

Proportional Pie-Cutting

Steven J. Brams¹, Michael A. Jones², Christian Klamler³

¹ Department of Politics, New York University, New York, NY 10003 USA; steven.brams@nyu.edu

² Department of Mathematical Sciences, Montclair State University, Montclair,
NJ 07043 USA; jonesm@mail.montclair.edu

³ Institute of Public Economics, University of Graz, A-8010 Graz, AUSTRIA;
christian.klamler@uni-graz.at

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Abstract David Gale (1993) was perhaps the first to suggest that there is a difference between cake and pie cutting. A cake can be viewed as a rectangle valued along its horizontal axis, and a pie as a disk valued along its circumference. We will use vertical, parallel cuts to divide a cake into pieces, and radial cuts from the center to divide a pie into wedge-shaped pieces. We restrict our attention to allocations that use the minimal number of cuts necessary to divide cakes or pies.

For an extended definition of envy-freeness to unequal entitlements, we provide a counterexample to show that a cake cannot necessarily be divided into a proportional allocation of ratio $p : 1 - p$ between two players where one player receives p of the cake according to her measure and the other

receives $1 - p$ of the cake according to his measure. In contrast, for pie, we prove that an efficient, envy-free, proportional allocation exists for two players. The former can be explained in terms of the Universal Chord Theorem, while the latter is proved by another result on chords. We provide procedures that induce two risk-averse players to reveal their preferences truthfully to achieve proportional allocations. We demonstrate that, in general, proportional, envy-free, and efficient allocations that use a minimal number of cuts may fail to exist for more than two players.

1 Introduction

It would seem that dividing a pie is not much different from dividing a cake. If we represent a pie by a circle, cutting it at any point and “straightening out” the circle gives a line segment, which can be used to represent a cake. So isn’t a cake just a pie that has been cut?

In fact, they are quite different objects, and the division of each may give very different results. We may think of a cake as a rectangle valued along its horizontal axis, and a pie as a disk valued along its circumference. We will use vertical, parallel cuts to divide a cake into pieces, and radial cuts from the center to divide a pie into wedge-shaped pieces. A pie-cutting procedure can be used to divide a shoreline on a lake into three connected pieces, whereas a cake-cutting procedure produces disconnected pieces of shoreline (Figure 1). Note that the minimal number of cuts necessary to

divide a cake among n people is $n - 1$, one less than the minimal number n necessary to divide a pie.

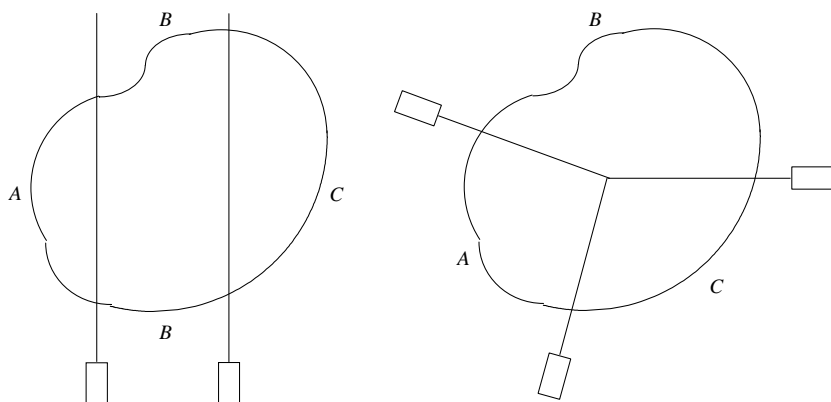


Fig. 1 Using a cake-cutting procedure to divide lake-front property among players A , B , and C results in a disconnected shoreline for B (left), whereas each player gets connected pieces using a pie-cutting procedure (right).

Gale (1993) posed the following question: If there are n players, does there always exist an *envy-free* allocation of a pie (no player desires another player's wedge-shaped piece) that is *undominated* (no other allocation is better for at least one player and not worse for the others) using the minimal number of n cuts? When there are just two players that have equal claims or entitlements to the pie, not only is the answer to Gale's question "yes," but there is also a procedure that produces an envy-free, undominated allocation among diameter cuts (Barbanel and Brams, 2005). For two players with equal entitlements, we also provide a "yes" answer to this question, where the allocation is undominated among those formed by two radial cuts.

But what if they don't have equal entitlements — say, one player is entitled to twice as much as the other player, and the pie must therefore be divided into unequal proportions. We show that the answer is still “yes” for two players if we extend the notion of envy-freeness to the case of unequal entitlements. However, the answer is “no” if there are three or more players, which we demonstrate with two 3-player examples. Thus, proportional pie-cutting by three or more players is not always possible.

