Citizen Candidates Under Uncertainty[*]

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Abstract

In this paper I add uncertainty about the total vote count to a "citizen candidate" model of representative democracy. I show that in a society with a large electorate, where the outcome of the election is uncertain and where winning candidates receive a large reward from holding office, there will be a two-candidate equilibrium and no equilibria with a single candidate.

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1 Introduction

In a representative democracy, citizens elect representatives who in turn, choose policies for the society. Traditional models divide the members of the society into two classes: voters, whose only political role is to vote, and politicians or political parties (often just two of them), who compete in the election. Citizen candidate models of the electoral process, on the other hand, explain how politicians emerge from the class of voters. In these models, some citizens become politicians by choosing to run as candidates in an election. The number and policy preferences of the candidates who run in equilibrium are determined by three factors: the policy preferences of every citizen, the benefits of holding office, and the cost of running for election.

The standard citizen candidate models suffer from a simplistic assumption that leads to unrealistic predictions. The assumption is that candidates can perfectly anticipate the outcome of the election and forecast exactly how many votes each candidate will receive. This assumption leads to the prediction that two candidates will run against each other only if they have the exact same number of supporters in the electorate. In models with a finite electorate, this implies that typically there will be a two-candidate equilibrium only if the number of citizens is even, because if it is odd, one of the candidates will generally have at least one more supporter than the other.

I solve this problem and assure political competition in the model by introducing uncertainty in the electoral outcome, and then showing that in a society with a large electorate, equilibria with one candidate do not exist, while equilibria with two candidates always exist. I model the uncertainty as in Myerson [5]: Citizens choose which candidate to support, but each citizen has a small probability of failing to convert her intention to support a candidate into an actual valid, counted vote for the candidate -perhaps the voter is unable to make it to the polling station, or she misuses the voting equipment and casts an invalid ballot. This individual probability of being unable to cast a valid vote generates an aggregate uncertainty about the total vote count.

The aggregate uncertainty about the electoral outcome crucially affects elections with a large electorate. Candidates decide to run based on the support they have in the electorate at the time they make the decision to run. However, elections are not deterministic: the aggregate uncertainty makes the outcome stochastic and a candidate who initially had less support may ultimately collect more votes and win the election. If the most popular candidate enters the race, another citizen with slightly less support will also run, in the hope of an upset victory. As a result, in a society with a large enough electorate, one-candidate equilibria do not exist, whereas two-candidate equilibria always exist, regardless of the exact number of citizens. These predictions are similar to those in Osborne and Slivinski's [6] model of citizen candidates, but Osborne and Slivinski work only with an infinite number of sincere voters, who have rather restricted preferences. My model, by contrast, allows for any finite number of citizens, who vote strategically and may have quite general forms of preferences.

The uncertainty I introduce in my model has a large effect unless the electorate is very small. In the case of a very small electorate the predictions of my model are similar to those of Besley and Coate [1]: The existence of equilibria with one or two candidates depends on whether the number of citizens is odd or even.

There are two other papers incorporating uncertainty to a model of citizen candidates: In Riviere's paper [7], a group of citizens learn their policy preference only after candidacies are announced, so the location of the median is uncertain at the time of the announcement. My model differs from Riviere's both in the assumptions I use and the results I obtain. In words of the author, Riviere's assumptions are "very restrictive" and in her citizen candidate model, two-candidate equilibria are "very rare."[1] I relax and generalize most of Riviere's assumptions to show that under mild conditions, equilibria with two candidates exist.

In a short note, Roemer [8] tackles the problem of indeterminacy of equilibria in Besley and Coate's model, advocating a particular refinement (Party-Unanimity Nash Equilibrium or PUNE) that yields a smaller but non-empty set of equilibria. Uncertainty is only a side issue in this note, and Roemer assumes that each candidate wins with a probability equal to the candidate's share of votes.

2 The Model

Let $\mathcal{N}$ be a society formed by N citizens labeled $i \in \mathcal{N} = \{1, ..., N\}$, with $N \geq 3$. This society must elect a representative, who will receive a benefit b from being elected and

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[1] Riviere also considers a game in which like-minded citizens can share the cost of running, forming a political party. In this setting, equilibria with two parties are no longer rare, but for some values of the cost of running, only equilibria with one or three candidates exist.

will also get to choose a policy in a unidimensional policy space $[0,1]$. The implemented policy I denote by p. Citizens have different preferences over the chosen policy; let $v_i(p)$ be a bounded function measuring the utility that citizen i derives from policy $p \in [0,1]$.

Each agent in the society can run as a candidate in the election, but doing so entails a cost c, which is small compared to the rewards from holding office. Formally, I assume that $b > 2c > 0$.

Let $I_i = 1$ if i runs as a candidate and zero otherwise, and let $W_i = 1$ if i runs as a candidate and wins the election and zero otherwise.

