of ethnic or non-ethnic attributes, these distributions should threaten democratic stability by producing a class of permanent winners and permanent losers. Other distributions, on the other hand, can generate either none, or multiple possible minimum winning categories (although only one will be activated at any given time). Such distributions are likely to produce fluidity in election results over time. Regardless of whether they consist of ethnic or non-ethnic attributes, these distributions are likely to generate fluidity in election results and therefore are likely to safeguard democratic stability.

Section II the distinction between attribute and category, and redefines an identity category as a *combination of attributes*. Section III uses the tools of combinatorial mathematics to show how we can use information about the distribution of individual repertoires of attributes across a population to predict the number of minimum winning categories that should result from a given distribution. Section IV discusses the implications of these predictions for theories of democratic stability.

II. Vocabulary

We illustrate the vocabulary throughout this section using examples of ethnic identity categories, which are our main concern in this paper. However, the logic should in principle be applicable also to identity categories of other kinds.

i. Attribute and Category

The basic building block of our model is the distinction between an attribute and an identity category. By an identity *category*, we mean a social group in which an individual is eligible for membership. Later, we use the term *activated* identity category to refer to that identity category in which an individual actually declares membership.

By an *attribute,* we mean a characteristic that qualifies an individual for membership in that social group, or signals such membership. The correspondence between attributes and categories is not objective but socially constructed. The defining attributes associated with each category will vary depending on who is doing the defining. But an identity category is always associated with *some* defining attributes.

<u>Ethnic</u> identity categories are simply those identity categories in which membership is defined by attributes that are, on average, either acquired through descent, or believed to be acquired through descent.[1] For instance, the attributes associated with the category "West-Indian" in New York, as identified by Mary Waters' respondents, include one or more of the following: birth in one of the English-speaking islands of the Caribbean (Jamaica, Barbados, Trinidad, Grenada etc) or Guyana; dark skin; language; and accent. (Waters 1999). The category "African-American," similarly, is commonly associated with some of the following attributes: descent from African American parents; dark skin; physical features; hair type; etc.

The distinction between attribute and category is relational, not absolute.[2] For any given domain of analysis, we can distinguish between an object that is a category and one or more objects that are attributes in relation to that category. However, objects that are attributes in relation to a category in one domain of analysis may themselves be categories in relation to other attributes in other domains of analysis. Take for example "dark skin," which we describe above as a qualifying attribute for either the category "West-Indian" or "African-American." In another domain of analysis, the term "dark skin" can itself be thought of as a category, defined by a discrete interval on some continuum of shades of skin colour (dark brown to black). And these "attributes" can themselves be thought of as categories in other domains of analysis. Our use of this distinction therefore, is always contingent upon a domain of analysis.

ii. Type and Value of Attribute

For the purpose of describing the distribution of attributes across individuals, it is useful to make a distinction between a type of attribute and the range of values on a type.

---

[1] For a fuller discussion of this definition and its relation to previous definitions of ethnic identities, see Chandra-Metz (2002).

[2] Ian Lustick's written comments made this point clear to us.

By "type" of attribute, we mean a class of attributes with mutually exclusive values. Values on an attribute can in principle be discrete or continuous. But throughout this paper, we assume the existence of discrete values on a type. Types of attributes with continuous values, such as height or skin colour, are accommodated by expressing ranges of values as discrete intervals.

For example, one type of attribute is "skin colour." The values on this type might include "black" and "white." We depict the type of attribute and the range of values on each type as follows:

Skin colour: {black, white}

Another type of attribute might be country of origin. The values on this type might include "foreign" and "native," represented as follows:

Origin: {foreign, native}

Each individual in a population possesses exactly one value on each type of attribute.

A "type of attribute" is related to what constructivist scholarship refers to as a "dimension" or "cleavage." The term "dimensions" and cleavage are most commonly used to describe an array, or family, of mutually exclusive *categories* that include 100% of the population (Posner, forthcoming). The term "type of attribute" describes a mutually exclusive family of *attribute-values* that go into the definition of those categories, and includes 100% of the population. In other words, given some domain of analysis, the term "type of attribute" describes the *inputs* in the production of categories, while the term "dimension" is normally used to describe the *output*. In order to underline the distinction, we use the terms "attribute-type" or "attribute-dimension" when referring to the input, and the term "category dimension" when referring to the output.

iii. Repertoire of Salient Attributes for a Population

By a population, we mean the collection of individuals in a given country. In principle, individuals within and across populations possess a vast range of characteristics, any of which might potentially be the defining attributes for membership in some category: height, weight, length of nose, eye colour, hair type, length of hair, occupation, dress, last name, income, etc. We stipulate, however, that despite this very large set of possibilities, populations are distinguished by the possession of a small repertoire of salient attributes. The process by which these attributes come to be salient is exogenous to our model. Throughout the remainder of this paper, the term attribute refers to only to salient attributes.

The list below represents a simple example of a repertoire of salient attributes. We will use this as our running example throughout this paper. Each row describes a type of attribute, associated with a set of values. The two types of attributes listed here are skin colour and origin, each with two values.

Skin colour: {black, white}
Origin: {foreign, native}

In general, the repertoire of salient attributes for a population, with j types of attributes and n_i values for attribute type A_i, can be represented in the following way:

$$
\begin{aligned}
A_1&: \{a_{1,1}, a_{1,2}, \ldots, a_{1,n_1}\}\\
A_2&: \{a_{2,1}, a_{2,2}, \ldots, a_{2,n_2}\}\\
&\ldots\\
A_j&: \{a_{j,1}, a_{j,2}, \ldots, a_{j,n_j}\}
\end{aligned}
$$

iv. Repertoire of Attributes for An Individual

Above, we described the repertoire of attributes for a population. The repertoire of attributes for an individual in this population consists of one value on each type of attribute in the population repertoire. In other words, we assume that all individuals possess one value on each type of attribute: e.g. every person has some skin colour; every person has some hair type etc.

Consider our running example of a population with the repertoire of two types of attributes with two values on each.

Skin colour: {black, white}
Origin: {foreign, native}

This population repertoire can generate the following four individual repertoires: Black and Foreign (BF); Black and Native (BN); White and Foreign (WF); and White and Native (WN).

In general, the larger the population repertoire, the greater the number of individual repertoires it will generate. Where we have j types of attributes and n_i values for attribute type A_i, the total number of distinct repertoires is $n_1 n_2 \ldots n_j$**.** ***(See Proof 1 in the Appendix.).***

v. Distribution of individual repertoires of attributes across populations

So far, we have argued that populations differ from each other based on their repertoires of salient attributes. However, populations with an identical population repertoire can also differ from each other based on the distribution of individual repertoires within them.

One population generated from the example above, for instance, might have a very high proportion of individuals with BN repertoires and low proportions of individuals with BF, WN and WF repertoires; another might have more individuals with a WN repertoire than any other; and so on.

These distinct distributions can be represented in Table 1, in which the proportion of individuals with different repertoires is represented by the values a, b, c and d:

Table 1

	Black	White
Foreign	a	b
Native	c	d

Here, a is the proportion of individuals in this population with the repertoire BF; b is the proportion of individuals with the repertoire WF; c is the proportion of individuals with the repertoire BN; and d is the proportion of individuals with the repertoire WN. There can be as many distinct populations are there are values of a, b, c and d, subject to the restriction that $a+b+c+d = 1$.

If we have more than two values on each type, we can still place the individual repertoires in such a table. The number of rows and columns in the table will correspond to the number of values on each of the two types.

If we have more than two types of attributes, representing the distribution using a table is more difficult, since the table will have as many dimensions as there are types of attributes. An alternate way to think about distributions in this case is to list the distinct individual repertoires that correspond to the population's repertoire of attributes and the proportion of the population associated with each.

Consider for instance a population with the following repertoire of three types of attributes with two values on each.

Skin colour: {black, white}
Origin: {foreign, native}
Height: {tall, short}

This population repertoire will generate the following 8 individual repertoires: Black and Foreign and Tall (BFT); Black and Foreign and Short (BFS); White and Foreign and Tall (WFT); White and Foreign and Short (WFS); Black and Native and Tall (BNT); Black and Native and Short (BNS); White and Native and Tall (WNT); White and Native and Short (WNS). Table 2 represents the proportion of individuals with each repertoire in this population as follows:

Table 2

Repertoire	Proportion
BFT	a
BFS	b
WFT	c
WFS	d
BNT	e
BNS	f
WNT	g
WNS	h

There can be as many distinct populations in this case are there are values of a, b, c, d, e, f, g and h, subject to the restriction that $a + b + c + d + e + f + g + h = 1$. Similarly, we can represent the range of distributions of individual repertoires that correspond to a population repertoire with any number of types of attributes and any number of values on each type in a two dimensional table.

Note that there is only an indirect link between the distribution of individual repertoires of attributes and the repertoire of attributes for the population. The distribution of individual repertoires of attributes depends upon how many distinct repertoires there are, and the proportions of each in the population. The number of individual repertoires of attributes is derived from the repertoire of attributes for a population. However, the proportions are not. For this reason, we say that the repertoire of attributes for a population constrains but does not determine the distributions of individual repertoires of attributes.

vi. Identity categories are combinations of attributes

Identity categories can now be defined as combinations of attributes using the "and" or the "or" operator, or some combination of the two.

A few of the identity categories that can be constructed from our running example include {Black}; {Black and Foreign}; {Black or White}.

Identity categories so constructed can be of varying degrees of complexity. They might for instance correspond to only one value on an attribute as the category "Black" does in our example. They might combine several values on one type of attribute as the category {Black or White} does. They might combine one or more values across types, as the category {Black and Foreign} does. And, while the examples from the small repertoire above all use either the "and" or the "or" operator, identity categories from a larger repertoire of attributes can be constructed by using a combination of "and" and "or" operators across and within types.

For example, the category "West Indian" in Waters' study might be thought of as a combination of values on three types of attributes (skin colour, native language and place of birth), using both the "and" and the "or" operator: {Dark skin, and English-speaking and (Born in Guyana or Barbados or Jamaica *or* Trinidad ……..)}. The category "Russian-speaking population" in Laitin's study of Estonia might be written as a combination of values on two types (language and nationality), using both the "and" and the "or" operator: {Russian-speaker and (Russian or Ukrainian or Belarusian or Polish or Jewish or Finn} (Laitin 1998). The category "Nubian" in Idi Amin's Uganda might be thought of as a simpler combination of values on one type (tribe), using only the "or" operator: {Baganda or Basoga or Banyoro or Acholi or Langi or Batoro or Kakwa…} (Kasfir 1979). The category "Other Backward Caste" in India might be thought of a comparably simple combination of values on a single type (caste), also using only the "or" operator: {Yadav or Kurmi or Patel or Lodh or Nai or Saini…}.