On the other hand, there are envy-free procedures for cutting a cake among three players, using the minimal number of 2 cuts, that yield an undominated (also called *efficient*) allocation (Barbanel and Brams, 2004 and Stromquist, 1980). Envy-free allocations of cake for four players exist, but no finite procedure with a uniform bound on the number of steps uses the minimal number of 3 cuts. For overviews of the cake-cutting literature, see Barbanel and Brams (2005), Brams and Taylor (1996), Robertson and Webb (1998), and Thomson (2007).

Because most disputes are between two parties that have equal claims to a disputed item or items (*e.g.*, spouses in a divorce), dividing objects when the parties have unequal claims has received relatively little attention in the cake-cutting literature (an exception is Robertson and Webb (1998), pp. 35-48). But there is a more important reason for this neglect: It is not always possible to use the minimal number of cuts to divide a cake between players if their entitlements are unequal, which we will demonstrate with a 2-player example. By contrast, we will prove that there always exists a

division of a pie using the minimal number of 2 cuts that reflects the unequal entitlements of the two players. More precisely, if the two players are entitled to p and $1 - p$ proportions of a pie, such that $0 < p < 1$, then we show that there exists an envy-free and efficient allocation in which the players' pieces are in the ratio $p : 1 - p$, as each values the pieces (players' valuations are assumed to be subjective).

Proportional allocations of cake and pie for two players can be viewed from a real analysis perspective by phrasing the results in terms of horizontal chords. For two players with unequal entitlements, the nonexistence of an undominated, proportional allocation of cake in which each player receives a piece valued at least as much as her entitlement is related to the Universal Chord Theorem. For pie, such an allocation exists because measures on pies can be viewed as periodic functions, just as angles that are equivalent modulo 2π are considered the same. We make these connections explicit in section 2.

In section 3, we provide two procedures that induce risk-averse players to reveal their preferences truthfully. The first procedure requires players to divide the pie into equally valued pieces and gives the players pieces that are exactly equal to their entitled shares, according to their preferences, but it may leave a surplus. The second procedure, which leaves no surplus, requires the players to submit their valuations as probability measures. This additional information improves the outcome by giving an envy-free and efficient allocation of the entire pie in the ratio $p : 1 - p$ of the players'

entitlements, according to their respective measures. If the players' preferences are not absolutely continuous with respect to each other, then the proportional allocation may be efficient only with respect to the ratio because a different allocation that violates the ratio, but which gives more to one player, may exist. We illustrate this with an example. In section 4, we demonstrate that, in general, proportional, envy-free, and efficient allocations that use a minimal number of cuts may fail to exist for more than two players.

2 Unequal Entitlements between Two Players: Existence Results

Assume that players' preferences for pie are defined by finitely additive, nonatomic probability measures over a disk. Finite additivity ensures that the value of a finite number of disjoint pieces is equal to the value of their union. It follows that no subpieces have greater value than the larger pieces that contain them. Because a piece of pie is defined by the wedge-shaped piece between two angles, represented by two points on the circumference, we require that the measure be non-atomic when projected onto the circumference of the pie. This ensures that a radial cut, which defines one border of a piece, contains no value. At times, we require players' measures to be absolutely continuous. Measures on a pie are *absolutely continuous* with respect to each other if, whenever a piece of pie has positive measure to one player, it has positive measure to all players.

Because we normalize the circumference of the disk so that it is of length 1, player i 's measure is defined by $v_i : 2^{[0,1]} \rightarrow [0, 1]$ where $2^{[0,1]}$ is the set of subsets of $[0, 1]$ and $v_i(S)$ is the value of the piece(s) given by $S \subseteq [0, 1]$. For example, for $\alpha < \beta$ the subset $[\alpha, \beta]$ is viewed as the wedge-shaped piece counterclockwise between the radii connecting α and β on the circumference to the center of the disk. For $\alpha < \beta$, let $[\beta, \alpha]$ be the wedge-shaped piece given by $[\beta, 1] \cup [0, \alpha]$. If $\beta > 1$, then let β refer to $\beta \bmod 1$. We assume that the value placed on an entire pie by each player is 1, *i.e.*, $v_i([0, 1]) = 1$ for all players i .

