Hollow Victory: The Minimum Winning Coalition*

RUSSELL HARDIN

University of Maryland, College Park

Introduction

More than a century ago, in 1868, former Representative Ransom Gillet (Democrat of New York, 1833–37) recalled that

the way to enact a tariff bill was to provide protection for the local interests of enough representatives to ensure the bill’s passage. “Interest and not principle,” he said, “determines what shall be done. If votes from Louisiana and Texas are needed, sugar will come in for favor. If support is needed from Illinois, Wisconsin, Minnesota and Michigan, lead, copper and pine lumber are provided for. If the votes of Pennsylvania are wanted, coal and iron receive full attention. . . . The principle of protection under a tariff never expands beyond the objects necessary to carry a bill.”

In The Theory of Political Coalitions, William Riker proposed his “size principle,” according to which: “In n-person, zero-sum games, where side-payments are permitted, where players are rational, and where they have perfect information, only minimum winning coalitions occur.”† Riker’s proof of the

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principle uses a clever incentive argument which allows him to escape from the morass of game theoretic solution theory. Over the past decade, the size principle has entered theoretical analyses as a basic assumption, and it has spawned numerous empirical studies. It


Further references are given in Riker and Ordeshook, Positive Political Theory, pp. 192–194. In addition, Riker presents anecdotal evidence in support of the size principle from the collapse after victory of the wartime grand coalitions of 1815, 1917, and 1945; from the American presidential elections of 1824 and 1844; and from the collapse after victory of recent national independence movements in colonial states. See further, Positive Political Theory, pp. 188–191, 194–196, and 199–201; and Riker, Political Coalitions, chap. 3.

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has also been subjected to serious criticism.\textsuperscript{5} In what follows I will argue that, contrary to Gillett's common sense and Riker's game theory, the size principle should be expected to fail in application. The first part of the argument will be that the range of the theoretical validity of the principle is far more restricted than is apparent from Riker's proofs and discussions of it. In particular, it is valid only for a very narrow class of all zero-sum symmetric games (the class of supersymmetric games, which will be defined below) and for a similarly narrow class of all zero-sum asymmetric games (which are vaguely the asymmetric equivalent of supersymmetric games). The incentive argument for the size principle fails for the more general classes of all symmetric and asymmetric games.

The second part of the argument will be more general and hence more succinct. It appears that any model which predicts a tendency toward minimum winning coalitions would be theoretically insignificant and empirically untestable for games in which \( n \) is very large, as for instance in the House of Representatives. This follows because in such games almost all winning coalitions which are possible will fall within a few per cent of minimal size merely by chance.

After briefly discussing the game theoretic context of the size principle, I will first address theoretical issues and then empirical issues. Throughout, the concern is exclusively with zero-sum games, but the zero-sum condition will not be constantly reiterated.

An \textit{N-Person Zero-sum Game}. In order to facilitate discussion of the derivation of the size principle, it will be useful to refer to an actual game. On the other hand, in order to maintain the reader's patience, I will restrain the urge to create ever more beautiful games to demonstrate ever more beautiful points. Hence, the reader will be burdened with only one tedious game matrix.

Consider the 5-person zero-sum game whose payoffs are allotted by majority decision from the set of outcomes presented in cursonry form in Game 1:

<table>
<thead>
<tr>
<th>Player</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outcome I</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Outcome II</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>-10</td>
<td>-50</td>
</tr>
<tr>
<td>Outcome III</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>-50</td>
<td>-10</td>
</tr>
</tbody>
</table>

If all possible permutations on Outcome II are included, the game is symmetric.\textsuperscript{6} The only restrictions on the payoffs are that they sum to zero; that the majority coalition will allot to its members payoffs which sum to the value of the coalition; and that the payoffs be subject to the constraint of individual rationality, that is, no player will accept less than his security level. The \textit{security level} of a player is the best payoff which he can unilaterally guarantee for himself even against a coalition of all the other players. The security level of players in Game 1 is -50. The \textit{value}, or \textit{characteristic function}, of a coalition is the total payoff which the coalition can guarantee its members no matter what players outside the coalition do. The possible majority coalitions in Game 1 include ten 3-player coalitions, five 4-player coalitions, and one 5-player coalition. The value of each 3-player coalition is 60. The value of each 4-player coalition is 50. And the value of the 5-player grand coalition, of course, is zero. The value of coalition ABC is represented as \( v(ABC) \).

\textbf{Symmetry in Games.} In general a game is said to be \textit{symmetric} if the value of a coalition is a function only of the number of players in it. For instance, in Game 1 above, all 3-person coalitions have a value of 60. All 1-person coalitions have a value of -50. The condition for symmetry will be met if the payoff configuration faced by one player is identical to that faced by every other player, in which case the game will be called \textit{identically symmetric}. The condition of symmetry can also be met under other payoff configurations, as for example if three players must choose by majority agreement among the three outcomes \((1,1,-2), (2, -2,0), \) and \((-2,1,1)\). In this game, the 3-person coalition has the value of 0; all 2-person coalitions have symmetric values.


6 All possible permutations can be generated by switching the labels of the players through all orderings. With five players, these number 5!, or 120 outcomes.
tions have the value of 2; and all 1-person coalitions have the value of –2. In this game, the payoffs are not symmetric, but the values of coalitions are. In order to determine whether a game is symmetric, it is necessary to know only the characteristic function form of the game, i.e., the values of coalitions. To determine whether a game is identically symmetric, it is necessary to know much more — the full matrix of payoffs.