In general, the greater the use of the "and" operator, the more restrictive the definition of a category. The greater the use of the "or" operator, the more permissive the definition of a category.

vii. Repertoire of Identity Categories for a Population

We conceptualize the repertoire of identity categories for a population as a set of combinations that can be generated from its repertoire of salient attributes. This repertoire describes the whole range of identity possibilities for individuals in that population.

The total number of different combinations, with no restrictions, that can be generated from our running example above is 16. These are: ∅; Black and Foreign; Black and Native; White and Foreign; White and Native; Black; White; Foreign; Native; Black or Foreign; White or Foreign; Black or Native; White or Native; (Black and

Native) or (White and Foreign); (Black and Foreign) or (White and Native); the entire population.

In general, the total number of different combinations, with no restrictions, that can be generated from a repertoire of j types of attributes and n_i values for attribute type A_i is $2^{n_1 n_2 \ldots n_j}$. Note that this includes both the empty set and the entire population as categories. ***(See Proof 2 in the Appendix)***

In modeling the "operative" repertoire of identity categories for populations, we may want to place further restrictions on which combinations are permissible. In the model that follows, we impose one simple restriction: we stipulate that individuals will consider only those combinations of attributes that use the "or" operator across values and the "and" operator across types instead of an arbitrary combination of the two. Our reasoning is that individuals have limited cognitive capacities and are not likely to process the entire range of possible combinations. This restriction produces a simpler range of identity possibilities. Instead of considering all possible combinations, individuals are allowed only to consider each type of attribute one at a time, deciding which values are allowed. They are not permitted to consider combinations in which the range of allowed values on one type of attribute is contingent upon allowed values on other types. We may also want to impose additional restrictions on permissible combinations to reflect institutional and historical legacies. We do not do this here but this is the subject of our current work on extensions to this model.

Given this restriction, our running example generates an identity repertoire of 10 categories, including the trivial examples of categories that include everyone and categories that include no one: ∅; Black and Foreign; Black and Native; White and Foreign; White and Native; Black; White; Foreign; Native; the entire population. In general, the total number of different combinations, with this restriction, that can be generated from a repertoire of j types of attributes and n_i values for attribute type A_i is $\prod_{i=1}^{j}(2^{n_i}-1)+1$. *(See Proof 4 in the Appendix.)*

The concept of an identity repertoire is perhaps the most important concept generated by constructivist scholarship but at the same time the most underdeveloped. Previous constructivist scholarship, including our own, typically uses the term "identity" in the term identity repertoire vaguely to mean attributes, categories or attribute-dimensions and category-dimensions. The only exception to this rule is Posner (forthcoming), who provides a precise definition of an identity repertoire as including identity <u>categories</u>. Further, it restricts the number of identity possibilities that individuals face to an arbitrarily low number, ranging between two and three categories (Sahlins 1989, Laitin 1986, Waters 1990, Waters 1999, Chandra 2003, Posner forthcoming). Finally, it does not allow for the possibility that elements in some existing identity repertoire might be recombined to produce still more identity possibilities. The result is that our theories of identity choice apply only to highly simplified worlds.

Our definition builds upon previous constructivist work by increasing its degree of precision and generality: First, it provides a precise conceptualization of the identity repertoire as consisting of *categories*. But it adds a new level of analysis by stipulating that these categories are generated by an underlying repertoire of *attributes*. Second, it allows for the possibility that an identity repertoire may consist of any number of categories, produced by any number of salient attribute dimensions (although our running example focuses on a simple 2*2 case for purposes of exposition). Third, by conceptualizing the repertoire of identity categories as composed of combinations and recombinations generated from an underlying repertoire of attributes, it captures more accurately the large range of identity possibilities that face individuals in the real world. (Note, however, that because the bounds and content of this repertoire of attributes are arbitrarily determined, the size of the repertoire of categories generated from it also remains arbitrarily determined in our model).

viii. Repertoire of Identity Categories for an Individual

An individual repertoire of identity categories consists of those categories for which she is eligible for membership based on her individual repertoire of attributes.

For example, an individual with the individual repertoire of attributes BF would be eligible for membership in four categories, namely: Black; Foreign; Black and Foreign; entire population. In fact, in our example, every individual will be eligible for membership in exactly four categories.

In general, the restriction that individuals will consider only those combinations of attributes that use the "or" operator across values and the "and" operator across types, will yield an individual repertoire of categories of size $\prod_{i=1}^{j}(2^{n_i-1})$. *(See Proof 5 in the Appendix).* Without any restrictions an individual repertoire of categories has size $2^{n_1 n_2 \dots n_j -1}$ *(See Proof 3 in the Appendix).*

The size of an individual's repertoire of categories depends upon the number of types and values of attributes in the repertoire of the population. However, as in the case of the repertoire of categories for a population, this conceptualization of individual repertoires of categories allows us to impose definite bounds on the identity options for any one individual. It indicates, further, that the identity options for any individual are typically larger than the repertoire of attributes that she begins with.

ix. Summary

In the next section, we use the concepts developed here to predict the number of minimum winning categories that can emerge across populations. But before we do that, it would be useful to summarize four key relationships derived from these concepts that will turn out to be especially relevant.

.

1. Repertoire of Attributes for a Population → Determines → Number and Composition of Individual Repertoires of Attributes

2. Repertoire of Attributes for a Population → Constrains but does not determine → Distribution of Individual Repertoires of Attributes

3. Repertoire of Attributes for a Population → Determines → Number and Composition of Categories in Repertoire of Categories for that Population

4. Distribution of Individual Repertoires of Attributes in the Population. → Determines → Size of the Categories in Repertoire of Categories for that Population

III Predicting the Number of Minimum Winning Categories for Different Distributions of Attributes

We assume that the electorate in our democracy is constituted by rational individuals playing an n-person, constant sum, winner-take-all game, in which they have to decide which identity category to join. The winner is defined as the majority ethnic category, where the majority exceeds some size threshold k ($.5<k<1$). The value of k is exogenously determined. The payoff is individually divisible among members of the majority ethnic category. Individuals have perfect information about the distribution of individual repertoires of salient attributes in the population.

Given a choice, all rational individuals will want to belong to a category that is of minimum winning size. We assume, for the purposes of completeness, that those who are not eligible for membership in any category of minimum winning size will join the largest category that they can of those remaining. However, this particular assumption is not necessary to the model. The results remain the same regardless of the assumption governs their identity choices.

The basic insight of the model can be summed up in a single proposition: when there is a single minimum winning category, the pattern of self-classification in the population will be stable, but the democratic system will be unstable. But when there are none or multiple minimum winning categories, the pattern of self-classification in the population will be unstable, while the democratic system will be stable. The logic underlying this proposition is as follows:

When only one minimum winning category exists, those eligible should declare membership in it. Having done so, they have no incentive to reclassify themselves, since such a reclassification will not bring rewards with it. Those who are not eligible for membership in the winning category will choose the largest of their remaining options.

However, they will be permanently locked out of power and so will have an incentive to subvert the system.

But, if the distribution of individual repertoires of ethnic attributes generates either none or more than one minimum winning category then while we should see instability in self-classification, but should see a stability in the democratic system. When a distribution of attributes does not generate any minimum winning category, we might expect individuals to experiment with different forms of self-classification and form temporary alliances with those from different categories in the search for a minimum winning coalition. In a world of such fluid self-identifications, the democratic system as a whole should be stable since no individual need fear permanent exclusion. When there is more than one possible minimum winning category, those who are losers when one identity category is activated have the option of reversing their situation if they can successfully induce potential members of an alternative winning category to declare this alternate membership. The possibility of reversing a disadvantageous situation within the existing rules of the game by should ensure that this democracy is stable.

The foundation for this model is Riker's theory of minimum winning coalitions, which predicts that in "in n-person, zero-sum games, where side-payments are permitted, where players are rational, and where they have perfect information, only minimum winning coalitions occur." (Riker, 32). This model is an application of Riker's theory of minimum winning coalitions to the study of identity change, conceptualizing identity categories as coalitions, with one modification: the restrictions on the possible coalitions that can form are not simply the payoff structure or the "weights" associated with individuals as is the case in Riker's formulation, but the distribution of individual repertoires of attributes in the population. For reasons that should now be clear, the remainder of this paper uses the terms "category," "coalition" and "combination" interchangeably. In conceptualizing ethnic categories as coalitions entered into by rational individuals in pursuit of individually divisible payoffs, we follow previous constructivist texts (Bates 1974; Posner forthcoming; Chandra forthcoming). The main innovation in this model, however, is its ability to incorporate a greater degree of complexity.

i. The Set of Possible Ethnic Categories For a Population

The set of possible ethnic categories for a population of n individuals corresponds to the set of possible combinations generated by its repertoire of salient ethnic attributes.

Table 3 lists the categories generated from the population repertoire that is our running example and the membership of each. It lists all 16 categories that would result if we did not impose any restrictions on the combinations that individuals are willing to consider. The shaded cells describe those categories that would be excluded if we impose the restriction that individuals consider only "and" combinations across types of attributes and only "or" combinations for values within types).

Table 3

Definition of Category	Allowed individual repertoires
∅	None
Black and Foreign	BF
Black and Native	BN
White and Foreign	WF
White and Native	WN
Black	BF, BN
White	WF, WN
Foreign	BF, WF
Native	BN, WN
Black or Foreign	BF, BN, WF
White or Foreign	BF, WF, WN
Black or Native	BF, BN, WN
White or Native	BN, WF, WN
(Black and Native) or (White and Foreign)	BN, WF
(Black and Foreign) or (White and Native)	BF, WN
Entire Population (Black or White; Foreign or Native)	BN, WN, BF, WF

A population with more than two types of attributes and two values on each attribute would generate a larger set of categories. A population with three types of attributes and two values on each, for instance, would generate 256 categories in the unrestricted case and 28 categories with the restriction above. To see why, recall from Section II. (vii) that the total number of different combinations that can be generated from a repertoire of j types of attributes and n_i values for attribute type A_i , with no restrictions, is $2^{n_1 n_2 \dots n_j}$ and, with the restriction we impose, is $\prod_{i=1}^{j}(2^{n_i}-1)+1)$. A population with two types of attributes and three values on each, by a similar logic, would generate 512 possible categories in the unrestricted case, and 50 possible categories with the restriction that we impose.

ii. The Size of Each Possible Ethnic Category

The size of each possible ethnic category depends upon the proportion of individuals in the population who possess each attribute that is included in the definition of that category.

Recall that this proportion can be represented in our simple example as follows:

Table 4

	Black	White

Foreign	a	b
Native	c	d

The size of each category that can be generated by this population, for the unrestricted and restricted cases then, are given in Table 5. As before, the shaded cells describe those categories included in the unrestricted case only.