For n players, entitlements to the pie are defined by (p_1, \dots, p_n) such that $p_i > 0$ and $\sum_{i=1}^n p_i = 1$. An allocation is denoted by a vector $S = (S_1, \dots, S_n)$ where S_i is player i 's piece. An allocation S is *proportional* if $\frac{v_i(S_i)}{p_i} = \frac{v_j(S_j)}{p_j}$ for all i and j . It follows that the proportions of the players' values $v_1(S_1) : v_2(S_2) : \dots : v_n(S_n)$ are equal to the proportions $p_1 : p_2 : \dots : p_n$ reflecting the entitlements. An allocation S is *envy-free* if $\frac{v_i(S_i)}{p_i} \geq \frac{v_i(S_j)}{p_j}$ for all i and j . For unequal entitlements, it means that no player thinks another player received a disproportionately large piece, based on the latter player's entitlement. Hence, no player would envy another player if it had its entitlement. For equal entitlements in which $p_i = \frac{1}{n}$ for all players i , our definition of envy-freeness reduces to the standard definition in which no player desires another player's piece. An allocation is *acceptable* if each player i receives a piece S_i valued at least as much

as its entitlement: $v_i(S_i) \geq p_i$. For two players, an acceptable allocation is envy-free.

An allocation S is *dominated* by allocation T if $v_i(T_i) \geq v_i(S_i)$ for all i and $v_i(T_i) > v_i(S_i)$ for some i . An allocation is *efficient* if it is not dominated by any other allocation with the minimal number of cuts. For two players, if there exists a proportional allocation that maximizes $v_1(S_1)$, then the allocation is efficient. For two players, we refer to the players as A and B .

For cake, probability measures satisfy the same properties as they do for pie, but we define them over a square instead of a disk. The measure must be non-atomic when projected onto a side of the square. The following example shows that proportional allocations that guarantee players their unequal shares of cake may not be possible using the minimal number of cuts. Following this example, we relate the result to the Universal Chord Theorem.

Example 1 (Dividing cake between two players with unequal entitlements)

We assume a cake given by the unit square is cut perpendicularly to the x -axis, and players' measures are over $[0, 1]$ along this axis. Let player A 's value for the piece $S \subseteq [0, 1]$ be given by the integral of the uniform density function (dotted line in Figure 2) over S or $v_A(S) = \int_S 1 dx$. Let player B 's value for the piece S be given by $v_B(S) = \int_S f_B(x) dx$ where

$$f_B(x) = \begin{cases} 4x & \text{for } x \leq \frac{1}{2} \\ 4 - 4x & \text{for } x > \frac{1}{2}, \end{cases}$$

is a triangular distribution (dashed lines in Figure 2).

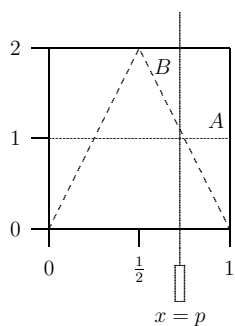


Fig. 2 Cutting the cake at $x = p$ for the measures of players A and B in Example 1.

Assume that players A and B are entitled to unequal portions of the cake given by p and $1 - p$, respectively, where $\frac{1}{2} < p < 1$. Due to the symmetry of the probability measures, it is sufficient to consider the case where player A cuts the cake at $x = p$, giving it $[0, p]$ that it values at p . Player B gets the remainder $(p, 1]$, which it values at $\int_p^1 (4 - 4x) dx = 2(1 - p)^2$. Because $2(1 - p)^2 < 1 - p$ when $p > \frac{1}{2}$, player B receives less than its entitled share, $1 - p$.

A proportional allocation is possible by solving for the cut-point x in $x : 2(1 - x)^2 = p : 1 - p$. However, in this case neither player receives a piece that it values as much as its entitlement. Thus, the players cannot both get their entitled shares. For both players to receive acceptable pieces, player B must get a piece in the middle of the cake, but one cut precludes this division.

For the above example where $p > \frac{1}{2}$, proportional outcomes are not acceptable. A result from real analysis, the Universal Chord Theorem, can

be applied to show that, in general, we cannot expect proportional allocations of cake to be acceptable. To understand how to view proportional allocations from this perspective, we must first define a horizontal chord. A continuous function $f : I \rightarrow I$ has a *horizontal chord* of length p if there exists an x and $x + p$ in I and $f(x) = f(x + p)$; the horizontal segment that connects $(x, f(x))$ to $(x + p, f(x + p))$ is the horizontal chord.