Riker’s Assumption of Symmetry. Riker tacitly introduces symmetry when he presents the graphs of Figure 1, which plot the values of coalitions against their sizes. On the horizontal axis are the sizes of coalitions ranging from \( m \) to \( n \), the total number of players in the game. On the vertical axis are the values, or characteristic functions, of the coalitions. One should not be confused by the multiplicity of curves actually plotted — each curve simply represents a typical game. For example, the flat curve labeled II represents a game in which every winning coalition has the value (or collective payoff) \( \gamma \), except that the grand coalition of all \( n \) players has the value 0 (because the game is zero-sum). The payoff to the corresponding losing “coalition” is always the negative of the payoff to the winning coalition.

A conspicuous point of these graphs is that they can be plotted as single-valued curves only if all coalitions of a given size have the same value, i.e., if the game is symmetric. If in Game 1 the only outcome possible were that presented as Outcome II without allowing permutations, then possible 3-person coalitions would have the values 60, 30, –10, and –40; and 4-person coalitions would have the values 50, 10, and –40. Hence, this restricted game cannot be fitted into Riker’s graphs.

Butterworth’s and Shepsle’s Assumptions of Symmetry. Butterworth likewise tacitly introduces the condition of symmetry when he presents symmetric counterexamples to Riker’s size principle, and when he refers to “games with a nonpositively-sloped characteristic function” such as those represented by curves I and II in Figure 1. In his proof of his own “maximum number of positive gainers” principle, he does not invoke or need the symmetry condition. But he concludes by suggesting that “a most desirable line of development is to generalize the analysis of [his] principle beyond simple symmetric games.” A simple generalization is easily deduced and will be noted below. Indeed all that is required is to restate Butterworth’s principle in words as general as his elegant proof.

In a recent defense of the size principle, Kenneth Shepsle explicitly assumes symmetry — “without seriously jeopardizing the argument,” he says. He is right insofar as he means to say that an assumption which is necessary to an argument cannot sensibly be said to jeopardize it. But the assumption does jeopardize the applicability of any size principle which is based upon it, since it is an assumption which may not be met in contexts which interest us. Shepsle calls the assumption of symmetry “an anonymity condition,” in that “the labels identifying the players in a coalition are not relevant to a determination of a coalition’s worth . . . .” The notion has a persuasive ring, but it is not the commonplace “anonymity condition” of conventional game theory, which is meant to apply even to games which are not at all symmetric. In its conventional version, anonymity is a condition on players’ abilities, not on the structure of payoffs in the game. Shepsle’s condition is strictly a condition on the latter, whereas the conventional condition is a way of saying that we are not interested in the differential skill of the players but only in the logical structure of the game. For example,

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Figure 1. Size and Value of Winning Coalitions


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\( m, y \) (1) (3)

\( m, y \) (1) (3)

---

\( \gamma \) 2\( \gamma \)

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(0) (2) (3) (4)

---

0 \( m \) \( m+1 \) \( m+2 \) \( n-2 \) \( n-1 \) \( n \)

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\( m, y \) (1) (3)

---

(0) (2) (3) (4)
for a given 2-person game payoff matrix, our analysis of the rationality of various strategies is anonymous in that it does not take into consideration whether the row player is Buster Keaton, Anatol Rapoport, or a sophomore psychology student at a large state university.10

Theoretical Issues

In what follows I will first show that Riker's proof is insufficient. Then I will show that for the narrow class of supersymmetric games, Riker's incentive argument yields a valid proof of the size principle. Then I will generalize from the condition of supersymmetry to show that the principle is also valid for a highly restricted class of asymmetric games. Finally, I will comment on Butterworth's critique of the principle and his modification of it, on Shepsle's recent "favorable impression of the size principle's validity,"11 and on von Neumann and Morgenstern's simple games.

Insufficiency of Riker's Proof. Riker's proof of the size principle is based on a consideration of the possible shapes of the graphs in Figure 1. He suggests that it is necessary to corroborate the principle for three cases, according as

I the slope is always negative,
II the slope is zero or negative,
III the slope is positive in part and negative in part.12

Riker concludes that the principle cannot be proved for case III, but that it is definitive for cases I and II. The actual proof for these two cases, however, is strictly algebraic rather than geometric. For case I, if S is a coalition which is winning, then by definition

\[ v(S) > v(S + i). \]

Riker claims that "it is always advantageous, regardless of the division of [the payoffs to the members of a larger than minimum winning coalition], (S + i), for the s members of S to eject person i."13

There are two points to be made about this claim before we turn to the proof of the size principle. First, the claim is not quite accurately worded, and as stated it may well be false.14 The correct wording is:

If \( v(S_{r-1}) \geq v(S_r) \), then some \((r-1)\)-member subcoalition of \( S_r \) has incentive to evict the other member.

Lemma 1

In subsequent discussion, it will be assumed that Riker's proof is based on the statement in Lemma 1, which is proved in the following paragraph.