Then the utility of agent i is:

$$U_i(p, I_i, W_i) = v_i(p) + bW_i - cI_i.$$

The game has four stages:

In the first stage or entry stage, each citizen simultaneously decides whether or not to enter the race and become a candidate. If no candidate enters the race, then the game ends and a default policy p_0 is implemented. If only one candidate enters the race, she is automatically elected, she implements her ideal policy, and the game ends.

If at least two candidates compete for office, in stage two or support stage, each citizen decides to support one of the candidates. Citizens who are indifferent toward all the candidates choose randomly which candidate to support.

In stage three, each citizen i who supports candidate j casts a valid vote for j with probability $(1-\mu)$. With probability $\mu \in [0,1)$, citizen i is unable to cast a valid vote for any candidate, so $i's$ support is lost. This probability μ is the same for each citizen and it is uncorrelated among citizens. The intuition is that citizen i supports j and intends to vote for j, but with probability μ, some random factor prevents citizen i from casting a valid vote; perhaps i cannot make it to the polls, or i votes but somehow the ballot is cast incorrectly and is later declared invalid and does not add to the total vote count.

Once all valid votes are counted, a winner is chosen by plurality rule: the candidate who has most valid votes (not necessarily the one with most intended support) wins, provided that she obtains at least two valid votes. If no candidate obtains more than one valid vote, the game returns to the beginning of stage three and citizens are called to vote again for the candidates they had decided to support in stage two.[2] In case of

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[2] This mild assumption prevents a candidate winning with her own vote only and no support in the

a tie for victory between two or more candidates with at least two votes, the winner is determined by a random draw that assigns equal probability to each of the candidates that are tied with most votes.

Finally, in stage four, the winner implements her ideal policy.

Citizens know the ideal policy of each candidate and they correctly anticipate that the winner will implement her ideal policy; candidates cannot commit at the entry stage to implement any other policy if they win the election.

The uncertainty about the vote count in stage three captures the idea that candidates cannot anticipate the outcome of the election, because they do not have enough information about the voting population. The candidates cannot anticipate whose vote will count and whose will not, so even though they anticipate the support intentions of the whole electorate, they are still uncertain about the electoral outcome.

There are several ways to incorporate uncertainty to the model. Roemer [8] lets each candidate win with probability equal to the candidate's share of votes. This assumption may seem crude but keeps his model simple and tractable. A more elegant approach would be to model turnout, assuming that the probability that a citizen votes increases with the expected benefit for the citizen from voting. With costless voting, every rational citizen who is not indifferent about every candidate should vote, but according to Quantal Response theories (see McKelvey and Palfrey [4]) citizens with little to gain or lose are more likely to make the mistake of abstaining than those with a higher stake in the outcome of the election. In this paper I model uncertainty as in Myerson [5] and I do not consider turnout decisions: The parameter μ is merely a simple way to capture the uncertainty faced by the candidates, who in reality, as in the model, do not know the exact distribution of preferences of the voting population.

The strategy of each citizen i has two components: An entry strategy, determining whether to run as a candidate or not, and a support strategy, which determines for any possible set of candidates which one will citizen i support.

Let $I_i \in \{0,1\}$ denote citizen i's pure entry strategy, where $I_i = 1$ denotes entry, and let $I = (I_1, ..., I_N)$ be the pure entry strategy profile of all the citizens. The set of candidates resulting from the entry strategy I is $C \subseteq \mathcal{N}$. Let $\gamma_i \in [0,1]$ be the mixed

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rest of the electorate, and it is technically convenient. However, if every citizen ran as a candidate and supported herself, no candidate would ever get two votes. We assume that in this unlikely scenario the game ends with a negative utility payoff that makes every agent worse off than in any other outcome.

entry strategy by citizen i, indicating the probability that i enters the race, and let γ be the entry strategy profile of all citizens.

Let $s_i : 2^N \longrightarrow \mathcal{N}$ denote the pure support strategy used by citizen i, which determines for each possible set of candidates, which one citizen i will support. For a given set of candidates C, $s_i(C)$ denotes the candidate supported by citizen i. Let $s = (s_1, ..., s_N)$ be the pure support strategy profile of every citizen in the society. Let σ_i be the mixed support strategy profile of citizen i, and let σ be the mixed support strategy profile of every citizen, so that $\sigma(C)$ denotes the mixed support strategy profile of every citizen given the set of candidates C. Let the subscript $_{-i}$ denote "every citizen in $\mathcal{N}$ except for i."

The equilibrium concept I use is Undominated Subgame Perfect Nash Equilibrium, ruling out weakly dominated strategies. An equilibrium is defined by an entry strategy profile γ^* and a support strategy profile σ^* such that:

(i) Given any set of candidates $C \subseteq \mathcal{N}$ of size at least two, $\sigma^*(C)$ is an Undominated Nash equilibrium of the support stage subgame.