Let player A 's and player B 's measures be projected onto the side of the cake $[0, 1]$. For $t \in [0, 1]$, define $x(t)$ by $v_A([0, x(t)]) = t$, so that $x(t)$ indicates the point to cut the cake so that the piece from 0 to $x(t)$ is valued at t by player A . The function $g(t) = v_B([0, x(t)])$ represents player B 's value of the same piece. The function $f(t) = g(t) - t$ is continuous (because the players' measures are non-atomic and absolutely continuous) on $[0, 1]$ with $f(0) = f(1) = 0$. The Universal Chord Theorem states that a continuous function with $f(0) = f(1) = 0$ has horizontal chords of lengths $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$, but does not necessarily have a horizontal chord of any given length that is not the reciprocal of a positive integer (Levy, 1934). For the function f , the endpoints of a horizontal chord of length p indicate the endpoints of a piece of cake that both players value at p . Obviously, they both value the complementary piece(s) at $1 - p$.

The Universal Chord Theorem states that there may not necessarily be a chord of length p for $p > \frac{1}{2}$. However, for cake, this result is more general than needed. Single-cut allocations of cake yield pieces of the form

$[0, c]$ and $[c, 1]$, thereby restricting the type of chords needed to provide a proportional, efficient, and acceptable allocation of cake.

Using radial cuts, it is possible to divide a pie between two players wherein each receives a wedge-shaped piece that it values as equal to at least its entitled share, according to its measure. The existence of such an allocation follows from the following lemma on horizontal chords of continuous, periodic functions (*e.g.*, Boas, 1960, p.78). The periodic function is defined from the players' measures. The periodicity arises from winding around the circumference and identifying the radii associated with 0 and 1.

Lemma 1 *A continuous function $f : [0, 2] \rightarrow \Re$ such that $f(t + 1) = f(t)$ for all $t \in [0, 1]$ has horizontal chords of all lengths $p \leq 1$. For $p \in [0, 1]$, there exists a $c \in [0, 1)$ such that $f(c + p) - f(c) = 0$.*

Proof For $p \in (0, 1)$, the integral $\int_0^1 f(t + p) - f(t)dt = \int_p^{1+p} f(t)dt - \int_0^1 f(t)dt = \int_0^1 f(t)dt - \int_0^1 f(t)dt = 0$ by the periodicity of f . By the Mean Value Theorem for integrals, there exists a $c \in [0, 1]$ such that $f(c + p) - f(c) = 0$.

The existence of a proportional allocation of pie for two players with entitlements p and $1 - p$ is shown in Theorem 1. Although Example 1 demonstrates that, in general, proportional, acceptable allocations of cake may not be possible with a minimal number of cuts, Robertson and Webb (1998) consider unequal entitlements of cake and the unequal division of discrete objects.

Theorem 1 *For any $p \in [0, 1]$, the pie can be cut with two radial cuts such that A values one piece at p and B values the other at $1 - p$, according to their respective measures.*

Proof Let v_A and v_B be the measures for players A and B , respectively. For $t \in [0, 1]$, let $x(t)$ be the first point counterclockwise on the circumference such that A values the corresponding wedge-shaped piece at t ; that is, $v_A([0, x(t)]) = t$. Moreover, let $g(t) = v_B([0, x(t)])$ be player B 's value for the same wedge-shaped piece.

Define $f(t) = g(t) - t$. The conditions on the players' measures ensure that f is continuous and $f(0) = f(1) = 0$. Extend f to the interval $[0, 2]$ by $f(t + 1) = f(t)$ for all $t \in (0, 1]$. Hence, f is periodic and satisfies the conditions of Lemma 1. Therefore, for any entitlements p for player A and $1 - p$ for player B , there exists a c in $[0, 1]$ such that $f(c + p) = f(c)$. Because the point $x(c + p)$ may be outside of $[0, 1]$, recall that we identify $x(c + p)$ with $x(c + p) \bmod 1$, and define

$$[x(c), x(c + p)] = \begin{cases} [x(c), x(c + p)] & \text{if } x(c) \leq x(c + p) \\ [x(c), 1] \cup [0, x(c + p)] & \text{if } x(c) > x(c + p). \end{cases}$$

Cutting the pie from the center to the points $x(c)$ and $x(c + p)$ on the circumference yields a wedge-shaped piece that both players A and B value at p ; that is, $v_A([x(c), x(c + p)]) = v_B([x(c), x(c + p)]) = p$. Give A the piece $[x(c), x(c + p)]$. Player B receives the remaining piece that it values at $1 - p$. The allocation has the desired entitlements.