PROOF OF LEMMA 1. Suppose \( S_r \) is a larger than minimum winning coalition. If the game is symmetric, then every coalition of \( r \)-1 players is also winning. Suppose \( v(S_{r-1}) \geq v(S_r) \) by hypothesis. Let A, B, \( \ldots \) be the players in \( S_{r-1} \) and let their payoffs in \( S_r \) be \( p_A, p_B, \ldots \). Suppose that the payoff to A is at least as great as the payoff to any other player in \( S_r \). Since \( v(S_r) > 0 \), it follows that \( p_A > 0 \). Now consider the coalition \( S_{r-1} \) whose members are B, C, \( \ldots \), i.e., which is \( S_r \setminus A \). Let the payoffs to the players in \( S_{r-1} \) be \( p_B^i, p_C^i, \ldots \)

We have,

\[ p_A^i + p_B^i + \ldots = v(BCD \ldots) \geq v(ABCD \ldots) = p_A + p_B + p_C + \ldots \]

or,

\[ p_A^i + p_B^i + \ldots \geq p_A + p_B + p_C + \ldots \]

Hence, \((p_A^i + p_B^i + \ldots) - (p_B + p_C + \ldots) \geq p_A > 0 \). That is, the sum of the payoffs to the players B, C, D, \( \ldots \) in \( S_{r-1} \) is greater than the sum of their payoffs in \( S_r \). Hence, it is in their collective interest to evict A in order to form \( S_{r-1} \).

It follows that, in any symmetric game in which \( S_r \) is a larger than minimum winning coalition and \( v(S_{r-1}) \geq v(S_r) \), some winning subcoalition, \( S_{r-1} \), of \( S_r \) has incentive to reduce the size of \( S_r \) by evicting the extra player. Note that not every winning subcoalition has such an incentive, because a single player, A, whose payoff in \( S_r \) is \( p_A \), can profitably be evicted only if

\[ p_A > v(S_r) - v(S_{r-1}). \]

10 In this respect, the interest of mathematical game theory is contrary to the interest of what are normally called games, such as basketball or chess. In chess, Boris Spassky will find it profoundly important to know whether it is Bobby Fisher or I who sits across from him. If the game theorists ever solve the game of chess (as tic-tac-toe has been solved), chess will lose its interest except among the ignorant (as tic-tac-toe can be played with interest only by the very young and the daft). Of course, game theorists have long been trying, and they may yet succeed. (Also of course, the class of the ignorant and daft may be much larger in the case of chess than in the case of tic-tac-toe.)

11 Shepsle, p. 515.

12 Riker and Ordeshook, Positive Political Theory, p. 182.

13 Ibid., emphasis added.

14 Assume by hypothesis that \( v(S) - v(S^i) = \Delta \), where \( \Delta > 0 \). It is possible that the security level, \( q_1 \), of player i is less than -\( \Delta \). Hence, it is possible that i was making side payments in excess of \( \Delta \) to the members of S. If S now evicts i, its members collectively lose this excess over \( \Delta \). In Game 1 for example, if coalition ABCD has formed with payoffs (26, 26, 26, -28), then ABC cannot afford to evict D, although ABC would be of winning size and \( v(ABC) > v(ABCD) \).
This condition can be put into words which make the point clearer: If, in a larger-than-minimum-winning coalition, \( S \), the sum of the payoffs to the members of some winning sub-coalition is less than the value of the subcoalition, then that subcoalition has incentive to form in the place of \( S \). The conditional of this statement will always be fulfilled for symmetric games in Riker’s Cases I and II, and it can be met even in games which fall into Riker’s Case III (the slope of the characteristic function is positive in part).

The second point to be made about Riker’s claim (Lemma 1) is that, although he does not say so, Riker evidently assumes that demonstration of the validity of this claim is prima facie sufficient to prove the size principle (for Case I games). It should be clear, however, that the validity of Lemma 1 is at best only necessary for proving the size principle. According to Lemma 1, some winning subcoalition of any larger-than-winning coalition has incentive to reduce the size of the coalition. But then it might still happen that any coalition of minimum winning size has incentive to expand its size. In order finally to prove the size principle with his line of argument, Riker would have to show that this latter is not the case. But it may well be the case, as can easily be shown for Game 1.

Because Game 1 is a majority rule game, coalition ABC can by itself determine which of the three represented outcomes will occur. If ABC selects Outcome III, D loses 50 units. Hence, ABC might make D an offer: we will select Outcome II if you will make us a side payment of roughly half the difference in your payoffs at these two outcomes. That difference is 40 units. Suppose ABC lets D have 22 units of that difference and asks for 6 units to each of its three members. The full set of payoffs (with side payments calculated in) would then be \((26,26,26,28,50)\). Note that this set of payoffs requires the cooperation in coalition of four players, one more than minimum winning, and all four members of the coalition are better off than they might have been outside the coalition.

Since Game 1 is identically symmetric, it follows that every minimum winning coalition in the game has incentive to expand from three players to four. That is to say that the incentive argument which Riker uses to demonstrate that only minimum winning coalitions occur can likewise be used in this game to demonstrate that only larger than minimum winning coalitions occur. Hence, this proof is insufficient reason to accept the size principle.  

**The Size Principle in Supersymmetric Games.**

Riker turned to his incentive argument in order to escape the endless circularities of the von Neumann-Morgenstern solution theory, and still there is circularity. But I have only demonstrated that for at least one game which meets the definition of Riker’s Case I, the incentive argument fails because it produces an infinite circular chain of expansions and contractions of coalitions. The question remains whether the demonstration generalizes to all games. The answer is that for one narrowly defined class of games, which can be called supersymmetric, Riker’s incentive argument does not lead to circularity, but for apparently all other symmetric games it does. Hence, the size principle can be applied only to supersymmetric games, as will be argued below.

If for some game the incentive argument is not to lead to endless circularity, it must happen that in that game minimum winning coalitions do not have incentive to expand. In general there will always be incentive for a winning coalition to expand except when the value of that coalition is equal to the negative sum of the security levels of the individual losers. That is, always except when there is nothing more to be squeezed out of the losers.