(ii) Given σ^*, the entry strategy profile γ^* is a Nash equilibrium at the entry stage of the game.

If in an equilibrium $\{\gamma^*, \sigma^*\}$ every citizen uses a pure entry strategy, I let C^* denote the set of equilibrium candidates.

I assume that each agent i has a unique favorite policy $p_i = \underset{p \in [0,1]}{\arg\max} v_i(p)$ and I label and order individuals according to their favorite policy, so that for all $i, j \in \mathcal{N}$, $i < j$ implies $p_i \leq p_j$. Given this ordering of citizens, I assume that the preference profile satisfies the Strict Single-Crossing property:

Definition 1 *A preference profile satisfies the Strict Single-Crossing property if for all* $x, y \in [0, 1]$ *and all* $i, j \in \mathcal{N}$ *such that* $y > x$ *and* $j > i$, $v_i(y) \geq v_i(x) \Longrightarrow v_j(y) > v_j(x)$.

This property implies that if a left-leaning citizen d prefers the right-most of two policies, then every citizen who is more right-leaning than d also prefers the right-most policy. Given two policies, one more liberal, one more conservative, there cannot be any overlapping so that "conservative" citizens support the liberal policy and some more "liberal" citizens support the conservative policy. With strictly single-crossing preferences, given any two policy positions $p' < p''$, there exists a cut-off point $\widehat{p}(p', p'')$ such that every citizen with an ideal policy below $\widehat{p}(p', p'')$ prefers p' to p'' and every citizen with an ideal policy above $\widehat{p}(p', p'')$ prefers p''.

Strict Single-Crossing, and the slightly weaker Single Crossing property in which $v_i(y) \geq v_i(x) \implies v_j(y) \geq v_j(x)$ and $v_i(y) > v_i(x) \implies v_j(y) > v_j(x)$, instead of $v_i(y) \geq v_i(x) \implies v_j(y) > v_j(x)$ as in Definition 1, are restrictions on the profile of preferences of all agents in the society. Unlike the Single-Peaked condition, the Single-Crossing conditions do not place a restriction on the shape of the preference relation of an individual agent, they limit only the heterogeneity of preferences across agents. An individual preference relation is Single-Peaked if given any pair of alternatives both to the left or both to the right of the ideal policy of the agent, she prefers the alternative closer to her ideal policy or peak. Then, a preference profile satisfies the Single-Peaked condition if every agent in the society has Single-Peaked preferences. By contrast, Single-Crossing is a condition defined directly on the whole profile, and not over the preference relation of any given individual.

Although Single-Crossing neither implies nor is implied by the Single-Peaked property, Gans and Smart [3] use the findings of Rothstein [9] to show that if the policy space has only three alternatives, then Single-Crossing is strictly weaker than Single-Peaked. Note as well that if agents' preferences are given by translations of a common, single-peaked function $\widetilde{v}$ defined over $[-1, 1]$ and with maximum at zero such that $v_i(p) = \widetilde{v}(p - p_i)$, then Single-Crossing also holds. In any case, the main results of this paper hold for any Single-Peaked profile of preferences, as I discuss below. For the interested reader, Gans and Smart [3] provide a more detailed discussion of Single-Crossing and Strict Single-Crossing, and their relation to other similar conditions in the literature.

I introduce the following notation.

Let m denote the median voter if N is odd, and let m_l and m_h denote the two medians if N is even. Labelling citizens by the relative position of their ideal policy from lowest to highest, m is the citizen in position $\frac{N+1}{2}$ if such fraction is an integer; otherwise m_l and m_h are respectively the citizens in positions $\frac{N}{2}$ and $\frac{N+2}{2}$.

Let $S_i(C, \sigma)$ denote the support for candidate i, that is, the number of citizens whose support strategy is to support candidate i, given that the set of candidates is C and the joint support strategy profile is σ. If all citizens use a pure support strategy, $S_i(C, \sigma) = \#\{j : s_j(C) = i\}$. If citizens use mixed support strategies, S_i is a random variable that can take different values depending on the support actions taken by the agents who mix.

Let V_i denote the number of valid votes for i. The difference $S_i - V_i$ corresponds to the number of citizens who support i but are unable to cast a valid vote for i and can be interpreted as the number of "lost votes" for i.

In any two-candidate race with $C = \{i, j\}$, let

$$L_{ij} = (S_i - S_j) - (V_i - V_j) = (S_i - V_i) - (S_j - V_j)$$

denote the "shift" from candidate i to candidate j in the difference of voting totals for the two candidates, compared to the original difference in support for the two candidates prior to the distortion introduced by the loss of votes. In short, L_{ij} is equal to the number of votes lost by i minus the number of votes lost by j, which we can interpret as a "net loss" of votes for i.

