Supplemental Appendix for “Challenger Entry and Voter Learning”

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In this appendix, we provide a formal definition of the D1 requirement in our model and show that the conjunction of requirements of sequential equilibrium and D1 rules out the equilibria such that some information sets are never reached on the path of play, including the equilibrium in which no challengers enter, but have no additional bite on the equilibrium we analyze, in which all information sets are reached with a positive probability.

Let $T$ be the set of type-pairs $(t_i, t_c)$, following $C$. Then the set of voter’s best responses, given beliefs $p$ and challenger’s action $C$, $BR(p, C)$, can be written as

$$BR(p, C) \equiv \arg\max_{(\mu, \rho)} \sum_{(t_c, t_i) \in T} p(t_c, t_i|C)E[u_v(t_c, t_i, C, M, R)].$$

The set of all possible voter best responses given any beliefs over set $T$, $BR(T, C)$, is, then, given by

$$BR(T, C) \equiv \bigcup_{\{p: p(T) = 1\}} BR(p, C).$$

Fix an equilibrium with expected payoff to challenger of $u^*_c(t_c, t_i)$. For each $(t_c, t_i, C)$, let $D_{(t_c, t_i)}$ be the set of best responses by the voter that make the challenger of type $t_c$ strictly prefer defection:

$$D_{(t_c, t_i)} = \{(\mu, \rho) \in BR(T, C) : u^*_c(t_c, t_i) < E[u_v(t_c, t_i, C, \mu, \rho)]\},$$

and let $D_{0(t_c, t_i)}$ be the set of best responses by the voter that make $t_c$ indifferent between defecting and not:

$$D_{0(t_c, t_i)} = \{(\mu, \rho) \in BR(T, C) : u^*_c(t_c, t_i) = E[u_v(t_c, t_i, C, \mu, \rho)]\}.$$

The D1 refinement (Cho and Kreps 1987) says: let $C$ be an action off the equilibrium path of play (i.e., an action no challenger takes with positive probability). Then posterior beliefs $p(t_c, t_i|C) > 0$
only if there exists no \((t'_c, t'_i)\) s.t. \(D^0_{\{t_c, t_i\}} \cup D_{\{t_c, t_i\}} \subseteq D_{\{t'_c, t'_i\}}\).

All perfect Bayesian equilibria such that all information sets are on the path of play are sequential equilibria. If all information sets are on the path of play, then sequential equilibrium and D-1 (and other off-the-path of play) refinements have no bite. The following lemma shows that the perfect Bayesian equilibria such that some information sets are never reached on the path of play do not survive the combination of the D-1 and sequential refinement. It follows that the equilibrium we characterize exhausts the set of sequential equilibria with D-1 refinement.

**Lemma 1** In any sequential equilibrium of this game that satisfies D1, every information set is on the path of play (i.e., for some \((t_c, t_i)\), \(C = 1\) with positive probability).

**Proof.** The proof is by contradiction. Suppose there exists a sequential equilibrium that satisfies D1 in which \(C = 0\) for all \(t_c\). \(C = 0\) is an optimal choice for all \(t_c\) if and only if \(\rho = 1\) and \(\mu = 0\) (insuring 0 probability of electing a challenger). Sequential equilibrium requires that \(\rho = 1\) and \(\mu = 0\) be sequentially rational choices, that \(p(t_c|C = 1, M = 0) = p(t_c|C = 1)\) and that \(p(t_i|C = 1, M = 0) = p(t_i|C = 1)\). Thus, \(\rho = 1\) in sequential equilibrium requires beliefs such that \(E[t_c|C = 1] \geq E[t_i|C = 1]\).

Suppose there exists \((\mu, \rho)\) and \((t_c, t_i)\) such that \((\mu, \rho) \in D^0_{\{t_c, t_i\}}\). Because \(C = 1\) entails a cost \(k\), \(t_c\)’s indifference between \(C = 0\) and \(C = 1\) implies (a) that her probability of winning is strictly positive and (b) that \(t_c + b - k \geq t_i\). Suppose \(t'_c > t_c\). Given \(\mu, \rho\), and \(t_i\), the probability that \(t'_c\) is elected is weakly higher than that of \(t_c\), and the latter’s benefit from holding office is strictly greater, while the payoff from losing or \(C = 0\) is the same. Thus for all \(t'_c > t_c\), \((\mu, \rho) \in D_{\{t'_c, t_i\}}\). Suppose \(t'_i < t_i\). Given \(\mu, \rho\), and \(t_c\), \(t_c\) is weakly more likely to be elected against \(t'_i\) than against \(t_i\) and the payoff from losing and from \(C = 0\) is strictly less against \(t'_i\), while the payoff from winning is the same. Thus for all \(t'_i < t_i\), \((\mu, \rho) \in D_{\{t_c, t'_i\}}\). Similarly, if \((\mu, \rho) \in D_{\{t_c, t_i\}}\), then \((\mu, \rho) \in D_{\{t'_c, t_i\}}\) for all \(t'_c > t_c\) and \((\mu, \rho) \in D_{\{t_c, t'_i\}}\) for all \(t'_i < t_i\). The D1 condition on beliefs requires that \(p(t_c|C = 1) > 0\) only if there exists no \(t'_c < t_c\) and \(p(t_i|C = 1) > 0\) only if there exists no \(t'_i < t_i\). Thus, for any beliefs satisfying D1, \(E[t_c|C = 1] > E[t_i|C = 1]\) - a contradiction. ■