Table 5

Definition of Category (Category)	Allowed individual repertoires	Size
∅	0	0
Black and Foreign	BF	a
Black and Native	BN	c
White and Foreign	WF	b
White and Native	WN	d
Black	BF, BN	a + c
White	WF, WN	b + d
Foreign	BF, WF	a + b
Native	BN, WN	c + d
Black or Foreign	BF, BN, WF	a + b + c
White or Foreign	BF, WF, WN	a + b + d
Black or Native	BF, BN, WN	a + c + d
White or Native	BN, WF, WN	b + c + d
(Black and Native) or (White and Foreign)	BN, WF	b + c
(Black and Foreign) or (White and Native)	BF, WN	a + d
Entire Population (Black or White; Foreign or Native)	BN, WN, BF, WF	a + b + c + d = 1

We do not, for purposes of brevity, list the size of the categories associated with larger population repertoires of attributes. However, the logic is identical.

iii. The Poset of Possible Categories and their Sizes

Since some of the possible categories are contained in each other, they form a partially ordered set. A partially ordered set (poset) consists of some elements which can be ordered in relation to each other and some that cannot. (See the appendix for a formal definition.) Figures 2(a) and 2(b) show the posets which corresponds to our running example.

Figure 2(a): The Unrestricted Case

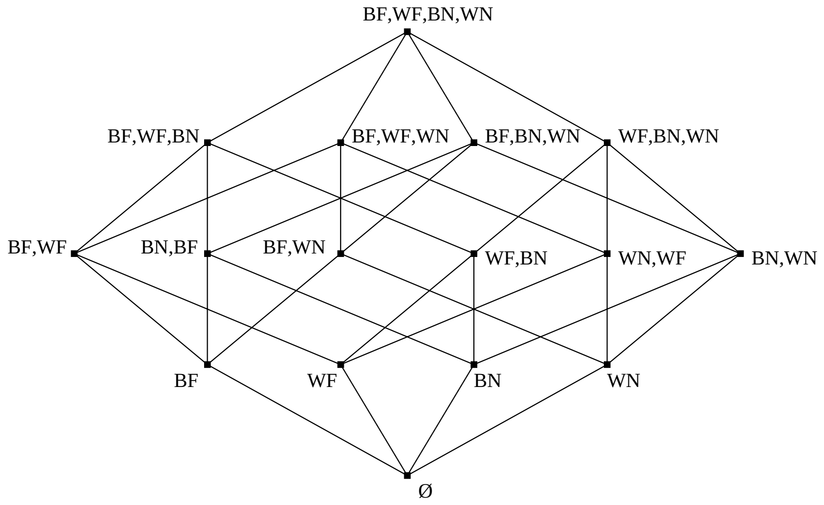
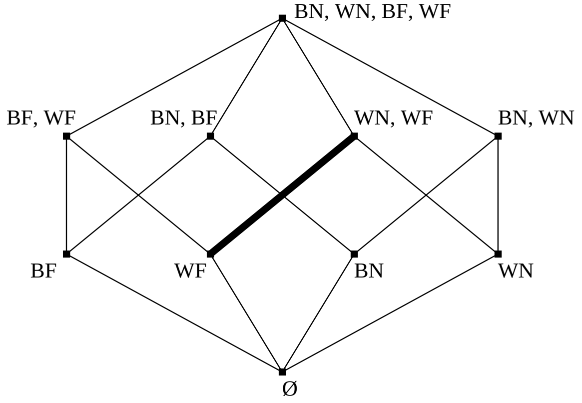


Figure 2(b): The Restricted Case



Each point in the diagram represents a category, described by the individual repertoires that are eligible for membership. In Figure 2a, the 16 points represent the membership of the 16 possible categories. In Figure 2b, the 10 points represent the membership of the 10 possible categories. At the lowest level we have ∅ and at the highest level we have the entire population.

Categories arrayed at a higher level in the diagram contain some of the categories at a lower level. Each line connecting two points represents containment: when a line connects a lower point to a higher point, the lower category is contained in the higher category. For example, the thick line in Figure 2b, shows that the category White and Foreign, corresponding to the individual repertoire WF, is contained in the category White, which corresponds to the individual repertoires WF and WN. When two categories are not linked by a line, the lower category is not contained by the higher category.

In general, any set of possible categories generated from a repertoire of attributes for a population can be represented as a poset. The structure of the poset depends upon the number of types of attributes in the population repertoire and the number of values. It will have as many nodes as there are categories derived from this population repertoire of attributes. The larger the number of categories, then, the larger and more complicated the poset will be.

For instance, the poset corresponding to a repertoire of attributes for a population that has three types of attributes and two values on each, for instance, will have 256 nodes (in the unrestricted case) and 28 nodes (with the restriction that we impose). The poset corresponding to a repertoire of attributes for a population that has two types of attributes and three values on each will have 512 nodes (in the unrestricted case) and 50 nodes (with the restriction we impose). We do not draw out posets corresponding to these larger and more complex population repertoires. But the key point here is that we can use this two dimensional diagram to describe populations with any number of types of attributes and ranges of values on each.

The appendix describes this poset further. For example, we show it is a lattice. We also determine its structure for any number of types of attributes in the population repertoire and any number of values.

iv. The Set Of Minimum Winning Categories

The set of minimum winning categories for any population varies with the context. We can now formally define a context, to mean all of the following information:

- the repertoire of salient attributes for the population,
- the distribution of the individual repertoires over the population, and
- some value k, $.5 < k < 1$, defined as the minimum winning threshold. We take the value of k to be exogenously determined in the short term.

A category is minimum winning in a context if it does not include the entire population, and fulfils two further conditions:

- its size is $\geq k$, and
- it does not contain any other possible category whose size is also $\geq k$.

The first condition gives us all categories that are larger than the minimum-winning threshold. We call this the set of all *winning* categories. The set *of winning* categories is *a subset of the set of all possible* categories generated by a population repertoire of attributes. However, as noted above, some winning categories may contain other, smaller ones. Individuals eligible for membership in two winning categories, one of which is contained by the other, should prefer the smaller one. The second condition captures this logic by including within our set of *minimum winning categories* only those categories which are “minimal by containment.” The set of minimum winning categories *is then a subset of the set of winning categories*.

v. Identifying The Set of Minimum Winning Categories in a Poset

The representation of all possible categories using a diagram allows us to identify easily whether a possible category both exceeds the minimum winning threshold k and is minimal by containment. In the examples below, we draw the posets that describe the possible categories generated by three different population repertoires and identity the number of minimum winning categories in each. For ease of exposition, each of these examples describes the set of possible categories in only the restricted case (where we assume that individuals will only consider “and” combinations across types and “or” combinations within types). The logic for the unrestricted case would be identical. The only difference is that the poset, and the number of minimum winning categories, would be larger for the unrestricted case.

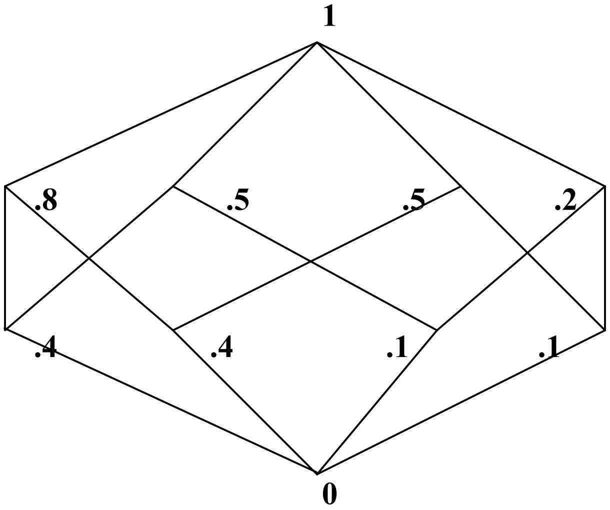
Suppose k = .6 and suppose that the values of a, b, c and d in our running example are as follows:

Table 6

	Black	White
Foreign	a(.4)	b(.4)
Native	c(.1)	d(.1)

Figure 5 shows us immediately that this population produces only one possible minimum winning category, defined as the category “Foreign”, with membership BF, WF.

Figure 5



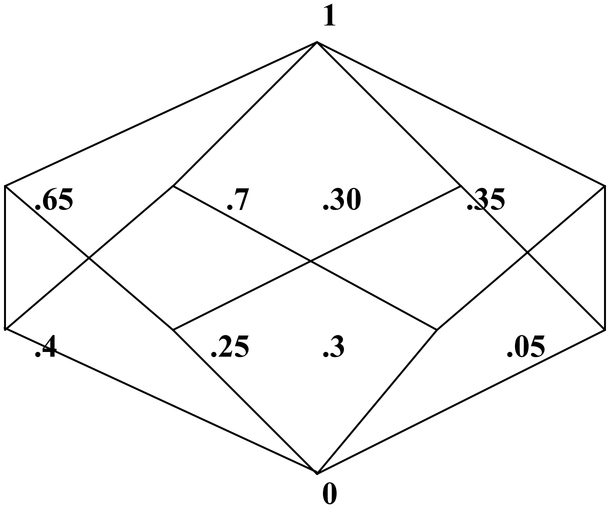
With the same k, consider now the following set of values of a, b, c, and d.

Table 7

	Black	White
Foreign	a(.4)	b(.25)
Native	c(.3)	d(.05)

Representing this population in Figure 6 shows us immediately that this population produces two possible minimum winning categories: one defined "Foreign", has the membership BF, WF and is of size .7; the other, defined as "Black" has the membership BF and BN and is of size .65.

Figure 6



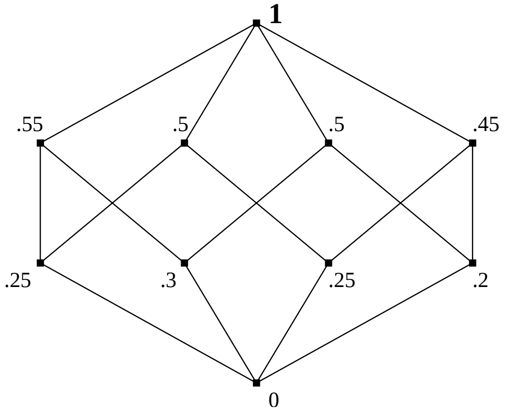
Consider, finally, the following set of values of a, b, c, and d, again with k=.6.

Table 8

	Black	White
Foreign	a(.25)	b(.3)
Native	c(.25)	d(.2)

Representing this population in Figure 7 shows us immediately that this population does not produce a single minimum-winning category.

Figure 7



IV. Predicting Democratic Stability

i. Predicting Democratic Stability for Particular Distributions of Attributes (Holding the value of k constant)

Earlier, we defined a stable democracy as with more than one minimum winning category. The three examples above nicely illustrate variation in the probability of democratic stability.