If the number of players in a game is \( n \), the number of players in a winning coalition \( S \) is \( r (r < n) \), the payoff to player \( j \) is \( p_j \), and the security level of player \( i \) is \( -q_i \), then the sum of the winnings of the winners is

\[
v(S) = \sum_{j \in S} p_j \leq \sum_{i \in S} q_i.
\]

In symmetric games, all players have the same security level, \(-q\), so that eqn. 1 reduces to

\[
v(S) = \sum_{j \in S} p_j \leq (n-r)q.
\]

Hence, if

\[
\sum_{j \in S} p_j < (n-r)q,
\]

the players in \( S \) have incentive to expand their coalition.

Note that there may be an incentive for winning coalitions of some sizes to expand and not for others. For example, there is obviously no in-

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15 Norman Frohlich has presented a related but substantially more elegant and more general demonstration of this conclusion in “The Instability of Minimum Winning Coalitions.”
centive for a coalition of \( n-1 \) players to expand, because its value is \( q_i \), and the single loser has nothing more to give. For symmetric games, the condition under which no winning coalition has an incentive to expand is

\[
v(S_r) = (n-r)q_i
\]

for all \( r \) for which \( S_r \) is winning. It is easily shown that this reduces simply to \( v(S_m) = (n-m)q_i \), where \( m \) is the minimal winning number.\(^{16}\)

We can call a game which meets the condition in equation 2 \emph{supersymmetric}. For such games, Riker's incentive argument for the size principle has great appeal: in a supersymmetric game no winning coalition has incentive to expand, but every larger than minimum winning coalition has incentive to contract down to minimal winning size. The condition of supersymmetry is \emph{sufficient} for the size principle to hold. It is also \emph{necessary} to prevent the incentive argument from leading us into endless circles.\(^{17}\)

A game can be supersymmetric without being identically symmetric, and vice versa. An easily represented class of supersymmetric games, however, is the class of identically symmetric majority rule games whose matrices of outcomes include all permutations on the outcomes \( (n-m, n-m, \ldots, -m, -m, \ldots) \), where the payoff \( n-m \) accrues to \( m \) players, the payoff \(-m\) accrues to \( n-m \) players, and

\[
\frac{n}{2} < m \leq \frac{n+1}{2}
\]

(In order to let the world move on to other concerns in the event of extended lack of agreement, it is sensible also to include in the matrices an outcome in which each player's payoff is 0).

A Supersymmetric Example. To show the appeal of Riker's argument for a case in which the condition of eqn. 2 is met, consider a game which is a modification of Game 1. Let the game be identically symmetric and majority rule, but let the payoffs which are negative in any outcome be equal at -30 (instead of the -10 and -50 as shown). Every non-zero outcome is therefore a permutation on the payoffs \((20, 20, 20, -30, -30)\). Suppose the coalition ABCD has formed with payoffs to its members \((20, 20, 20, -30)\). The value of the coalition is 30. Rather than create a new term, let us follow Butterworth and refer to a member of a winning coalition whose payoff is negative as a \emph{negative gainer}.

Now, according to Riker's argument, three members of ABCD have incentive to expel the fourth member. At first glance, it might seem that ABC should expel D, the negative gainer, but this first glance is deceiving. Since the distribution of payoffs to the coalition is \((20, 20, 20, -30)\), the expulsion of D will gain ABC nothing. ABC has no incentive to expel D. Oddly, however, any subcoalition of D with two positive gainers has incentive to expel the third positive gainer. For example, if B, C, and D collude to expel A, they will gain 50 units to divide among themselves, so that all three can easily be made substantially better off.\(^{18}\)

As the new winning coalition, however, BCD has no incentive to expand by adding A or E.

It is easy to see that, beginning from any initial winning coalition in this game, and given any division of the spoils within that coalition, Riker's incentive argument will lead to a point at which there is no longer any incentive to expand or contract, and that point will always be when a three-person or minimum winning coalition has formed. Of course, it would still and would always be the case that each (or at least \( n-1 \)) of the individual players has incentive to strive for a larger payoff through coalitional realignment or threat of realignment.

Asymmetric Games. Eqn. 1 suggests another possibility, which is that

\[
v(S_r) = \sum_{i \in S_r} q_i
\]

in a game which is \emph{not} symmetric. As an example of an asymmetric game, consider again Game 1 under majority rule with the \emph{only} outcomes being those actually given and not including permutations on these. In this game, players A, B, and C have security levels of 0; D and E have security levels of -50. The value of ABC is 60; the negative sum of the security levels of D and E is 100. Clearly, ABC has incentive to expand.

But any three-player coalition which includes D and E has no incentive to expand. Consider CDE, which has the value of 0. The losers, A and B, have security levels summing to 0. In this game Riker's incentive argument leads any
initial winning coalition through an inexorable series of expansions and expulsions which ends at a three-player coalition including D and E which has no incentive to do anything further. Hence, the size principle applies to this game. And there is Schellingesque irony in the end result: the two apparently weakest players are sure to be in the winning coalition.

As a rule, it is more difficult to give general analyses of asymmetric than of symmetric games. For symmetric games it was moderately simple to characterize the class of all games to which the size principle evidently applies. For asymmetric games, the task is more tedious. Clearly, a necessary condition for the size principle to hold in an asymmetric game is that eqn. 3 be fulfilled for some coalition of minimal winning size. A sufficient condition is that it be fulfilled for every coalition of winning size. The example in the foregoing paragraph, however, shows that this latter sufficient condition is not a necessary condition.