If the allocation from Theorem 1 is not efficient, then 3 radial cuts may be used to give the two players pieces valued at exactly their entitlements, leaving a surplus piece (white space in Figure 3). For example, if A receives $[0, \alpha]$ valued at p according to its measure but B values the complement at more than $1 - p$, then it can be trimmed to give B a piece $[\beta, 1]$ valued at $1 - p$. This can be done as in Figure 3 so that the surplus is a wedge-shaped piece. Then this surplus can be divided by cutting it at some θ (between α and β in Figure 3) that mirrors the entitlements of the players; adding these portions of the surplus to the players' pieces gives them larger pieces, but still in the proportion of their entitlements. Theorem 3 formalizes this idea to show the existence of an envy-free, efficient, and proportional allocation using 2 radial cuts in which both players receive acceptable pieces.

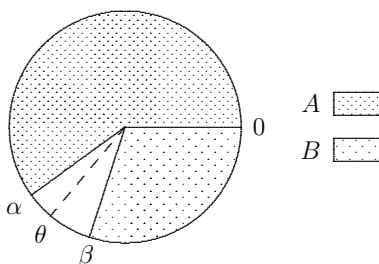


Fig. 3 Player A receives $[0, \alpha]$ valued at p while player B receives $[\beta, 1]$ valued at $1 - p$. The surplus (white space) can be divided in proportion to the players' entitlements by cutting at some θ between α and β .

Theorem 2 *For two players with entitlements p and $1 - p$, there exists an envy-free, efficient, and proportional allocation using 2 radial cuts such that*

each player receives at least its entitled share, given the players' measures are absolutely continuous.

Proof Let v_A and v_B be the measures for players A and B , respectively. For t on the circumference $[0, 1]$, let $y(t)$ be the point on the circumference in which the ratio of $v_A([t, y(t)]) : v_B([y(t), t]) = p : 1 - p$.

Because the players' measures are nonatomic and absolutely continuous, player A 's value of the piece between t and $y(t)$ is a continuous function that achieves a maximum for some $t^* \in [0, 1]$. Cut the pie with radial cuts at t^* and $y(t^*)$, giving A the piece $[t^*, y(t^*)]$ and B the complement. Because A 's piece is maximized and the allocation is proportional, this allocation is efficient.

Theorem 1 shows the existence of an allocation whereby each player receives at least its entitlement. It follows that the above maximization gives A a piece that it values at least as much as p , and B a piece that it values at least as much as $1 - p$. Because each player finds the allocation acceptable, it is envy-free.

Theorem 2 requires that the players' measures be absolutely continuous. Without absolute continuity, an initial allocation could be modified by giving more pie to one player, thereby increasing the value that that player receives, without decreasing the value of the piece that the other player receives. The following example demonstrates the necessity of absolute continuity.

Example 2 (If preferences are not absolutely continuous, there need not exist a proportional, efficient allocation.)

Assume that players A and B are equally entitled to a pie. Let player A 's measure be uniformly distributed over the disk. Hence, player A values the piece $S \subseteq [0, 1]$ by $v_A(S) = \int_S 1 dx$. Assume that player B values the piece $S \subseteq [0, 1]$ by $\int_S f_B(x) dx$ where

$$f_B(x) = \begin{cases} 4 & \text{for } x \in \left[\frac{3}{16}, \frac{5}{16}\right] \text{ and } x \in \left[\frac{11}{16}, \frac{13}{16}\right] \\ 0 & \text{otherwise,} \end{cases}$$

That is, player B places equal value on each of two sectors, centered around 12 o'clock and 6 o'clock, and this value is uniformly distributed over each sector (shaded wedges in Figure 4).

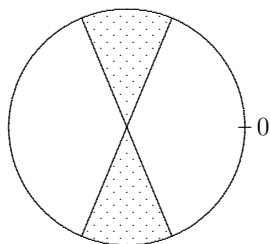


Fig. 4 Player B 's value is uniformly distributed over the sectors between the points $\frac{3}{16}$ and $\frac{5}{16}$ on the circumference and between the points $\frac{11}{16}$ and $\frac{13}{16}$ on the circumference.

The only proportional allocations occur when the pie is divided along any diameter. Under such a division, both players receive exactly half of the pie and the ratio of their pieces $\frac{1}{2} : \frac{1}{2}$ satisfies the equal entitlement

requirement. However, such an allocation is inefficient. This follows because there is an allocation in which player B receives $\frac{1}{2}$ and player A receives more than $\frac{1}{2}$: Give player B the top wedge (the pie between $\frac{3}{16}$ and $\frac{5}{16}$ on the circumference, centered around 12 o'clock) that it values at $\frac{1}{2}$, and give Player A the remainder, which it values at $\frac{7}{8}$. Although this efficient allocation is envy-free, it violates the players' equal entitlements.