Consider Figure 5, in which there is only one minimum winning category: the category "Foreign," with size .8 and membership of individuals with the repertoires WF or BF. With perfect information about the distribution of individual repertoires of attributes, all individuals will see that the category "Foreign" is the only winning category. Individuals with the repertoires WF and BF, then, will declare membership in the category "Foreign." Individuals with the repertoires WN or BN should declare membership in the category "Native," the largest category in which they are eligible, with size .2. Thus, a bipolar division should emerge in the electorate, with 80% of the population classifying itself as being of foreign origin, and 20% as native.

Given the winner take all assumption, the payoff for the members of the minority category will be 0. Even under such oppressive circumstances, however, individuals with the repertoires WN or BN will not reclassify themselves, since they simply do not have any ethnic identity category in their repertoire that will improve their lot. And individuals with the repertoires WF or BF will not reclassify themselves since they cannot do better under a different system of classification. As a result, this population should have an unstable democracy, under threat of subversion from this permanent minority, as long as the minimum winning threshold k remains constant.

But consider Figure 6, which generates two minimum winning categories: "Foreign", with size .65 and membership of individuals with repertoires BF or WF; and "Black", with size .7 and membership of individuals with repertoires BF or BN. In this scenario, the possibility for change in the electoral verdict does exist, driven by those who stand to lose under one scheme of categorization but can gain in another. This democracy, then, should be characterized by fluidity in electoral competition and therefore stable. Similarly, consider the example in which the distribution of individual repertoires fails to generate any minimum winning category, as in the case of Figure 7. In this scenario, we should also expect electoral fluidity and therefore democratic stability, as small categories, none of which are of minimum winning size, form temporary alliances with each other.

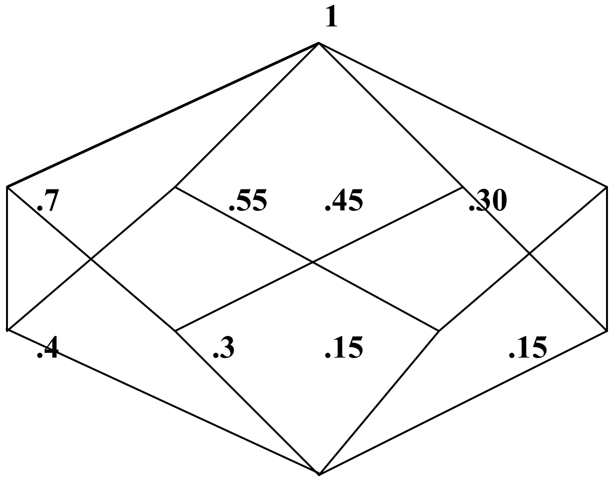
ii. Predicting Democratic Stability for Particular Distributions of Attributes (Allowing the Value of k to Vary)

Above, we predicted the probability of democratic stability for different distributions of attributes, holding the value of k constant. But another way to predict the probability of democratic stability is to explore the extent to which a democracy is robust to changes in the value of k.

For example, consider the population described in Figure 5. For this distribution of attributes, we see from Figure 5 that the minimum winning category will remain unchanged for a large range of values of k ($.5 < k \leq .8$). For any k value in this range, this population will have a “Foreign” majority of 80% and a “Native” minority of 20%. We might think of the k value as representing the electoral system, or some combination of electoral, party and legislative institutions that determine the size of the requisite electoral coalition. To the extent that this population makes it impossible for losers to turn the situation to their advantage for a large range of k values, this population should produce an unstable democracy under a wide range of institutional conditions.

But the population described in Figure 8 should produce a stable democracy with only a small change in k values.

Figure 8



0

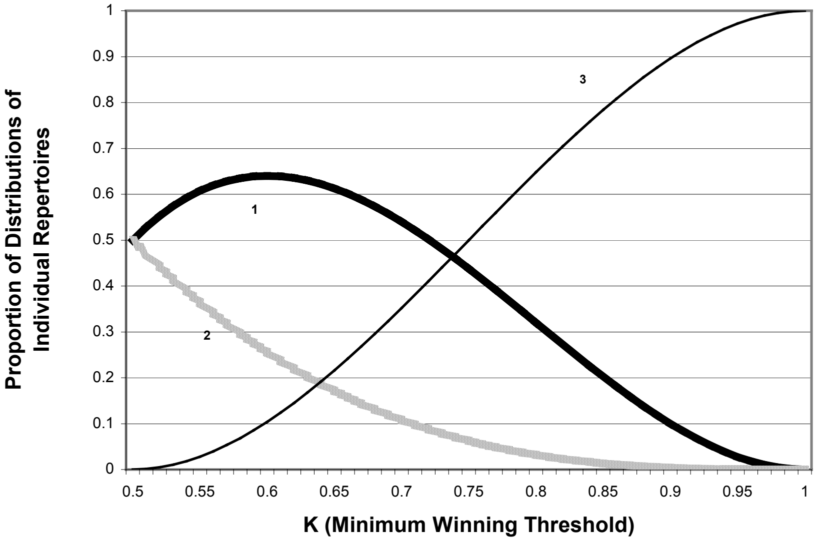
When k = .6, the population represented in Figure 8 will generate only one minimum winning category: “Foreign,” of size .7. However, a small change in the value of k, from .6 to .55 generates two minimum winning categories: “Foreign” of size .7, and “Black” of size .55. Slight changes in institutional design should safeguard democratic stabilility for this population.

iii. Predicting Democratic Stability in General for Different Distributions of Attributes and Different Values of k

Can we say anything about the prospects for democratic stability associated with population repertoires of different sizes and composition? Are large population repertoires more likely to be associated with stable democracies than small ones? Are population repertoires with a small number of types but a large range of values over these types more likely to produce stable democracies than population repertoires with a large number of types but a small range of values over these types?

Let us first discuss the possibility of democratic stability with the population repertoire in our running example, with 2 types of attributes and two values on each type. Four distinct individual repertoires can be generated from this population repertoire, and these four repertoires can be distributed in varying proportions across a population. Chart 1 summarizes the proportion of these distributions that produce, for different (discrete) values of k, exactly one minimum winning category (Line 1); more than one minimum winning category (Line 2); and no minimum winning category (Line 3).

Chart 1
Population Repertoire: 2 Types, 2 Values on Each
No. of Individual Repertoires: 4



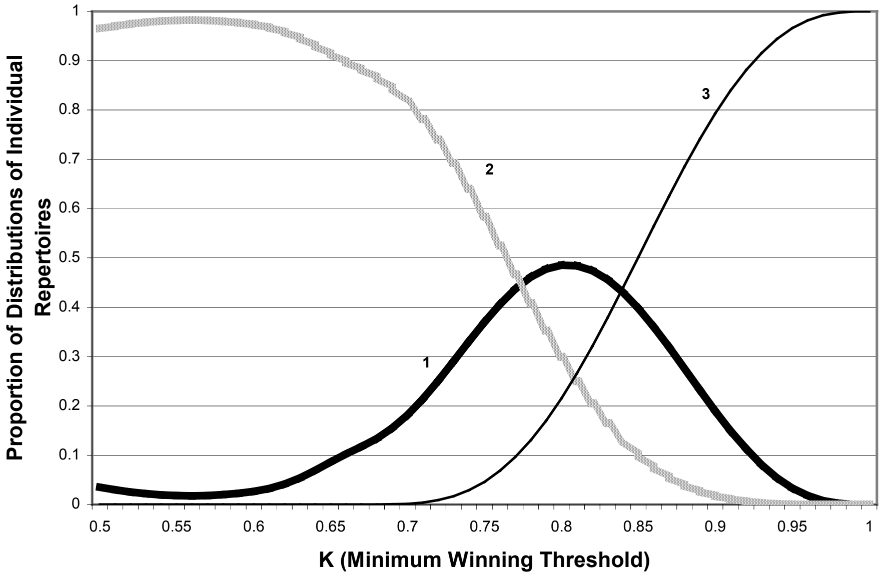
For low to moderate values of k (.5 < k ≤ .73), this population repertoire is more likely to produce a single minimum winning category than any other outcome. However, for high values of k, (.73 < k <1), no minimum winning category becomes the more likely outcome. The proportion of distributions that generates more than one minimum winning category also decreases with increasing values of k. However, it is always less likely than one of the other two outcomes.

Another way to interpret this chart is as follows: As the value of k increases, distributions of individual repertoires that generate any minimum winning category become increasingly rare. But when a distribution of individual repertoires does generate a minimum winning category, the probability that this will be a unique one is high.

What does this mean about the prospects for democratic stability in particular countries with a population repertoire of 2 types of attributes and 2 values on each type? In order to extrapolate from this graph to countries that have such a population repertoire, we need to know how different distributions of individual repertoires are distributed across these countries. In the absence of further information, we make the assumption that these distributions are randomly distributed across countries. Given this assumption, we can infer that the probability that these countries will generate stable democracies is low for low to moderate values of k (.5 < k ≤.73), but increases with higher values of k. (.73 < k <1). (Note, however, that this inference would not hold if the distribution of individual repertoire of attributes across countries were skewed.)

Consider now what happens when we hold constant the number of types of attributes in a population repertoire but increase the number of <u>values</u> on each type from 2 to 3. This population repertoire generates a total of 9 distinct individual repertoires that can be distributed in different proportions across a population. Chart 2 summarizes the proportion of these distributions that produce, for different values of k, exactly one minimum winning category (Line 1), more than one minimum winning category (Line 2), and no minimum winning category (Line 3).

Chart 2
Population Repertoire: 2 Types, 3 Values on Each
No. of Individual Repertoires: 9



The first thing to note about this population repertoire is that it generates a much higher likelihood of a stable democracy than the previous one. For low to moderate values of k (.5 < k ≤ .77), this population repertoire is most likely to generate multiple minimum winning categories. For a small interval (.78 < k ≤ .83, the proportion of distributions that generate a single minimum winning category than any other outcome. And for high values of k (.83 < k < 1), the absence of a minimum winning category becomes the most probable outcome. Making as above the assumption that these distributions are randomly distributed across countries, we can infer that countries with this type of population repertoire should be likely to have stable democracies for all values of k except for a brief window when k is moderately high.

Consider, finally the case of a population repertoire in which we increase the number of <u>types</u> from 2 to 3, keeping the number of values on each type constant. This population repertoire, with 3 types and 2 values on each, generates a total of 8 distinct individual repertoires which can be differently distributed to produce distinct populations.

Chart 3 summarizes the proportion of these distributions that produce our three different outcomes for different values of k.