The class of games which meets the condition of eqn. 3, either for all winning coalitions or merely for some minimum winning coalition, is conspicuously narrow compared to the class of all games (it is trivially easy to construct asymmetric games which do not meet the condition, and a random zero-sum matrix generator might work for days before it cranked out a matrix that met it). Perhaps this class is as peculiarly narrow in the space of all majority rule games as the class of supersymmetric games is in the space of all symmetric majority rule games. In any case, it is at least clear that Riker’s incentive argument produces endless circularity in an enormous array of asymmetric games, but that it leads to the size principle in a narrowly defined class of asymmetric games.

Butterworth’s Modification of the Size Principle. Robert Butterworth has perceived from counterexamples that the size principle is not well founded, and he has suggested a modification which would salvage much of its import. He recommends and proves a principle of the “maximum number of positive gainers,” according to which: “In n-person, zero-sum games with a nonpositively-sloped characteristic function, where side-payments are permitted, where players are rational, and where they have perfect information, the largest possible number of positive gainers that can occur is the number of players necessary to comprise a minimum winning coalition.”

Despite the geometric notion of “a nonpositively-sloped characteristic function,” Butterworth demonstrates his principle algebraically rather than geometrically. If we generalize the assumptions involved in his proof, we can restate his principle in a fuller form without implicitly restricting it to symmetric games:

The number of players necessary to compose a minimum winning coalition sets the upper limit on the number of positive gainers in any n-person zero-sum game where side-payments are permitted, where players are rational and have perfect information, and where, if S is a winning coalition, then for every i not in S,

$$v(S) \geq v(S+i).$$  (4)

The final condition in this statement can be met by some games which are not strictly symmetric, that is, whose characteristic functions are not single-valued functions of coalition size. Games which meet this condition can be called “quasi-symmetric.”

In general, to assume that a genuine social interaction is analogous to a symmetric game is to assume that much of social structure is irrelevant to explanations of coalition formation. Unless we live in a state of anarchy, we are likely to play symmetric games only in the laboratory. Hence, it is useful to ask just how asymmetric a game can be while still qualifying as quasi-symmetric and thus coming under Butterworth’s principle.

Consider a perfectly symmetric game of n players whose payoff structures are identical. Now let various payoffs be altered marginally while maintaining the zero-sum condition. If some players enjoy marginal gains while others suffer marginal losses, winning coalitions of a given size, say r, will no longer all have identical values. Rather, there will be a marginal spread from the maximum, $$v_{\max}(R)$$, to the minimum, $$v_{\min}(R)$$. Butterworth’s logic would now not be violated so long as the minimum value at size r is greater than or equal to the maximum at size r+1, i.e.,

$$v_{\min}(R) \geq v_{\max}(R+1).$$

This condition will determine the amount of variance from symmetry permitted in individual players’ payoffs. Other things being equal, one would expect that the larger the number of players in the game, the less the variance which

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18 The 50 units are the difference between the value of coalition BCD (which is 60) and the sum of the payoffs to B, C, and D in coalition ABCD (which is 19).

would be permitted. For situations in which social structure matters very much, the condition of quasi-symmetry might be met in very small committees but not in large committees. Hence, Butterworth's version of the size principle might apply to coalition formation in such small bodies as the United States Supreme Court, but not to such large bodies as the United States House of Representatives (see further the discussion in the section on empirical issues).

Shepsle's Defense of the Size Principle. Endless circularity such as that implicit in the demonstration of the "Insufficiency of Riker's proof" (above) is the bane of game theory. All too often, if there is a second outcome which is superior to a first, then there is sure to be a third which is superior to the second but inferior to the first. Shepsle's apparent recognition of such paths of reasoning has led him to seek justifications for the size principle other than Riker's argument from incentives. Shepsle adduces an argument for a perfectly symmetric 5-person game (the game is not supersymmetric). But the argument is extraordinarily tenuous and abstruse: it is based on the notion of the "bargaining set" of Aumann and Maschler. Riker's argument from incentives is substantially more compelling, as is perhaps even Representative Gillet's assertion from common sense.

When it comes to generalization beyond the 5-person case, Shepsle notes that "although existence results have been provided for stable sets in an impressive variety of games, the results, to a nongame theorist, appear highly incomplete and idiosyncratic." The results are impressive for their variety only because such variety indicates how ungeneral and ungeneralizable the results are. And this pessimistic realization is likely to impress game theorists far more strongly than nongame theorists. Yet Shepsle concludes that, if "the usual assumptions about n-person zero-sum coalition processes are supplemented with assumptions about coalition intentions and capabilities, there are good reasons to expect minimum winning coalitions in all but the most extreme instances." This claim is not supported by Shepsle's paper or any of his references. The strongest claim which is supported is that there are weak reasons to expect minimum winning coalitions in the instance of perfectly symmetric games with not more than five players. (With fewer than five players the problem is uninteresting: In majority rule games of fewer than five players, the losers cannot number more than one.)