When a pie is cut with two radial cuts, the first cut can be viewed as selecting the endpoints of the interval defining a cake, whereas the second cut divides the interval into two pieces. We return to Example 1 to illustrate how cutting a cake with 2 cuts can yield a proportional, efficient allocation, by viewing the cake as a pie.

Example 3 (The cake-cutting example with two cuts: A continuation of Example 1) Because player B values the cake near $\frac{1}{2}$ more than near 0 and 1, player B needs to receive the middle piece of cake for its division to be envy-free and efficient. Due to symmetry, this division occurs by cutting the cake at x and $1 - x$ for some $x \in (0, \frac{1}{2})$. Figure 5 demonstrates how the cake can be viewed as a pie: Connecting the endpoints of the line segment (cake), 0 and 1, to form a circle (pie), we can cut the pie at x and $1 - x$ to obtain an envy-free and efficient allocation.

Player A values the sum of the left and right pieces at $2x$. Player B values the middle piece at $1 - 4x^2$. Proportionality requires cutting the cake at x and $1 - x$ such that x satisfies $2x : 2(1 - 4x^2) = p : 1 - p$.

For players to receive acceptable pieces in ratio $p : 1 - p$ for $p > \frac{1}{2}$, then

$$x = \frac{(p-1) + \sqrt{5p^2 - 2p + 1}}{4p}.$$

It follows that the allocation is envy-free.

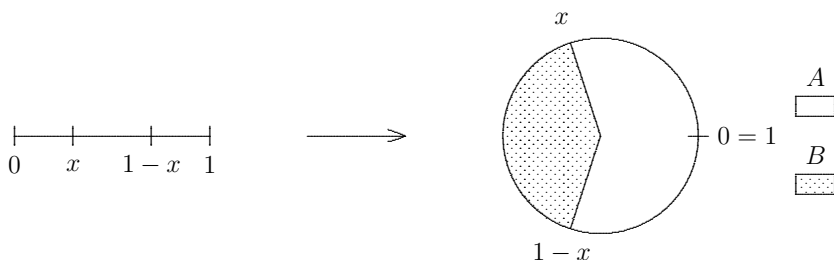


Fig. 5 By identifying the endpoints of the unit interval, the cake in Example 3 can be viewed as a pie. Player B receives the cake between x and $1 - x$, which forms a wedge-shaped piece of pie.

3 Proportional Pie-Cutting Procedures for Two Players

The existence of an allocation with specific properties does not imply a procedure whereby the players would truthfully reveal their preferences to arrive at the allocation. For example, Jones (2002) shows the existence of a one-cut allocation of cake for two players such that the allocation is envy-free, equitable (both players receive pieces that they value the same according to their measures), and efficient (with respect to single-cut allocations). But Brams *et al.* (2006) prove that there does not exist a procedure to induce the truthful revelation of preferences that would lead to this allocation, although there is a procedure that can approximate this allocation.

We find procedures that ensure that each player receives a proportional, envy-free, and efficient piece, regardless of the actions of the other player. We assume that players try to maximize the minimum-value pieces (*maximin pieces*) that they can guarantee for themselves, regardless of what the other players do. In this sense, the players are risk-averse and never strategically announce false measures if it does not guarantee them more-valued pieces. Further, we assume that the players do not know each other's preferences. This uncertainty, coupled with risk-aversion, induces the players to reveal their preferences, or measures, truthfully under the procedures.

We next present a moving-knife procedure that guarantees an allocation wherein each player receives a piece exactly equal to its rational-number entitlement. We show that risk-averse players will truthfully divide the pie into n equally valued sectors to maximize the minimum value of the pieces they receive. The main drawback of the procedure is that the allocation may be inefficient, because there is almost always a surplus that goes unallocated to the players. Even though the procedure may leave no surplus, it does not guarantee an efficient outcome, as the proof of Theorem 2 demonstrates. We describe the location of a knife from the center of the disk to a point on the circumference by its angle. This is equivalent to allowing the circumference to be of length 2π .

Proportional Pie-Cutting Procedure for Rational-Number Entitlements

Suppose that player A is entitled to $\frac{k}{n}$ of the pie and player B is entitled to $\frac{n-k}{n}$ of the pie.

1. Select a point on the circumference of the pie at random. Denote the radius from the center of the pie to this point as 0 radians. Let this angle be angle 0.

2. Player A , unobserved by player B , marks $n - 1$ additional angles that, together with angle 0, divide the pie into n sectors (dashed lines in Figure 6a).