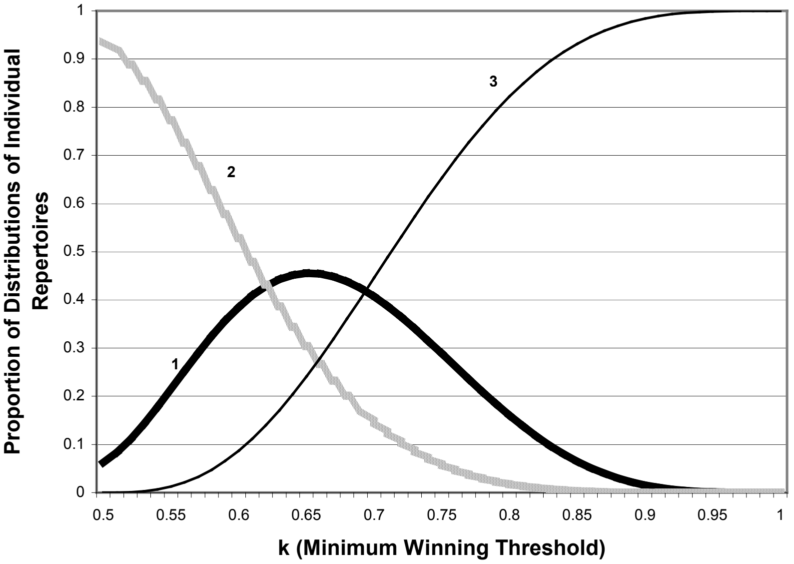


Chart 3
Population Repertoire: 3 Types, 2 Values on Each
No. of Individual Repertoires = 8

Here, for low values of k (.5< k <.63), we are more likely to see multiple minimum winning categories than any other outcome. For a brief window, (.63 ≤ k ≤ .69), a single minimum winning category becomes more likely than the other two alternatives. And for moderate to high values of k (.69 < k < 1), we are most likely to see the absence of a minimum winning category. Making as above the assumption that these distributions are randomly distributed across countries, we can infer that countries with this type of population repertoire should have stable democracies at low and high k values.

We have not experimented with larger increases either in the number of values on each type, keeping the number of types constant, or in the number of types. Our expectation is that increases in either will dramatically increase the probability of fluidity in identities and therefore democratic stability.

IV. Conclusion

We have tried to show that there is no necessary link here between the politicization of ethnic categories and the likelihood of democratic instability. Instead, the key variable is the underlying distribution of attributes that constitute ethnic identity categories: some distributions will generate unstable multi-ethnic democracies while others will not. While we have focused in this paper mainly on ethnic categories, the same logic should apply also to non-ethnic ones. When the distribution of non-ethnic attributes, such as landownership, or income, or rural-urban residence or ideological affiliations and so on generate only a single minimum winning identity category, we should expect to see democracy under strain. But we should expect a stable democracy, other things equal, when the distribution of non-ethnic attributes produces none or multiple minimum winning categories.

In future work, we expect to explore some key extensions to this model. First, we impose further restrictions on the attributes and combinations of attributes that individuals are most likely to consider, introducing weights to differentiate between attributes that have been historically and institutionally privileged from those that have not. Second, whereas this model applies only to winner-take-all situations in which the outcome is determined by majority rule, we have begun to explore the possibility of change associated either with other decision rules in a winner-take-all system (e.g. plurality rule) or with systems that are not winner-take all (e.g. in which the payoff is distributed proportionally among groups). A third extension would broaden the applicability of the model to scenarios in which individuals are motivated by the desire for collective benefits rather than individually divisible goods as they are here. Finally, in future work, we would like to shift the focus from the distribution of attributes to the design of institutions. We are especially interested in the institutions that determine the value of k. Holding the distribution of individual repertoires of attributes constant, this model allows us to ask: What institutional designs will be optimal given the distribution of identity repertoires in a population? By comparing the outcomes that the same individuals will obtain under different thresholds, we can theorize about the institutions that they should find optimal.

REFERENCES

Ake, Claude. 1993. "Rethinking African Democracy." in *The Global Resurgence of Democracy*, eds. Larry Diamond and Marc. F. Plattner, 70-82. Baltimore: Johns Hopkins Press.

Bates, Robert. "Ethnic Competition and Modernization in Contemporary Africa." *Comparative Political Studies,* Vol. 6, No. 4 (1974): 457-483.

Chandra, Kanchan. Forthcoming, January 2004. *Why Ethnic Parties Succeed: Patronage and Ethnic Headcounts in India.* Cambridge University Press.

Chandra, Kanchan and Daniel Metz. "A Cross-National Database on Political Parties." Paper presented at the Annual Meeting of the Midwest Political Science Association, Chicago, April 2002

Chandra, Kanchan. "Ethnic Parties and Democratic Stability." Working Paper, M.I.T, 2003.

Dahl, Robert. 1956. *A Preface to Democratic Theory*. Chicago: University of Chicago Press.

Dahl, Robert. *Polyarchy* New Haven: Yale University Press, 1971).

Horowitz, Donald. 1994. "Democracy in Divided Societies." in *Nationalism, Ethnic Conflict and Democracy*, eds. Larry Diamond and Marc F. Plattner, 35-55. Baltimore: Johns Hopkins Press.

Horowitz, Donald. 1991. *A Democratic South Africa: Constitutional Engineering in a Divided Society*. Berkeley: University of California Press.

Horowitz, Donald. 1985. *Ethnic Groups in Conflict*. Berkeley: University of California Press.

Kasfir, Nelson. 1979. "Explaining Ethnic Political Participation." *World Politics* 31: 3, 365-484.

Laitin, David. *Hegemony and Culture*. Chicago: University of Chicago Press, 1986.

Laitin, David. 1998. *Identity in Formation*. Ithaca: Cornell University Press.

Lijphart, Arend. *Democracy in Plural Societies*. New Haven: Yale University Press, 1977.

Posner, Daniel, forthcoming. *The Institutional Origins of Ethnic Politics in Zambia*. Cambridge University Press.

Rabushka, Alvin and Kenneth A. Shepsle, *Politics in Plural Societies* (Columbus: Charles E. Merrill, 1972).

Riker, William. *A Theory of Political Coalitions*. New Haven: Yale University Press, 1962.

Rustow, Danwart, "Transitions to Democracy," *Comparative Politics*, 12, no.3 (April 1970): 337-364.

Sahlins, Peter, 1989. *Boundaries: The Making of France and Spain in the Pyrenees*. Berkeley: University of California Press.

Waters, Mary, 1990. *Ethnic Options*. Berkeley: University of California Press.

Waters, Mary. 1999. *Black Identities*. Cambridge MA: Harvard University Press.

Appendix: Mathematical Aspects of the Model

version dated: August 21, 2003

The purpose of this appendix is to discuss the mathematical aspects of our model. The appendix is divided as follows. The relevant definitions are presented in Section 1. In Section 2, we count the number of possible coalitions overall and in an individual's repertoire of possible coalitions. It is natural to consider a partial ordering on the set of possible coalitions by containment. We determine the structure of this poset and explain how to draw it in Section 3. Additional properties of the poset are discussed in Section 4. Section 5 uses this framework to calculate the probability of having zero, one or many minimum winning coalitions given population repertoires of attributes of different sizes.

1 Definition of our Model

Let P be a finite set which we call the population. We call the elements of P individuals.

Let $\mathcal{A} = \{A_1, A_2, ..., A_j\}$ be a finite set of types of attributes. For each type $A_i \in \mathcal{A}$ we associate a set $V_i = \{a_{i1}, a_{i2}, ..., a_{i,n_i}\}$ of values for that types. Let $V = \cup V_i$. We call V the set of all attributes.

Now, each individual takes on exactly one value for each type of attribute. This can be expressed as follows. We have a function $f : P \times \mathcal{A} \rightarrow V$ such that $f(p, A_i) \in V_i$ for each $p \in P$ and $A_i \in \mathcal{A}$. We call the set $f(p, \mathcal{A})$ the repertoire of individual p. There are at most $n_1 n_2 \dots n_j$ different repertoires.

Proof 1. Since an individual takes on exactly one of n_i values on type A_i, there are n_1 possibilities for the value on the first type, n_2 possibilities for the value on the second type, and so on. Multiplying gives us at most $n_1 n_2 \dots n_j$ different repertoires. $\square$

Let $\mathcal{R} = \{f(p, \mathcal{A}) : p \in P\}$, the set of all repertoires achieved by an individual in the population P. By our previous remark we see that $\#\mathcal{R} \leq n_1 n_2 \dots n_j$.

We define a possible coalition as a set of individuals which satisfy some logical (ie. and, or, not) combination of the attributes. Recall that an individual's repertoire of attributes is a set $\{v_1, v_2, \ldots, v_j\}$ consisting of one value from each type of attribute. The set of all individuals with that repertoire is the set in which all individuals satisfy the condition "v_1 and v_2 and ... and v_j." Since individuals have exactly one value from each type of attribute, any logical combination of the attibutes can be written as an "or" combination of the repertoires. This allows us to give an equivalent definition of a possible coalition.

Definition 1. *Let $R \in \mathcal{R}$ be the subset of allowed repertoires. Then we let $\mathcal{C}_R = \{p \in P : f(p, \mathcal{A}) \in R\}$, the set of all individuals whose repertoire is in R. We call $\mathcal{C}_R$ a* ***possible coalition.***

In this first definition, we allow any combination of the repertoires. This may not be applicable to all situations. In applications of this model, we may need to restrict the kinds of combinations of attributes which are allowed to form coalitions. We propose an altenate definition of possible coalition to illustrate how a general type of restriction on which coaltions are allowed form affects our theory.

In the second definition, we only allow certain combinations of the attributes. The restriction we impose amounts to the following: the coalition is defined by specifying a set of allowed values from each type of attribute independent from the set of allowed values on another type. An individual can then belong to a coalition if she has an acceptable value on each type of attribute.

Definition 2. *For each type of attribute A_i, let $R_i \in V_i$ be the set of allowed values for A_i. Then we define $\mathcal{C}_{(R_1,\ldots,R_n)} = \{p \in P : f(p, A_i) \in R_i\}$. We also call these* ***possible coalitions.***

Note that if we let R be the set of repertoires whose value on attribute A_i is in R_i, then we get $\mathcal{C}_{(R_1,\ldots,R_n)} = \mathcal{C}_R$ as defined in the first definition. So the possible coalitions from the second definition are a subset of those arising from the first definition. In general, no matter which restriction is imposed on the formation of possible coaltions, we will always be considering a subset of the possible coalitions defined by Definition 1.

These are the only two definitions of possible coalitions that we use in this appendix. However, other types of restrictions could also be imposed when defining the posssible coalitions, and the analysis we present in the following sections could be carried out with alternate definitions.

Definition 3. *By a* ***context*** *we mean the following information: a population P, types of attributes $\mathcal{A}$, values on those attributes V, a function f telling us which individuals have take on which attributes, and a number k, $.5 \leq k \leq 1$, which we call the* ***winning threshold****.*

Definition 4. *A possible coalition, $\mathcal{C}$, is called a* ***winning coalition*** *if it satisfies the following condition:*

1. $\frac{\#\mathcal{C}}{\#P} \geq k$.