Against Shepsle's claim for our good reasons to generalize from the 5-person case, von Neumann and Morgenstern hint of awe when they discuss the enormous rapidity with which the complexity of the analysis of games grows as the number of players increases. With each additional player there is qualitative change. For three players there is a unique zero-sum game (it is symmetric). For four players, the space of all zero-sum games is 3-dimensional; for five players, 10-dimensional; for nine players, 246-dimensional — it seems absurd to contemplate the nature of such games. To turn to the symmetric games with which Shepsle deals, the 4-person symmetric game is unique. The space of the 5-person symmetric games is one-dimensional; of 9-person games, 3-dimensional; of 101-person symmetric games (the class which the United States Senate might be thought to play when in full attendance), 49-dimensional. Above five persons,
game theory virtually collapses. The results are so few and so trivially narrow that Riker had no hope of deriving a general theory of coalitions within extant game theory (for more than two persons, game theory is largely a theory of coalitions). He therefore intelligently abandoned game-theoretic solution theory and resorted to an ad hoc but intuitively plausible argument from incentives, an argument which is commonplace in economics. The thought of reverting to solution theory at its present state of development cannot be seriously entertained by anyone whose purpose is as ambitious as Riker's effort to generalize about an important, broad class of political phenomena.

Addendum on Simple Games. Supersymmetric games resemble von Neumann and Morgenstern's simple games, but they are not equivalent. In supersymmetric games, as in simple games, losing coalitions are flat (see footnote 16) — that is, every player in a losing coalition loses as much as if he were standing alone against a coalition of all remaining players. Simple games differ from supersymmetric games, however, because in simple games every coalition is either winning or losing. It follows that if $S$ is a coalition in a simple game and $T$ is its complement, then either $S$ or $T$ is winning and the other is losing.

Hence, straightforward majority voting games for even numbers of players cannot be simple, although they can be supersymmetric. Similarly, majority games which require over-large majorities (e.g., two-thirds) can be supersymmetric but not simple. Majority games for odd numbers of players are called direct majority games by von Neumann and Morgenstern. Such games can be both simple and supersymmetric.

In a phrase of von Neumann and Morgenstern, a losing player is "completely defeated" if his payoff is his security level. We can generalize this descriptive phrase into an instructive definition: a coalition is completely defeated if its value is the sum of the security levels of its members, i.e., if it is flat. Obviously, a single player coalition is completely defeated when opposed by a coalition of all other players. In a simple game, every losing coalition is completely defeated. In a supersymmetric game in which a winning coalition has formed, the losing coalition is completely defeated. If we plot the values of winning coalitions by size for a simple or a supersymmetric game, the points will fall on the topmost graph in Figure 1.

The very fact that losing coalitions in a game are completely defeated allows the size principle to be applied to the game. Among symmetric games it applies only to the supersymmetric games, which include the symmetric cases of simple games, i.e., the direct majority games. Since von Neumann and Morgenstern gave more than 80 pages over to simple games, one might expect that Riker would have pursued their discussion in his derivation of the size principle, as in fact he appears to have done. After briefly mentioning simple games, Riker comments that they "are probably rare in nature," and "hence, little of practical value is likely to result from studying them."

Riker's dismissal was perhaps too hasty, because the size principle is of little greater "practical value" than the study of simple games. Indeed, supersymmetric games are little more than an extension of the symmetric simple games to include games with even numbers of players and games in which winning coalitions must be larger than merely simple majority. The validity of the size principle is restricted to just such games (and to their asymmetric counterparts). But there seem to be no clear a priori grounds for declaring that such a restricted size principle is useless — as Riker's remark suggests it is.

Summary. In sum, both Riker's and Butterworth's claims appear to have analytical merit. But the merit of each is highly circumscribed. Under the terms of Riker's proof, his size principle can apply only to the rarefied class of supersymmetric games and their asymmetric counterparts. As Riker has well understood, even when the principle applies, there may be enormous coalitional instability. Butterworth's principle applies to the class of all asymmetric zero-sum games as well as to those asymmetric games which meet the condition of equation 4, i.e., to quasi-symmetric games. Again, there may be enormous coalitional instability.

It should be clear that both principles apply to some games (the supersymmetric games). And it can easily be shown that each principle applies to games to which the other does not apply. The size principle applies to those asymmetric games in which the number of positive gainers will necessarily be more than minimal winning size because their security levels are

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21 Ibid., p. 423.
22 See further, ibid., p. 428.
23 Ibid., pp. 420–503.
greater than zero. And Butterworth's principle applies to all symmetric games.

Butterworth's principle is somewhat more general. But it is concomitantly perhaps less interesting, since it is not about coalitional size but about the number of positive gainers in a game. Hence it requires a clear notion of what constitutes a positive gain. Ironically, one of the greatest appeals of Riker's initial statement of the size principle was that it specified characteristics of a winning coalition rather than the apportionment of winnings within that coalition. As Anatol Rapoport notes, the results of game theory have been almost exclusively concerned with apportionment; but behavioral scientists are more typically concerned with coalitional structure. In politics, coalitions are conspicuously more tangible than payoffs (there may be exceptions, such as in the politics of graft). In game theory, the reverse is generally true. "In fact, seldom if ever can the behavioral scientist define payoffs in a conflict situation precisely enough to be realistically concerned with the question of their apportionment." By modifying Riker's principle to require explicit measurement of payoffs, Butterworth has reasserted the inutility of game theory for generating predictive theories of political behavior.

It seems unlikely that Butterworth's principle can be applied outside the laboratory. It was a virtue of Riker's principle that it seemed testable in real situations. Alas, Butterworth's principle may be untestable, while the set of games to which Riker's principle can be applied may be too restricted to be of interest, as Riker's dismissal of simple games suggests.

**Empirical Issues**

Shepsle comments that "it is extremely unfortunate that much of the research on and criticism of the size principle is empirical in nature." On the contrary it would seem that this is the point of a contribution to "positive political theory": to be empirically researched and criticized. If there is misfortune in the empirical work, it is that there is enormous disagreement over the empirical results. In positive theory, factual claims should be either true or false — not both.