L_{ij} is a discrete random variable, whose distribution depends on the support for each candidate and the uncertainty parameter μ: Let $f_{ij}(l)$ be its probability mass function and let $F_{ij}(l)$ be its distribution function, so $f_{ij}(l)$ is the probability that $L_{ij} = l$ and $F_{ij}(l) = \sum_{k=-N}^{l} f_{ij}(k)$.

In any race with two candidates $C = \{i, j\}$ with ideal policies $p_i < p_j$, let $\widehat{p}_{ij}$ denote the cut-off point such that every citizen with an ideal policy less than $\widehat{p}_{ij}$ prefers candidate i, every citizen with an ideal policy above $\widehat{p}_{ij}$ prefers candidate j, and only citizens with an ideal policy equal to $\widehat{p}_{ij}$ are indifferent between i and j.

3 Existence of Equilibria

Let us first present the benchmark case in which there no uncertainty about the vote count ($\mu = 0$) and the candidates can anticipate the outcome of the election. This benchmark corresponds to the Besley and Coate [1] model with the additional assumptions of Strict Single-Crossing preferences in a unidimensional policy space, and $b > 2c$.

A single candidate equilibrium exists if and only if N is odd, whereas two candidate equilibria generically do not exist if N is odd and they exist if N is even. Equilibria with multiple candidates are also possible.

These differences depending on the exact number of citizens are plausible in an election with a small electorate, such as a vote in a committee. However, in any election with a very large electorate, there will be some uncertainty about the number of votes each candidate will get and the results of the model should not depend on whether the size of the electorate is odd or even.

In the rest of the paper, I capture the uncertainty assuming that $\mu > 0$ and I show that if the electorate is sufficiently large, whether N is odd or even is irrelevant for the existence of equilibria with one or two candidates in elections with a large reward for holding office.

If citizens use mixed entry strategies, the number of candidates who run may vary in different outcomes of the same equilibrium. I categorize equilibria according to the number of citizens who enter the race with positive probability. If citizens use only pure strategies at the entry stage, I say the equilibrium is *pure.*

Definition 2 *An n-candidate equilibrium is an equilibrium in which the number of citizens who run with positive probability is n.*

Definition 3 *An equilibrium is "pure" if every citizen uses a pure entry strategy.*

3.1 Single Candidate Equilibrium

In this subsection, I characterize existence of single candidate equilibria, and I show that they do not exist in a society with a sufficiently large electorate. In a unidimensional space, a unique median is a Condorcet winner and will have more support (and more expected vote share) than the other competitor in any two-candidate race.

Nevertheless, the uncertainty about the vote totals gives any other candidate challenging the median some positive probability of winning the election, and in equilibrium the median can only run alone if the probability of victory for any candidate running against her is too low.

Lemma 1 *There exists a one-candidate equilibrium if and only if N is odd, the median is unique, and for any citizen $j \in \mathcal{N} \backslash \{m\}$,*

$$(b + v_j(p_j) - v_j(p_m)) \Pr[W_j = 1 | C = \{m, j\}] \leq c.$$

This equilibrium is pure and unique among one-candidate equilibria, and m is the single candidate.

The proofs of all the results are in the appendix. The intuition is that the median(s) would enter and run against any other citizen who was running alone, so only a unique median can run unopposed, and even the median can run unopposed only if any challenger would have a small enough probability of victory.

A candidate trailing by a small number of supporters almost certainly loses if the electorate is also small, but as the electorate gets larger, the number of lost votes will increase and a candidate trailing by the same small number of supporters will have a better chance of victory. For instance, in an electorate with 5 citizens, a 3-2 split of support will give the weaker candidate a very small chance of victory. However, in an electorate with millions of citizens, a split of support in which the stronger candidate has only one more supporter is a virtual tie, and both candidates have an almost equal probability of victory.

Given a fixed parameter of uncertainty μ, consider a sequence of societies of increasing size. Consider also a corresponding sequence composed of a pair of candidates in each society such that the difference in support for the two candidates in each pair is constant along the sequence. For example, in society $\mathcal{N}_N$ of size N construct the pair with the median m_N and the next citizen, $(m+1)_N$; regardless of the distribution of preferences in each society, for every element of this sequence of pairs the median has one more supporter than the candidate in position $m+1$. The probability of victory converges to a half for both candidates as the size of the society increases. I use this result for the first theorem on existence of single candidate equilibrium in large societies.