This condition says that the proportion of the population which is in $\mathcal{C}$ is at least as large as the winning threshold.

Definition 5. *We call a winning coalition $\mathcal{C}$* ***non-trivial*** *if it is not equal to P, the entire population.*

Definition 6. *A possible coalition, $\mathcal{C}$, is called a* ***minimum winning coalition*** *if it satisfies the following two conditions:*

1. $\frac{\#\mathcal{C}}{\#P} \geq k$, *and*
2. $\frac{\#\mathcal{C}}{\#P} < k$ *for every* $\mathcal{C}' \subset \mathcal{C}$.

The first condition says the possible coalition is a winning coalition. The second condition says that, subject to being winning, the coalition is minimal by containment, i.e. it does not contain a smaller winning coalition.

2 Number of Possible Coalitions

Given some population repertoire of attributes, we can calculate the number of possible coalitions for both of the definitions proposed above. We can do so not only for small examples but in the most general situation, where we have j types of attributes and these have n_1, n_2, $\ldots$, n_j values each.

2.1 General Definition

In the first definition, possible coalitions are defined by listing the allowed repertoires without any restriction on the combinations. Therefore, the possible coalitions are in a natural one-to-one correspondence with the subsets of $\mathcal{R}$, the set of repertoires. There are $2^{\#\mathcal{R}}$ such subsets.

Proof 2. To obtain this number, consider each element of $\mathcal{R}$, ie. each repertoire. Either the repertoire is in a subset or it is not, giving us 2 choices. To choose a subset, we must decide whether each repertoire is in the subset. Therefore, we have to make $2 \times 2 \times \cdots \times 2 = 2^{\#\mathcal{R}}$ choices in all, giving us the number of subsets of $\mathcal{R}$. Hence we conclude that there are $2^{\#\mathcal{R}}$ possible coalitions. $\square$

In the case that there is at least one individual with each repertoire of attributes, we have $\#\mathcal{R} = n_1 n_2 \dots n_j$ and this gives $2^{n_1 n_2 \dots n_j}$ possible coalitions.

We may also calculate the size of an individual's repertoire of possible coalitions, that is the number of possible coalitions in which an individual is eligible for membership. For every individual, it is $2^{\#\mathcal{R}-1}$.

Proof 3. We use the same reasoning as above and notice that an individual p is contained in a possible coalition $\mathcal{C}_R$ if and only if $f(p, \mathcal{A}) \in R$. Therefore we want to count the number of subsets of $\mathcal{R}$ which contain $f(p, \mathcal{A})$. These are in one-to-one correspondence with the subsets of $\mathcal{R} - f(p, \mathcal{A})$, a set of size $\#\mathcal{R} - 1$ and hence there are $2^{\#\mathcal{R}-1}$ such subset. This is us $2^{\#\mathcal{R}-1}$ possible coalitions in an individual's repertoire of possible coalitions. $\square$

If there is at least one individual with each repertoire of attributes, the size of an individual's repertoire of possible coalitiosn is $2^{n_1 n_2 \dots n_j - 1}$ because $\#\mathcal{R} = n_1 n_2 \dots n_j$.

2.2 Restricted Definition

Now consider the second definition. We will determine the number of possible coalition only in the case where there is at least one individual with each repertoire of attributes. In this case, there are $\Pi_{i=1}^{j}(2^{n_i} - 1) + 1$ possible coalitions.

Proof 4. According to our definition, to determine a possible coalition we must specify allowed values for each type of attribute. As long as each subset of allowed values is non-empty, we get a non-empty possible coalition. So our plan is to count the number of non-empty possible coalitions first and then add 1 at the end. Also note that two different choices of non-empty subsets of allowed values on each type give a different possible coalition. Therefore, we simply need to count the number of ways in which we can specify the subsets of allowed values for each type of attribute. For type A_i, there are n_i values and therefore 2^{n_i} subsets of values, $2^{n_i} - 1$ of which are non-empty. Therefore we get $\Pi_{i=1}^{j}(2^{n_i} - 1)$ non-empty possible coalitions. Adding 1 for the empty coalitions, we see that in total we have $\Pi_{i=1}^{j}(2^{n_i} - 1) + 1$ possible coalitions. $\square$

Finally, we determine the size of an individual's repertoire of possible coalition under the restricted definition to be $2^{n_1 n_2 \ldots n_j - j}$, again working only in the case that there is at least one individual with each repertoire of attributes.

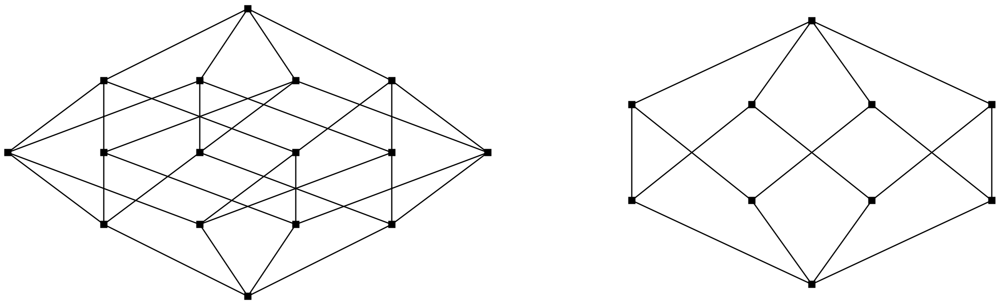
Proof 5. Consider individual $p \in \mathcal{P}$. Individual p is in the possible coalition $\mathcal{C}_{(R_1, R_2, \ldots, R_j)}$ if and only if $f(p, A_i) \in R_i$. Notice that this condition implies that R_i is non-empty for $i = 1, \ldots, j$. Now subsets $R_i \subseteq V_i$ that contain $f(p, A_i)$ are in one-to-one correspondence with subsets of $V_i - \{f(p, A_i)\}$. Since V_i has size n_i, $V_i - \{f(p, A_i)\}$ has size $n_i - 1$. Hence there are $2^{n_i - 1}$ such subsets of V_i containing $f(p, A_i)$. This implies that there are $\Pi_{i=1}^{j} 2^{n_i - 1} = 2^{n_1 n_2 \ldots n_j - j}$ possible coalitions in an individual's repertoire of possible coalitions. $\square$

3 Drawing the Poset of Possible Coalitions

Consider now the set of possible coalitions. Since some of the possible coalitions are contained in others, it is natural to consider their partial order defined by containment. Note that $\mathcal{C}_{R_1} \subseteq \mathcal{C}_{R_2}$ if and only if $R_1 \subseteq R_2$.

Formally, a partial ordered set (poset) is a pair $(P, \leq)$ where P is a set and $\leq$ is a relation of P satisfying the following conditions:

1. $p \leq p$ for all $p \in P$.
2. If $p \leq q$ and $q \leq r$ then $p \leq r$ for all p, q, $r \in P$.
3. If $p \leq q$ and $q \leq p$ then $p = q$ for all p, $q \in P$.



general definition

restricted definition

Figure 1: Example of the lattice of possible coalitions for the general and the restricted definition

In our case, the set P is the population and our relation is containment.

If the poset has an element which is less than or equal to every other element, we call it a $\hat{0}$. If it has an element which is greater than or equal to every other element, we it a $\hat{1}$. For basic information about posets we refer the reader to [1, 3, 4].

Posets may be draw by use of a Hasse diagram. In the main part of our text we repeatedly see the diagram for the poset of possible coalitions which arises in the case where we have two attributes each with two values. In a Hasse diagram, dots represent the element of P and lines represent containment. If $x < y$, then y is drawn above x and line is drawn between x and y if $x < y$ and there is no z such that $x < z < y$. If we draw a line from x up to y, we say y covers x. Figure 1 shows the Hasse diagram for the poset of possible coalitions which arise from each of our definitions, in the case where we have two types of attributes each having two values. Note that these two posets have different numbers of elements, as observed in the previous section.

Before discussing the properties of the poset of possible coalitions, we describe it concretely and provide an algorithm for drawing the poset when we are dealing with larger sets of attributes.

We need one more definition in order to give this description. If S and T are posets, then the direct product, $S \times T$, of S and T is $\{(s,t) : s \in S, t \in T\}$ with the order relation $(s,t) \leq (s',t')$ if and only if $s \leq s'$ and $t \leq t'$. There is a procedure for drawing the Hasse diagram for the product of two posets. Take the first poset S and draw it with large circles instead of dots for the

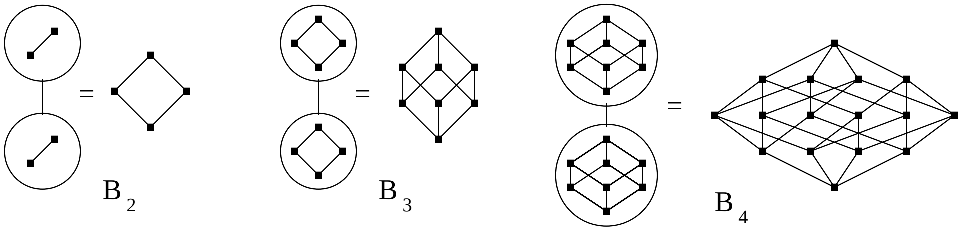


Figure 2: Drawing B_4

element of S. Now, in each circle, put a copy of T. This is called a nested line diagram for $S \times T$. To get the Hasse diagram for $S \times T$ we replace the lines between the circles with lines connecting corresponding elements of T in each circle. We will show examples of this procedure when we give the descriptions of the lattices of possible coalitions.

3.1 General Definition

In the general definition, where we allow any combination of repertoires to define a possible coalition, the structure of our lattice depends only on the number of repertoires which have at least one individual, $\#\mathcal{R}$. If there are $\#\mathcal{R} = n$ such repertoires, then our lattice is B_n, the boolean lattice on n elements. This follows simply by definition, since a boolean lattice is defined to be the poset of subsets of the set $\{1, 2, ..., n\}$ ordered by containment. It is known that $B_n = \mathbf{2}^n$ is the product of n chains of length 2. Combined with the description of how to draw the product of two posets, this gives a good way to draw out and visualize our poset.

Figure 2 shows how to draw $B_4 = \mathbf{2}^4$. First we construct $B_2 = \mathbf{2} \times \mathbf{2}$. Using our drawing of B_2 we construct $B_3 = \mathbf{2} \times B_2$. Similarly we obtain $B_4 = \mathbf{2} \times B_3$.