A principal reason for disagreement among researchers has been a general failure to propose what constitutes a fair test of the size principle. Those who assert that coalitions in legislatures tend to be of minimal winning size do not give us a base, or null model, for comparison. We need to know what might happen if the theory failed to apply. Since the size principle has the ring of a statistical assertion, it should be possible to find statistical null models against which to compare data from actual games such as legislative coalition formation. For instance, a null model which seems much too easy would be a mean winning coalition size halfway between minimal winning and unanimous size. If the data showed that the mean winning coalition size were merely closer to a bare majority than to unanimity, then the size principle would be supported; otherwise it would be rejected. Perhaps many of us would be surprised if even a Congress composed of all Democrats or all Republicans failed this test. Hence, we would want the size principle to survive a harsher test, such as the random voting behavior model which follows.

**A Random Legislature**. Assume for a moment that we have a legislature in which the conditions of Riker's principle are met and which is supersymmetric. Every vote by this body determines a winning coalition and a losing coalition. Each player, or legislator, faces the same matrix of outcomes as every other player. Assume also that each player is indifferent about which players join in his coalition — his concern is to be a winner and to gain as much as possible. (This assumption is implied in Riker's statement of his principle.) Under this set of conditions, the size principle is theoretically deducible. The assumption which Riker considers crucial in political analysis is that of individual rationality. It is this assumption which precludes larger than minimal winning coalitions. Let us therefore drop it for the moment and simply look at all winning coalitions which might form, including those which are not minimal winning.

If the legislature is not extremely small, the set of all winning coalitions in it will include many coalitions of minimal winning size, but only one coalition of the whole legislature. A plausible null model under the conditions of Riker's principle is one in which all winning coalitions are equiprobable. If every legislator flipped a coin to decide whether to join the heads or the tails coalition on each vote, the distribution of winning coalitions by size would fit this null model. This distribution is presented in Table 1 for legislatures ranging from

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33 Rohde, "A Theory of the Formation of Opinion Coalitions in the U.S. Supreme Court," is an exception.
Table 1. Percentage of Winning Coalitions by Size of Legislature and
Percentage of Members in the Winning Coalitions

<table>
<thead>
<tr>
<th>Size of Legislature</th>
<th>50-55%</th>
<th>50-60%</th>
<th>50-65%</th>
<th>50-70%</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>62.5%</td>
<td>62.5%</td>
<td>62.5%</td>
<td>82.0%</td>
</tr>
<tr>
<td>9</td>
<td>49.2%</td>
<td>49.2%</td>
<td>92.1%</td>
<td>97.3%</td>
</tr>
<tr>
<td>15</td>
<td>69.8%</td>
<td>69.8%</td>
<td>99.0%</td>
<td>99.9%</td>
</tr>
<tr>
<td>21</td>
<td>81.1%</td>
<td>92.1%</td>
<td>99.6%</td>
<td>**</td>
</tr>
<tr>
<td>41</td>
<td>88.3%</td>
<td>97.2%</td>
<td>99.9%</td>
<td>**</td>
</tr>
<tr>
<td>61</td>
<td>92.8%</td>
<td>99.0%</td>
<td>99.9%</td>
<td>**</td>
</tr>
<tr>
<td>81 (Senate)</td>
<td>95.5%</td>
<td>99.6%</td>
<td>**</td>
<td>**</td>
</tr>
<tr>
<td>101</td>
<td>97.2%</td>
<td>99.9%</td>
<td>**</td>
<td>**</td>
</tr>
<tr>
<td>201</td>
<td>99.7%</td>
<td>**</td>
<td>**</td>
<td>**</td>
</tr>
<tr>
<td>301</td>
<td>**</td>
<td>**</td>
<td>**</td>
<td>**</td>
</tr>
<tr>
<td>401 (House)</td>
<td>95.4%</td>
<td>**</td>
<td>**</td>
<td>**</td>
</tr>
</tbody>
</table>

* The smallest winning coalition has more than 55% of the membership.
** More than 99.95%.

5 to 401 members in which a simple majority wins. The United States Senate would correspond roughly to a body of 81 members, and the House of Representatives to a body of 401 members, depending on attendance. In Table 1, the columns present the percentage of all winning coalitions which include more than half but less than 55, 60, 65, and 70 per cent, respectively, of the members. So for instance, in an 81-member legislature, 99.6 per cent of all winning coalitions include less than 65 per cent of the legislators.

From Table 1, it is clear that in large legislatures, the overwhelming majority of all winning coalitions will be near the minimal size just by statistical chance in this simple-majority, random-voting legislative model. Admittedly, this is a crazy model of legislative voting behavior. Traditional discussions of congressional voting are concerned with those structural and attitudinal variables which prevent such random behavior. But the size principle abstracts from such traditional concerns to analyze voting as strictly an individualist effort to maximize utility. To test the size principle is to determine whether such an abstraction is valid in a particular instance, as is done for the House of Representatives below.