Theorem 1 *Given $\mu > 0$, there exists some n such that if $N > n$, there is no single candidate equilibrium.*

Suppose the benefit of holding office is three times the cost of running. Then any candidate with a one-third chance of victory will be willing to run. The probability of victory for a weaker candidate with one less supporter than the median is more than a third if $N \geq 103$ for $\mu = 0.05$; or if $N \geq 1087$ for $\mu = 0.005$. Since void ballots exceed 0.5% in most elections, these numerical examples show that the theorem applies for relatively small electorates. A higher degree of uncertainty about counting an individual vote has the same effect as increasing the size of the population: In either case it becomes harder for the agents to anticipate the exact outcome of the election and a challenger will have greater incentives to run against the median.

In an election with a small electorate the median can run unopposed because any other candidate would have only a very slim chance of beating the median. The median must be unique to run unopposed, so it is crucial that the electorate is odd. However, as the electorate gets larger, the uncertainty about the vote count gives other candidates challenging the median a better chance of winning, and for a sufficiently large society the probability of victory for citizens with an ideal point close to the median is high enough so that they will run and not let the median win unopposed, not even a unique, Condorcet winner median.

As a result, in any equilibrium at least two citizens will enter with positive probability. If citizens use mixed entry strategies, it could be that in a particular outcome only one candidate stands for election, but the positive probability of entry by other citizens must be part of the equilibrium.

3.2 Two-candidate equilibria

Two candidate equilibria are a common feature of plurality elections, yet they are generically non-existent in citizen candidate models without uncertainty if the number of citizens is finite and odd. I show that introducing uncertainty in the model guarantees the existence of two-candidate equilibria if the size of the society is sufficiently large.

To obtain this result, I add a restriction on the preference profile; I require that each citizen has a distinct ideal policy. Note that if the ideal points are drawn from any continuous distribution, this requirement is met with probability one. The formal condition is that for any $i, j \in \mathcal{N}$, $p_i \neq p_j$.[3]

Intuitively, this assumption rules out entry by multiple candidates with the same policy. As an illustrative extreme case, suppose all citizens but one share a common ideal policy, and suppose that the cost of running is negligible compared to the benefit of holding office. Then all the citizens with the same ideal policy run against each other. No equilibria with less than all of them running is possible. However, if the rest of the electorate is able to discriminate among the pair of candidates running, then in a large enough electorate, equilibria with two candidates exist.

Theorem 2 *Given $\mu > 0$, there exists some n such that if every citizen has a distinct*

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[3] As suggested by a referee, it suffices to assume that $m-1$ and $m+1$ (if N is odd) or the two medians (if N is even) have a distinct ideal policy.

ideal policy and $N > n$, a pure two-candidate equilibrium exists.

The proof is constructive. If there is a unique median, in the absence of uncertainty (or with little uncertainty in small electorates), two-candidate equilibria did not exist unless the median citizen is indifferent between the two candidates, an event that generically does not occur.

However, as the electorate grows, a positive uncertainty raises the probability that a candidate trailing by one supporter wins the election. If there are large benefits of holding office, the same intuition that made single candidate equilibria impossible guarantees the existence of two-candidate equilibria for large electorates. As the electorate grows, the probability that a weaker candidate trailing by a given number of supporters wins the election converges to one half, and $b > 2c$ guarantees that such a weaker candidate will want to run, and that equilibria with two candidates exist in large societies, whether the number of citizens is even or odd.

Note that Lemma 1, Theorem 1 and Theorem 2 all hold if preferences are Single-Peaked but not Single-Crossing. With either Single-Peaked or Single-Crossing preferences, a median running in a two-candidate race has at least as many supporters as her opponent, thus no other citizen but a unique median can run unopposed (Lemma 1). However, if citizen $m+1$ challenges the median m, with Single-Peaked or Single-Crossing preferences every citizen to the right of $m+1$ prefers candidate $m+1$ over the median m, so candidate $m+1$ trails the median m by only one supporter, and if the electorate is large enough the probability that candidate $m+1$ wins is high enough for $m+1$ to enter the race. Then, the median cannot run unopposed (Theorem 1) and a two candidate equilibrium exists (Theorem 2).

Two-candidate equilibria with mixed entry strategies may also exist,[4] but only the weaker candidate will mix: The stronger candidate has a probability of victory which is over one half, so she prefers to enter. Furthermore, if there is a two-candidate equilibrium in which candidate i enters for sure and candidate j mixes between running and not running in her entry decision, there is also a pure equilibrium in which both i and j enter with probability one.

Proposition 2 *If there exist a two-candidate equilibrium in which i and j run with*

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[4] Previous literature has paid only scant attention to equilibria with mixed entry. In a novel experimental paper, Cadigan [2] provides an example.

positive probability, then there exists a pure two-candidate equilibrium in which i and j run.

Only a weaker candidate with not so good odds of victory may be indifferent about running or not. The weaker candidate may only mix if she is indifferent about running. If she is indifferent, it is also a best response for her to run with probability one. The stronger candidate wants to run regardless of the probability of entry by the weaker one. If both candidates have the same support, both enter with probability one.