3. Player B places one knife along the radius at angle 0 and places $n - 1$ knives from the center of the pie to the circumference at $n - 1$ angles that, together with angle 0, divide the pie into n sectors (solid lines in Figure 6a).

4. Player B rotates the n knives counterclockwise in such a way that the knives continue to define n sectors.

5. Player A stops the rotation when one of player B 's knives is coincident with one of player A 's n angles (k_4 and 4 in Figure 6b), and there are k consecutive sectors in the counterclockwise direction from this knife, according to A 's angles, that do not intersect $n - k$ consecutive sectors in the clockwise direction from this knife, according to B 's knives.

6. Player A reveals its angles. The pie is cut in three places: the two radii defining the boundary of A 's k consecutive sectors; and at the knife that, together with the knife coincident with A 's angle, forms the boundary of the $n - k$ consecutive sectors, according to B 's knives (see Figure 6b).

7. If A does not call stop before player B 's knives traverse one sector – that is, when the knife at angle 0 reaches the position of the first knife counterclockwise from angle 0 (knife k_1 in Figure 6) – then neither player receives any of the pie.

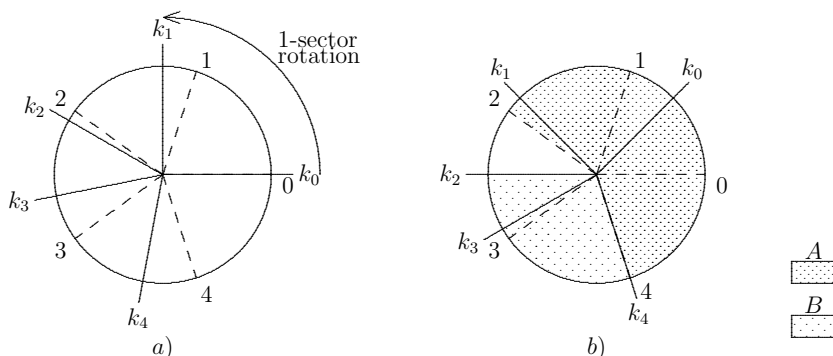


Fig. 6 Players A and B use the proportional pie-cutting procedure for rational-number entitlements to divide the pie in a ratio of $3 : 2$. *a)* Player A 's five marks divide the pie into five sectors, as do player B 's five knives. *b)* Player B rotates its knives until player A stops the rotation when knife k_4 coincides with A 's angle 4. Player A receives the 3 consecutive sectors counterclockwise from knife k_4 , and player B receives the pie between knives k_2 and k_4 .

Theorem 3 *Under the proportional pie-cutting procedure for rational entitlements, a risk-averse player A will truthfully submit $n - 1$ angles that, with angle 0, divide the pie into n equally valued sectors. A risk-averse player B will rotate its knives so as to keep the value between them equal. The players will receive pieces exactly to their entitlements, according to their respective measures.*

Proof Because neither player A nor player B is aware of which consecutive sectors it will receive, for A and B to ensure that they receive k and $n - k$ sectors, respectively, A will submit $n - 1$ angles to create, with angle 0, n equally valued sectors, according to its measure, and B will rotate its n knives to keep the value of pie between the knives equal.

It remains to show that the procedure terminates with the players receiving pieces valued at their entitlements when they follow their risk-averse strategies. Beginning with angle 0, number player A 's n angles 0 to $n - 1$ in a counterclockwise direction. Beginning with the knife k_0 at angle 0, if player B 's knives are numbered 0 through $n - 1$ in a counterclockwise direction, then a one-sector counterclockwise rotation moves knife k_i to the initial position of knife k_{i+1} (except knife k_{n-1} ends at angle 0), as shown in Figure 6a. One knife traverses each sector, so the entire pie is covered. Therefore, as player B rotates its n knives one sector, each of player A 's angles will coincide at some point with one of player B 's knives. We will show that at some angle j , there will be k consecutive sectors counterclockwise from angle j , according to player A 's angles, that do not intersect with the $n - k$ consecutive sectors clockwise from angle j , according to player B 's knives.

Define piece i to be the k consecutive sectors counterclockwise from angle i for $i = 0$ to $n - 1$, according to player A 's angles. Let u_i be the value of the complementary piece, according to player B 's measure. The complementary pieces for $i = 0$ to $n - 1$ cover the pie $n - k$ times, and $\sum_{i=0}^{n-1} u_i = n - k$. Because the average value of a complementary piece is

$\frac{n-k}{n}$, then there exists an angle j such that the complement of piece j is valued at least as much as $\frac{n-k}{n}$. It follows that when a knife is coincident with angle j , then the complement to piece j consists of $n - k$ consecutive sectors, according to player B 's knives, that do not intersect piece j , as in Figure 6b).