House Roll-call Voting. David Koehler has analyzed roll-call voting in the United States House of Representatives and claims to “demonstrate empirically that the members do tend to form minimal winning coalitions.” What he actually demonstrates is that the average size of winning coalitions on contested roll-call votes does not correlate very well with the size of the dominant party’s majority. In order to determine more directly whether the size prin-

Table 2. Frequency of Occurrence of Majorities by Size

<table>
<thead>
<tr>
<th>Size of majority</th>
<th>House of Representatives (83rd–90th Congresses)*</th>
<th>Random voting in a 401 member body</th>
</tr>
</thead>
<tbody>
<tr>
<td>%</td>
<td></td>
<td>%</td>
</tr>
<tr>
<td>50-55</td>
<td>347</td>
<td>21.8%</td>
</tr>
<tr>
<td>55-60</td>
<td>264</td>
<td>16.6%</td>
</tr>
<tr>
<td>60-65</td>
<td>211</td>
<td>13.3%</td>
</tr>
<tr>
<td>65-70</td>
<td>170</td>
<td>10.7%</td>
</tr>
<tr>
<td>70-75</td>
<td>145</td>
<td>9.1%</td>
</tr>
<tr>
<td>75-80</td>
<td>121</td>
<td>7.6%</td>
</tr>
<tr>
<td>80-85</td>
<td>120</td>
<td>7.5%</td>
</tr>
<tr>
<td>85-90</td>
<td>83</td>
<td>5.2%</td>
</tr>
<tr>
<td>90-95</td>
<td>132</td>
<td>8.3%</td>
</tr>
</tbody>
</table>

* In “The Legislative Process and the Minimal Winning Coalition,” Koehler presents the data graphically; hence, these figures are taken from David H. Koehler, “Coalition Formation and the Legislative Process” (paper presented at the American Political Science Association Annual Meeting, Chicago, 1971), p. 48. The total number of House roll-call votes listed is 1,593. In addition there were 450 votes in which the majority exceeded 95%; these votes have been omitted from the percentages above, which therefore sum to 100 per cent.

ciple fits House voting behavior, we would want to compare the actual frequencies of coalitions of various sizes with their expected frequencies under a plausible null model, such as the random voting model of Table 1.

If the size principle applies very well to the House, then we should expect that the frequency of larger coalitions would be reduced, and the actual vote distribution should be nearer minimal size than the null model distribution. Table 2 presents Koehler’s House roll-call data for eight Congresses and the null model distribution for a 401-member legislature. The size principle does not seem to apply. Whereas in actuality only 52 per cent of winning coalitions contained less than 65 per cent of those representatives voting, in a 401-member legislature run by flipping coins, 99.999999 per cent of all winning coalitions would contain less than 65 per cent of the membership. Furthermore, whereas very large coalitions are quite common in the House, they would almost never happen in a random legislature.

Perhaps the random legislature is far too severe a test. The House data, however, do not even do well against the extremely easy test above in which the mean winning coalition size is halfway between simple majority and unanimity. If all the data are included (see the note to Table 2), then the mean winning coalition in several years of House roll-call voting comprised between 72 and 77 per cent of those voting, as compared to the easy null model’s 75 per cent.

It appears that legislative voting, at least in the recent House of Representatives, does not sufficiently approximate a symmetric n-person zero-sum bargaining game to permit application of the postulated size principle. It might be that for certain classes of issues, the House would more nearly approximate the conditions of the principle than it does for the general class of all roll-call votes. Unless such a case is found, however, Koehler’s data suggest that what we wish to explain is just the reverse of what the size principle implies, i.e., we should wish to explain why there is such a high degree of agreement on roll-call votes in the United States Congress.

Conclusion. The data in Table 2 imply merely that the conditions for the size principle are not met in the House of Representatives. Suppose there were a legislature in which those conditions were more nearly met. The figures of Table 1 suggest that in a body as large as the House, even if the inordinately stringent conditions for the size principle were actually met, then the principle would be virtually useless on a priori grounds. In a 401-person zero-sum bargaining game with supersymmetry and perfect information, more than 95 per cent of all possible winning coalitions would be within five per cent of minimal size by chance. That is, less than five per cent of all possible winning coalitions include more than 55 per cent of the players. Hence, at best the size principle, if it operated, could reduce this five per cent slack.

It follows that, even for the restricted realm to which it is applicable, the size principle could at most produce a marginal improvement in our expectations of coalition behavior in such a large body. And direct statistical test of the principle in a large legislature would probably be inconclusive at best. Since the conditions for the principle are extremely stringent, and since its import decreases as the size of the body to which it is applied increases, it seems unlikely that the principle is either useful or testable in bodies much larger than committees.

This last point is a conclusion similar to that reached above for Butterworth’s modification of the size principle. Butterworth’s principle might be expected to apply to small bodies such as committees, but not to large bodies. Again, however, Butterworth’s principle requires a measure of gains and losses, so that empirical tests of it will be far more complicated than tests of Riker’s principle. Indeed, it seems utterly implausible that Butterworth’s principle could be tested against House roll-call voting behavior, or even against nine-man Supreme Court behavior. One of the attractions of Riker’s principle is that it is simply and directly testable even though the conditions which make for it are beyond easy test. The conceptual difficulties in testing Butterworth’s principle, on the contrary, seem as severe as those involved in testing the conditions for it.

Summary

The proof of Riker’s size principle is in-
adequate for the general class of zero-sum bargaining games (whether symmetric or asymmetric), and the principle is valid only for a very restricted class of games—the supersymmetric games and their asymmetric counterparts. Butterworth’s modification of the size principle (the maximum number of positive gainers principle) can be extended to cover games which are only approximately symmetric. Roll-call voting in the United States House of Representatives overwhelmingly violates the size principle; hence, the House does not generally play a supersymmetric zero-sum bargaining game. More generally, both Butterworth’s and Riker’s principles seem inapplicable to large bodies. Thus, game-theoretic coalition theory adds little to our ability to explain legislative coalition formation.