3.2 Restricted Definition

Now consider the restricted definition of possible coalitions. Let $S_1, S_2, ..., S_n$ be posets with a $\hat{0}$. Let $\hat{S_i}$ denote S_i with its $\hat{0}$ removed. Then we let $S_1 \star \cdots \star S_n = (\hat{S_1} \times \cdots \times \hat{S_n}) \bigcup \hat{0}$. Notice that $S_1 \star S_2 \star S_3 = (S_1 \star S_2) \star S_3 = S_1 \star (S_2 \star S_3)$. To draw the nested line diagram for $S_1 \star S_2$, we use the procedure for drawing the product of two posets that was described above and connect

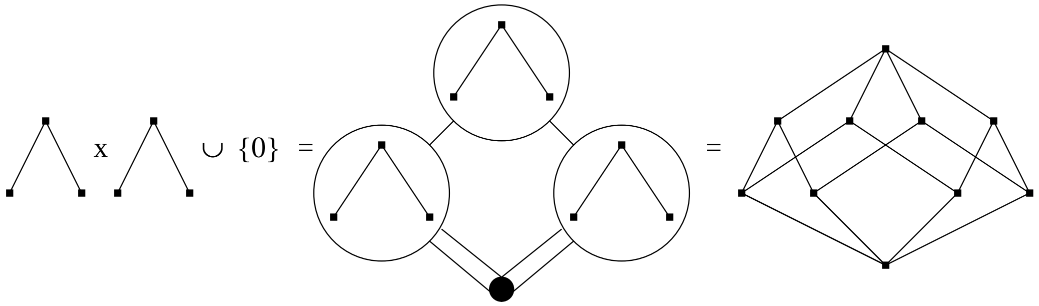


Figure 3: Lattice of possible coalitions (restricted definition) in the case with two attributes each having two values

the new element $\hat{0}$ to the bottom circles by double lines to indicate that $\hat{0}$ is below every element of the poset in that circle.

Now we may describe the structure of the lattice of possible coalitions that arises from the second definition. Suppose that we have j types of attibutes having $n_1, n_2, ..., n_j$ values. If there is at least one individual in each repertoire, then our lattice of possible coalitions is $B_{n_1} \star B_{n_2} \star \cdots \star B_{n_j}$ where B_{n_i} denotes the Boolean lattice on n_i elements.

Proof 6. We start by considering the non-empty possible coalitions. The non-empty possible coalitions correspond to a choice of non-empty subsets of values for each attribute. Therefore, since there is at least one individual that has each repertoire, the non-empty part of our lattice is simply the product of the posets of non-empty subsets of values for each attribute. In other words, we have $\hat{B}_{n_1} \times \hat{B}_{n_2} \times \cdots \times \hat{B}_{n_j}$. The only missing possible coalition is the empty coalition which is contained in every other possible coalition. Therefore our result follows by adding a $\hat{0}$ to $\hat{B}_{n_1} \times \hat{B}_{n_2} \times \cdots \times \hat{B}_{n_j}$. $\square$

To illustrate this procedure, Figure 3 shows the construction of the poset of possible coalitions when there are two attributes having two values each and Figure 4 shows the contruction of the poset when there are two attributes having two values and three values respectively.

4 Lattice Structure and Other Properties

This section discusses a few of the properties of posets of possible coalitions. These properties are not needed for the analysis found in the next section but

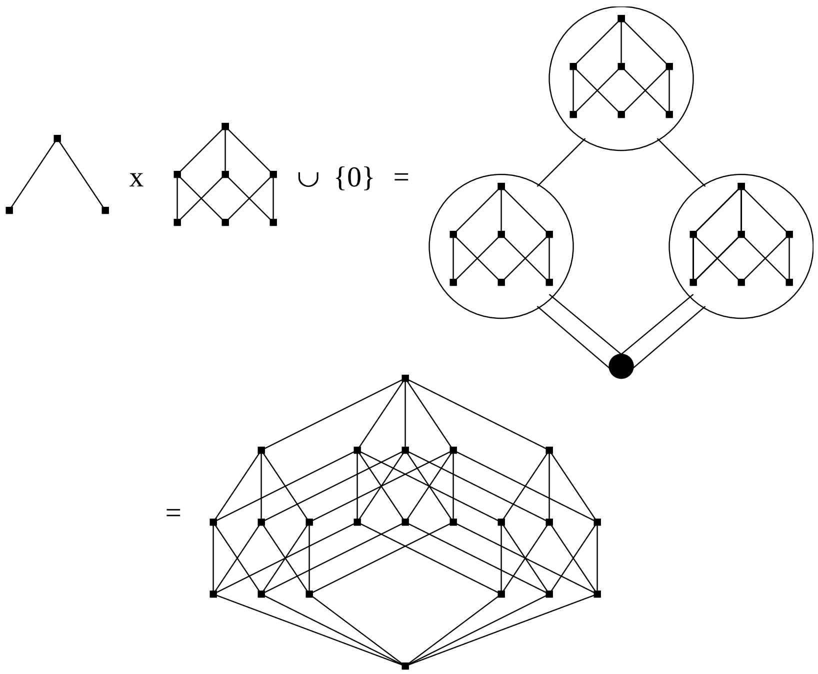


Figure 4: Lattice of possible coalitions (restricted definition) in the case with two attributes having two values and three values respectively

are included as interesting properties which may be useful for future analysis of the posets. At the end of the section we also mention their connection to Formal Concept Analysis.

The more interesting posets that we have encountered are the ones that arise from the restricted definition which we propose for possible coalitions. The posets arising from the first definition are all boolean and thus have been well studied. In particular, boolean lattices have all of the properties which we are going to present for the posets arising from the second definition.

The first fact to notice about the poset of possible coalitions is that it is a lattice. To be a lattice, every pair of elements, x and y, must have a greatest lower bound (a unique minimal element among all the elements which are greater than x and y in the partial order) and a least upper bound (a unique maximal element among all the elements which are less than x and y in the partial order). The greatest lower bound is denoted by $x \wedge y$ and the least upper bound is denoted by $x \vee y$. The greatest lower bound of two possible coalitions, $\mathcal{C}_{(R_1,\ldots,R_j)} \wedge \mathcal{C}_{(R'_1,\ldots,R'_j)}$, is simply the intersection of those possible coalitions, $\mathcal{C}_{(R_1,\ldots,R_j)} \cap \mathcal{C}_{(R'_1,\ldots,R'_j)} = \mathcal{C}_{(R_1 \cap R'_1,\ldots,R_j \cap R'_j)}$. The least upper bound is not quite the union of the two possible coalitions, because that union may not be a possible coalitions. However, we can still describe the least upper bound simply. It is obtained by taking the union of the allowed values over each type of attribute. We have $\mathcal{C}_{(R_1,\ldots,R_j)} \;\vee\; \mathcal{C}_{(R'_1,\ldots,R'_j)} = \mathcal{C}_{(R_1 \cup R'_1,\ldots,R_j \cup R'_j)}$. Since the lattice is finite, we immediately see that our poset is a complete lattice meaning that every subset of element of our poset has a greatest lower bound and a lowest upper bound.

There is one more property of this lattice that is worth mentioning. It is complemented. A lattice is complemented if for every $x \in P$ there exists a y such that $x \wedge y = \hat{0}$ and $x \vee y = \hat{1}$. The possible coalition $\mathcal{C}_{(R_1,\ldots,R_j)}$ has as a complement the possible coalition $\mathcal{C}_{(V_1 - R_1,\ldots,V_j - R_j)}$, obtained by allowing all the values on a type of attribute which are not allowed in the original possible coalition. Note that possible coalitions do not neccessarily have a unique complement.

This section would not be complete without mentioning the connection to Formal Concept Analysis. As we are doing here, Formal Concept Analysis studies the relationship between objects and (possibly multi-valued) attributes by considering a lattice, the formal concept lattice, whose elements are basically set of objects defined by sharing the same attributes. To obtain our poset of possible coalitions as a formal concept lattice, we must change our list of values on each type of attribute to include all subsets of values from that type. Regardless of what type of restriction is imposed on which coalitions can form, the set of possible coalitions will still be a formal concept lattice. For further information on Formal Concept Analysis, see [3].

5 Predicting the Number of Minimum Winning Coalitions

We want to apply our description of the poset to predicting when there will be exactly one minimum winning coalition and when there will be more. In this section, we present an algorithm for calculating the proportion of distributions of attributes over a population give rise to exactly zero, one, or more than one minimum winning coalition.

Notice that if $\mathcal{C}$ is a winning coalition, then all possible coalitions which contain $\mathcal{C}$ are also winning coalitions because they are at least as large as $\mathcal{C}$. In mathematical terms, this says that the set of winning coalitions forms a dual order ideal. The minimum winning coalitions are the mimimal elements of the dual order ideal of winning coalitions. Therefore, we have one minimum winning coalition exactly when the dual order ideal of winning coalitions has one minimal element, that is, it is a principal dual order ideal.

(An order ideal of a poset P is defined to be a subset $I \subseteq P$ such that if $i \in I$ and $j \leq i$ then $j \in I$. A dual order ideal of a poset P is defined to be a subset $I \subseteq P$ such that if $i \in I$ and $i \leq j$ then $j \in I$. A principal dual order ideal is defined to be an order ideal with exactly one minimal element.)

We would like a simple characterization of when the dual order ideal of winning coalitions is principal. Alternatively, we can consider a possible coalition $\mathcal{C}$ and ask when it is the unique minimum winning coalition. First it must be winning. That is $\#\mathcal{C} \geq k$. Second, it must be minimal in the dual order ideal of winning coalitions. For this second condition to be satisfied, it is enough to require that all $\mathcal{C}'$ which do not contain $\mathcal{C}$ are not winning. That is $\#\mathcal{C}' < k$ for all $\mathcal{C}' \not\supseteq \mathcal{C}$. However, the set of possible coalitions $\mathcal{C}' \not\supseteq \mathcal{C}$ form an order ideal. Therefore, when implementing our algorithm, it will be enough to verify that $\#\mathcal{C}' < k$ for $\mathcal{C}'$ which are maximal in this order ideal.

Let $r_1, r_2, ..., r_s$ denote the proportion of the population in each repertoire.

Note that the size of a possible coalition is simply a sum of the r_i's corresponding to repertoires which are allowed in that possible coalition. So we can express $\#\mathcal{C} \geq k$ and $\#\mathcal{C}' < k$ as linear inequalities in the r_i's.

Note also that by definition of the r_i's we have the following equations.

$$r_1 + r_2 + ... + r_s = 1$$

$$0 \leq r_i \leq 1 \text{ for } i = 1, 2, ..., s$$

Therefore, the r_i's form a $(s-1)$-dimentional polytope $\mathcal{P}$. The points of this polytope are the distributions of attributes over the population.