Since considering mixed entry does not expand the set of pairs of citizens who may run in a two-candidate race, I focus on equilibria which are pure at the entry stage. In the constructive proof of Theorem 2, the median(s) or a citizen very close to the median are the two candidates who enter for sure in equilibrium. However, there exist other equilibria involving citizens far from the median: Two very extreme candidates i and j can run in equilibrium, insofar as the cutting point $\widehat{p}_{ij}$ between those who support i and those who support j is very close to the median, (or to the medians if N is even).

Let i and j be the any two citizens, such that $p_i < p_j$. Let $q(x)$ be the minimum number of supporters that i must have in order for i to win with probability no less than x, given that every other citizen will support j. That is, $q(x)$ is the minimum k such that:

$$\Pr[W_i = 1 | S_i = k, S_j = N - k] \geq x.$$

In the following proposition I use the function q to specify the number of supporters each candidate must have in order to sustain a two-candidate equilibrium.

Proposition 3 *Suppose each citizen has a distinct ideal policy. For any $i, j \in \mathcal{N}$ such that $p_i < p_j$, let*

$$D = q\left(\frac{c}{b + v_i(p_i) - v_i(p_j)}\right) \text{ and } R = q\left(1 - \frac{c}{b + v_j(p_j) - v_j(p_i)}\right), \; D, R \in \mathcal{N}.$$

If $\widehat{p}_{ij} \in (p_D, p_R)$, there exists a pure two-candidate equilibrium in which i and j run.

There is a two-candidate equilibrium if both candidates get a similar number of supporters and thus they both have a sufficiently high probability of victory. In order for the electorate to split in roughly equal halves, the cutting point $\widehat{p}_{ij}$ between those who support i and those who support j must be close to the median. Proposition 3 shows it suffices that the cutting point lies in between the ideal policies of citizens D and R, where the function $q(x)$ specifies the identity of D and R, which depends on the identity of i and j.

A very slight weakening of the sufficient condition in Proposition 3 is already a necessary condition: If i and j run in a pure two-candidate equilibrium, then $\widehat{p}_{ij} \in [p_D, p_{R+1})$. Only equilibria in which one of the candidates is indifferent about running or not need not satisfy the sufficient condition.

Since $b > 2c$, each candidate is willing to run for a probability of victory less than a half and it follows that D is weakly to the left of the median m if N odd, or of the low median m_l if N is even. Similarly, H is weakly to the right of m or m_h. Suppose N is even. Then if m_l and m_h have different ideal policies, $\widehat{p}_{m_l,m_h} \in (p_{m_l}, p_{m_h}) \subseteq (p_D, p_R)$ and there exist a two-candidate equilibrium with pure entry strategies in which m_l and m_h run against each other.

Corollary 4 *Suppose each citizen has a distinct ideal policy. If the number of citizens is even, a pure two-candidate equilibrium exists.*

In an equilibrium without uncertainty, the two candidates must have equal support. With uncertainty, they must have similar, not necessarily equal support. In larger societies, the margin by which the weaker candidate trails in support may be bigger in absolute terms, but the fraction of the population that supports each candidate must converge to a half in a two-candidate equilibrium as the electorate gets larger. If the weaker candidate lags heavily in support, she would abandon the race.

The cutting point $\widehat{p}_{ij}$ will have to be close to the median in order for the candidates to have similar support. But the candidates themselves need not be close to the median. There is no convergence result in terms of the policy that will ultimately be implemented, but only in terms of the "undecided voter", the citizen who is indifferent between the two candidates. This citizen (if it exists) ought to be close to the median, splitting the electorate into two halves of roughly the same size, so that both candidacies are competitive.

The two candidates can be two moderates, or two extremists (one from each extreme), or anything in between so long as they split society into two groups of similar size.

4 Conclusion

I have introduced a model of representative democracy with endogenous candidates and uncertainty about the total vote count. I predict that the median is able to run a successful, unopposed campaign only if the number of citizens is small and odd. If the electorate is large, the exact number of citizens is irrelevant; no citizen can run unopposed and a two-candidate equilibrium exists.

I have characterized equilibria with one and two candidates in pure and mixed strategies and I have found that only candidates who run in some pure equilibrium may also run in a mixed equilibrium.

5 Appendix

5.1 Proof of Lemma 1

Proof. First, I prove the necessary condition. Suppose N is even and citizen $h \geq m_h$ runs unopposed. In equilibrium, support strategies for any race with two candidates are sincere. Then, if m_l enters the race and $p_{m_l} \neq p_h$, $S_{m_l} \geq S_h$ and the probability that m_l wins is at least one half, while if $p_{m_l} = p_h$, every citizen mixes support with equal probability and m_l wins with probability one half. Thus, $b > 2c$ implies that m_l in any case would prefer to run, and h running alone cannot be an equilibrium. Suppose $d \leq m_l$ runs unopposed. Then, by an analogous logic, m_h would prefer to run, and it cannot be an equilibrium that d runs unopposed. Therefore, there cannot be a one-candidate equilibrium if N is even.