Let us add the condition that tell us that $\mathcal{C}$ is a minimum winning coalition, so that we are considering the polytope defined by:

$$r_1 + r_2 + ... + r_s = 1$$

$$0 \leq r_i \leq 1 \text{ for } i = 1, 2, ..., s$$

$$\#\mathcal{C} \geq k$$

$$\#\mathcal{C}' < k \text{ for } \mathcal{C}' \text{ maximal in the order ideal of } \mathcal{C}' \not\supseteq \mathcal{C}.$$

Then we are considering the polytope, call it $\mathcal{P}(\mathcal{C})$, of all distributions of the attributes over the population such that $\mathcal{C}$ is the only minimum winning coalition. Moreover, the quotient $\frac{V(\mathcal{P}(\mathcal{C})}{V(\mathcal{P})}$, of the volumes of these polytopes, is the proportion of distributions of attributes over the population in which $\mathcal{C}$ is the only minimum winning coalition.

It is clear that $\mathcal{P}(\mathcal{C}) \cap \mathcal{P}(\mathcal{D})$ is empty for distinct possible coalitions $\mathcal{C} \neq \mathcal{D}$. Therefore the proportions of distibutions that produce only one minimum winning coalition is

$$\sum_{\mathcal{C} \text{ a possible coalition}} \frac{V(\mathcal{P}(\mathcal{C}))}{V(\mathcal{P})}.$$

The algorithm for calculating what proportion of distribution of the attributes yields only one and more than one minimum winning coalition can be summarized as follows. First we draw out the poset of possible coalitions as described in Section 3. Next we go through each possible coalition $\mathcal{C}$ and calculate $\frac{V(\mathcal{P}(\mathcal{C}))}{V(\mathcal{P})}$. Finally, $\sum_{\mathcal{C}} \frac{V(\mathcal{P}(\mathcal{C}))}{V(\mathcal{P})}$ give us the proportion of distributions which result in only one minimum winning coalition and $1 - \sum_{\mathcal{C}} \frac{V(\mathcal{P}(\mathcal{C}))}{V(\mathcal{P})}$ gives

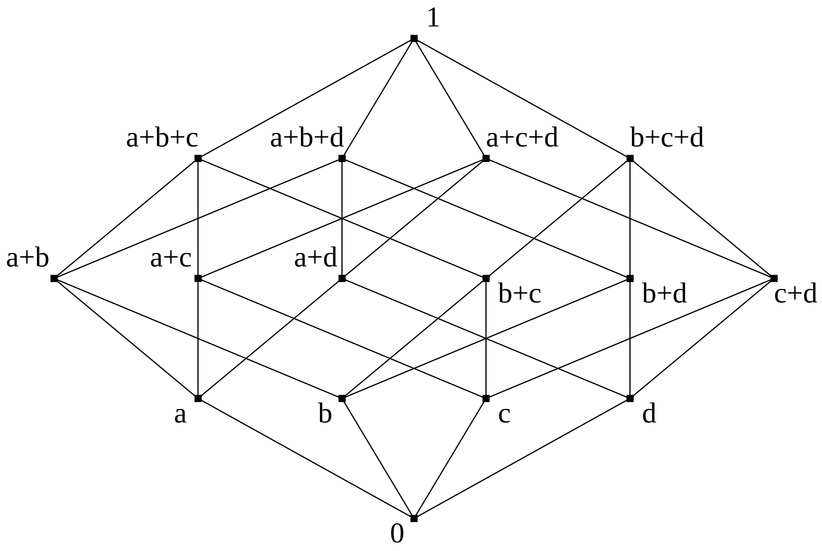


Figure 5: Proportion of the population in each possible coalition (general definition)

the proportion of distributions in which result in more than one minimum winning coalition. If we only want to consider non-trivial minimum winning coalitions, we can do so by taking the sum over all non-trivial minimum winning coalitions. This is done in the program that follows.

We were able to implement this algorithm on small examples using symbolic mathematical software, Maple 8.00 and Franz's Maple package for convex geometry [2]. Thankfully, due to symmetries of the poset of possible coalitions, the calculation of $V(\mathcal{P}(\mathcal{C}))$ for a specific possible coalition often amounts to a calculation which has been done previously. This greatly improves the computational complexity of our algorithm.

Writing out the details for a small example will make this clearer. We will do this for both the general and the restricted definitions of possible coalition in the case where we have two attibutes each with two values.

5.1 General Definition

Let a, b, c, and d be variables denoting the proportion of the population with each repertoire. See Figure 5. Then

$$a + b + c + d = 1 \qquad (1)$$

$$0 \leq a, b, c, d \leq 1 \tag{2}$$

This defines an 3-dimentional polytope and its volume is $\frac{1}{3}$.

By symmetry, we only need to consider three principal dual order ideals: the one where a denotes the size of the minimal element, the one where $a+b$ denotes the size of the minimal element, and the one where $a+b+c$ denotes the size of the minimal element. These correspond to the following sets of inequalities: for a we have equations 1 and 2 along with

$$\begin{aligned} a &\geq k \\ b+c+d &< k, \end{aligned}$$

for $a+b$ we have equations 1 and 2 along with

$$\begin{aligned} a+b &\geq k \\ a+c+d &< k \\ b+d+d &< k, \end{aligned}$$

and for $a+b+c$ we have equations 1 and 2 along with

$$\begin{aligned} a+b+c &\geq k \\ a+b+d &< k \\ a+c+d &< k \\ b+d+d &< k. \end{aligned}$$

Using the following Maple program we calculate the volumes of these polytopes and for $.5 \leq k \leq 1$ we compute the proportion of the distributions of attributes over the population in which there is exactly one non-trivial winning coalition.

```
> libname := libname, "/convex":
> with(linalg):
> with(convex):
> printlevel:=0:
> Q:=intersection([1,1,1,1]=1, [1,0,0,0]>=0, [0,1,0,0]>=0,
[0,0,1,0]>=0, [0,0,0,1]>=0):
> k:=100:
> for j from 50 to k do
```

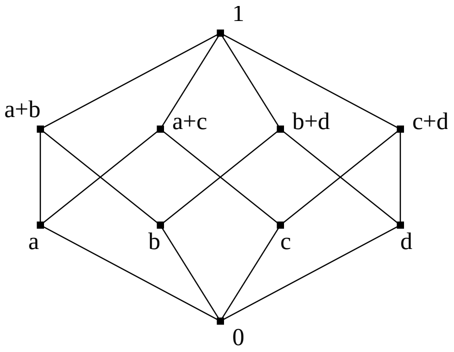


Figure 6: Proportion of the population in each possible coalition (restricted definition)

```
> Q1:=intersection([1,1,1,1]=1, [1,0,0,0]>=0, [0,1,0,0]>=0,
[0,0,1,0]>=0, [0,0,0,1]>=0, [k,0,0,0]>=j, [0,k,k,k]<=j):
> Q2:=intersection([1,1,1,1]=1, [1,0,0,0]>=0, [0,1,0,0]>=0,
[0,0,1,0]>=0, [0,0,0,1]>=0, [k,k,0,0]>=j, [0,k,k,k]<=j,
[k,0,k,k]<=j):
> Q3:=intersection([1,1,1,1]=1, [1,0,0,0]>=0, [0,1,0,0]>=0,
[0,0,1,0]>=0, [0,0,0,1]>=0, [k,k,k,0]>=j, [0,k,k,k]<=j,
[k,0,k,k]<=j, [k,k,0,k]<=j):
> print(j,evalf((4*surface(Q1)+6*surface(Q2)+4*
surface(Q3))/surface(Q)));
> od;
```

A similar program obtains the proportion of the distributions of the attributes over the population in which the only winning coalition is the entire population. From these two sets of output, it is easy to calculate the proportion of the distributions of attributes over the population which have more than one minimum winning coalition. The output of these programs is shown as a graph in the main part of the text.

5.2 Restricted Definition

Let a, b, c, and d be variable denoting the proportion of the population with each repertoire. See Figure 6. Then

$$a + b + c + d = 1 \tag{3}$$

$$0 \leq a, b, c, d \leq 1 \tag{4}$$

As in the case of the general definition, this defines an 3-dimentional polytope and its volume is $\frac{1}{3}$.

By symmetry, we only need to consider two principal dual order ideals: the one where a denotes the size of the minimal element and the one where $a+b$ denotes the size of the minimal element. These correspond to the following sets of inequalities: for a we have equations 3 and 4 along with

$$\begin{array}{rcl} a & \geq & k \\ b+d & < & k \\ c+d & < & k, \end{array}$$

and for $a+b$ we have equations 3 and 4 along with

$$\begin{array}{rcl} a+b & \geq & k \\ a+c & < & k \\ b+d & < & k \\ c+d & < & k. \end{array}$$

Using a Maple program very similar to the previous one, we calculate the volumes of these polytopes and for $.5 \leq k \leq 1$ we compute the proportion of the distributions of attributes over the population in which there is exactly one non-trivial winning coalition.

```
> libname := libname, "/convex":
> with(linalg):
> with(convex):
> printlevel:=0:
> Q:=intersection([1,1,1,1]=1, [1,0,0,0]>=0, [0,1,0,0]>=0,
[0,0,1,0]>=0, [0,0,0,1]>=0):
> k:=100:
> for j from 50 to k do
> Q1:=intersection([1,1,1,1]=1, [1,0,0,0]>=0, [0,1,0,0]>=0,
[0,0,1,0]>=0, [0,0,0,1]>=0, [k,0,0,0]>=j, [0,k,0,k]<=j,
[0,0,k,k]<=j):
> Q2:=intersection([1,1,1,1]=1, [1,0,0,0]>=0, [0,1,0,0]>=0,
[0,0,1,0]>=0, [0,0,0,1]>=0, [k,k,0,0]>=j, [k,0,k,0]<=j,
```

```
[0,k,0,k]<=j, [0,0,k,k]<=j):
> print(j, evalf((4*surface(Q1)+4*surface(Q2))/surface(Q)));
> od;
```

Again, a similar program obtain the proportion of the distributions of the attributes over the population in which the only winning coalition is the entire population. From these two sets of output, it is easy to calculate the proportion of the distributions of attributes over the population which have more than one minimum winning coalition. The output of these programs is shown as a graph in the main part of the text.

Similar programs where written for the other cases discussed in the main part of the text.

References

[1] Garrett Birkhoff, *Lattice theory*, 3 ed., American Mathematical Society, Providence, RI, 1967.

[2] Matthias Franz, *convex - a Maple package for convex geometry, Version 0.91*, Konstanz, 2001, available via internet at `http://www.mathe.uni-konstanz.de/~franz/convex/`.

[3] Bernhard Ganter and Rudolf Wille, *Formal concept analysis: Mathematical foundations*, Springer-Verlag, New York, NY, 1999.

[4] Richard P. Stanley, *Enumerative combinatorics*, vol. 1, Cambridge University Press, Cambridge, UK, 1997.