Suppose N is odd and the median is not unique and suppose citizen $h \in \mathcal{N}$ runs unopposed. Then, if any of the medians runs against h, the median wins with probability no less than a half, so the median prefers to run, and it cannot be an equilibrium in which h runs unopposed.

Suppose the median is unique and runs unopposed. If $\exists\ j \in \mathcal{N} \backslash \{m\}$ such that

$$(b + v_j(p_j) - v_j(p_m)) \Pr[W_j = 1 | C = \{m, j\}] > c,$$

then j would run against m.

Second, I prove the sufficient condition. If a unique median m runs and

$$(b + v_j(p_j) - v_j(p_m)) \Pr[W_j = 1 | C = \{m, j\}] \leq c \ \forall j \in \mathcal{N} \backslash m,$$

then no citizen j has an incentive to run against m, and since $b > c$, m prefers to run and the entry condition of the equilibrium is satisfied. To complete the equilibrium, I only need to construct a support equilibrium σ^* for each possible support subgame reached after any off-equilibrium outcome of the entry game. Since the number of players is finite and their strategy set is also finite (there are only a finite number of candidates to choose from), such an equilibrium exists for each subgame, possibly in mixed support strategies.

To show uniqueness, it suffices to note that m would prefer to run against any other citizen $h \in \mathcal{N}$ who was running alone. The equilibrium is pure because since $b > c$, m strictly prefers to run than to drop out, given that no other citizen enters the race. ■

5.2 Proof of Theorem 1

Proof. The only single candidate equilibrium is that in which a unique median is running (Lemma 1). Consider a sequence of societies $\{\mathcal{N}_N\}_{N=3}^{\infty}$ of size N odd, all of them with $\mu_N = \mu$ and satisfying all the assumptions in the model. Now suppose in each society $\mathcal{N}_N$ the median and the citizen immediately to her right run, so $C_N = \{m_N, (m+1)_N\}$, where the subindex denotes the society to which a citizen or a set of candidates belongs. Then in the equilibrium of the support subgame every citizen chooses support sincerely and $S_{m_N} - S_{(m+1)_N} = 1$. If m_N loses at least three votes more than $(m+1)_N$, $V_{(m+1)_N} - V_{m_N} \geq 2$ and $(m+1)_N$ wins the election. The probability that m_N loses at least three votes more than $(m+1)_N$ is:

$$1 - F_{m,m+1}(2).$$

Let $bi[n, p; k]$ be the probability that a binomial distribution with parameters (n, p) takes a value of k. Then

$$f_{m,m+1}(l) = \sum_{k=0}^{N} bi[\frac{N+1}{2}, \mu; k + l] bi[\frac{N-1}{2}, \mu; k].$$

As $N \to \infty$, $f_{m,m+1}(l)$ converges to zero for any given integer l, in particular for $f_{m,m+1}(2)$, $f_{m,m+1}(1)$ and $f_{m,m+1}(0)$, so $F_{m,m+1}(2) - F_{m,m+1}(-1)$ converges to zero. Since $S_{m_N} >$

$S_{(m+1)_N}$, the probability that the m_N loses less votes than $(m+1)_N$ is less than a half, so $F_{m,m+1}(-1) < \frac{1}{2}$ for all N. But $F_{m,m+1}(2) > \frac{1}{2}$ for all N, thus it must be $F_{m,m+1}(2)$ converges to $\frac{1}{2}$. Then, the probability that m_N loses at least three more votes than $(m+1)_N$ converges to $\frac{1}{2}$ and the probability that $(m+1)_N$ wins also converges to a half. Given $b > 2c$, this implies that if N is large enough, $(m+1)_N$ wants to run against m_N, and m_N cannot be the only citizen running with positive probability in equilibrium. ■

5.3 Proof of Theorem 2

Proof. If N is odd, I construct a pure equilibrium in which γ^* is such that $I^*_m = I^*_{m+1} = 1$, $I^*_j = 0$ for all $j \in \mathcal{N} \setminus \{m, m+1\}$, and σ^* is such that for any $C = \{m, m+1, h\}$ with $h \in \{\emptyset \cup \mathcal{N} \setminus \{m, m+1\}\}$, $s_i(C) = m$ for any $i \leq m$ and $s_i(C) = m+1$ for any $i > m$.

Given σ^*, no citizen would support a third candidate and therefore no third candidate wants to enter. Consider again a sequence of societies $\{\mathcal{N}_N\}_{N=3}^{\infty}$ indexed by their size, and for each society $\mathcal{N}_N$ select the pair $\{m_N, (m+1)_N\}$. Since $S_{m_N} - S_{(m+1)_N} = 1$ for all N, it follows that as $N \to \infty$ the probability of victory converges to one half for both m_N and $(m+1)_N$. Since $b > 2c$, m_N and $(m+1)_N$ want to run against each other if the electorate is N is large enough. Therefore, γ^* is an equilibrium of the entry stage given σ^*.

Dropping the subindex, for any society $\mathcal{N}$, the support strategy σ^* is such that citizens choose support sincerely among m and $m+1$, and that no citizen would support a third entrant. If an agent deviates from this support strategy to support a third candidate h, then $S_h = 1$; since victory in an election requires two votes, the probability of victory for h is zero, same as before the deviation. Since m and $m+1$ have a different ideal policy, it is then an undominated best response of each citizen not to support the entrant, and to choose support sincerely for either m or $m+1$, as dictated by σ^*. For any other set of candidates off the equilibrium path, the subgame at the support stage is finite and an equilibrium of the subgame exists.

If N is even, then m_l and m_h would run against each other in equilibrium regardless of the size of the society, for they each have a one half probability of victory. With a support strategy that assigns no support to an entrant, no third candidate will enter, by the same arguments as in the odd case. ■

5.4 Proof of Proposition 2

Proof. Suppose there exists a two-candidate equilibrium $\{\gamma^*, \sigma^*\}$ in which $\gamma_i^*, \gamma_j^* > 0$. Without loss of generality, suppose that given σ^*, the probability that i wins given that $C = \{i, j\}$ is at least one half. Then, $b > 2c$ implies that at the entry stage, running is the only best response of i. Thus, it must be that $\gamma_i^* = 1$. Suppose $p_i = p_j$. Then the probability of victory is at least one half for both i and j and it must be that $\gamma_j^* = 1$ too and $\{\gamma^*, \sigma^*\}$ is a pure equilibrium.

Suppose instead that $p_i \neq p_j$. Let $\gamma' = \{\gamma_j', \gamma_{-j}^*\}$, with $\gamma_j' = 1$ and let σ' be such that for any C containing $\{i, j\}$ and any $h \in \mathcal{N}$,

$$s_h'(C) = \underset{k \in \{i,j\}}{\arg\max} v_h(p_k)$$

and for any other set of candidates not including i and j, σ' defines an equilibrium of the support subgame. I want to show that $\{\gamma', \sigma'\}$ is an equilibrium. First, note σ' defines an equilibrium of any support subgame in which $\{i, j\} \in C$ because $p_i \neq p_j$ and no candidate with a single supporter can win the election (see proof of Theorem 2). Second, note that $\sigma'(C) = \sigma^*(C)$ for $C = \{i, j\}$. By assumption, given σ^* and γ_{-j}^* , $\gamma_j^* > 0$ was a best response for j at the entry stage. It must then be that $\gamma_j' = 1$ is also a best response given σ^* and γ_{-j}^*, and given σ' and γ_{-j}'. I assumed that given σ^*, the probability that i wins given that $C = \{i, j\}$ is at least one half. Thus, given σ' and γ_{-i}', it is still a best response for i to run with probability one. Therefore, a pure strategy of always entering by i and j, and no entry by any other candidate is an equilibrium of the entry stage given σ' and $\{\gamma', \sigma'\}$ is a pure two-candidate equilibrium in which $C^* = \{i, j\}$. ■

5.5 Proof of Proposition 3

Proof. Sufficient Condition: If $\widehat{p}_{ij} > p_D$, at least D citizens support i. Then i has a probability of victory no less than $\frac{c}{b+v_i(p_i)-v_i(p_j)}$ and i wants to run. If $\widehat{p}_{ij} < p_R$, candidate i has a probability of victory less than $1 - \frac{c}{b+v_j(p_j)-v_j(p_i)}$, and citizen j wants to run. If both i and j want run, and given that they have different ideal policies, there exist an undominated support strategy that would assign no support to any third entrant, and then no third citizen would want to deviate and enter. ■

I also proof the necessary condition enunciated in the text:

Proof. If $\widehat{p}_{ij} < p_D$, then at most $D-1$ citizens support i, and at least $N-L+1$ support j. Then, the probability of victory for citizen i is less than $\frac{c}{b+v_i(p_i)-v_i(p_j)}$ and i prefers to let j run alone and win. If $\widehat{p}_{ij} \geq p_{R+1}$, at least R citizens support i and $R+1$ at least mixes (or supports i too), and at most $N-R-1$ support j, with citizen $R+1$ at most mixing. Then, the probability that i wins is more than $1-\frac{c}{b+v_j(p_j)-v_j(p_i)}$, and citizen j does not want to run. ■

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