We next present, in four steps, a procedure that guarantees an envy-free and efficient allocation of the entire pie when the ratio of player A 's entitlement to player B 's entitlement is $p : 1 - p$ and p may be an irrational number. Because angle θ is chosen at random, the first procedure requires the players to know their measures in order to divide the pie into n equally valued pieces. In contrast, the second procedure requires the players to submit their measure functions; then the players or a referee must solve an optimization problem. In return for its increased informational demands and its increased computational complexity, the procedure gives an efficient allocation for any p . We realize that it may be difficult, if not impossible, for a player to articulate its measure for pie. However, the same procedure could be used if players were to approximate their measure functions, perhaps by evaluating many equally-angled pieces and assuming that the measure is uniform over these pieces.

Efficient Proportional Pie-Cutting Procedure

1. Each player submits its (possibly false) measure over the unit disk.
2. A referee places a first mark at $\theta = 0$ and a second at c_0 (see Figure 7); the position c_0 is chosen so as to make the ratio of A 's value $v_A([0, c_0])$

of the piece $[0, c_0]$ to B 's value $v_B([c_0, 0])$ of the remainder equal to $p : 1 - p$, according to the players' submitted measures.

3. A referee rotates the first mark θ counterclockwise between 0 and 2π , simultaneously moving c_θ to keep the proportion between the players' values $v_A([\theta, c_\theta]) : v_B([c_\theta, \theta])$ at $p : 1 - p$, according to their submitted measures.

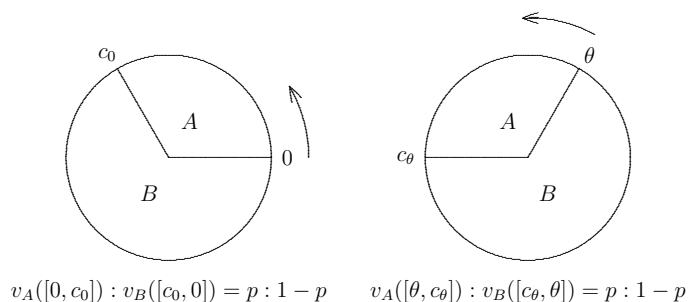


Fig. 7 A referee rotates θ counterclockwise from 0 to 2π , keeping $v_A([\theta, c_\theta]) : v_B([c_\theta, \theta]) = p : 1 - p$.

4. After a complete rotation, the referee selects an angle θ^* that maximizes value of player A 's piece $[\theta^*, c_{\theta^*}]$. This maximizes the value of B 's piece, according to B 's submitted measure, too.

This and the previous procedure are similar to the moving-knife procedures found in the cake-cutting literature. But the outcome under the second procedure can be determined without knives and without a referee. For example, a computer program could determine the angles θ^* and c_{θ^*} on the basis of the players' submitted measures. Robertson and Webb (1998) discuss the information requirements for moving-knife and other procedures in the cake-cutting literature.

Theorem 4 *Under the efficient proportional pie-cutting procedure, risk-averse players will truthfully reveal their measures. The procedure gives an envy-free and efficient allocation such that the ratio of the values of the wedge-shaped pieces that A and B receive is $p : 1 - p$, according to their submitted measures.*

Proof By Theorem 2, there exists an allocation that gives an envy-free and efficient allocation according to the submitted measures.

Suppose that player A submits the false measure g_A . Because A does not know B 's measure, it is possible that player B 's measure is g_A , too. Then player A will receive (at random) one of the infinitely many pieces valued at p according to B 's measure. Not all such pieces can be valued at more than p according to A 's measure, so A may receive a piece valued at less than p . However, A can guarantee p by being truthful. Hence, risk-averse players will truthfully reveal their measures.

4 Three or More Players

Surprisingly, increasing the number of players by just one – from two to three – may rule out proportional allocations for players with unequal entitlements using the minimal number of cuts such that each player receives a piece at least equal to its entitlement. The following two 3-player examples can be extended to more than three players. The first example is simple and intuitive, whereas the second example is more general; the latter

demonstrates that proportional allocations may fail to exist when player entitlements (rational or irrational) are not equal.

Example 4 (Measures and entitlements for which no 3-player proportional allocation exists.)

Assume that players A , B , and C are entitled to pie in the proportions $3 : 1 : 1$. The sectors representing the players' measures appear in Figure 8. Player A distributes $\frac{1}{10}$ uniformly on each of the ten sectors in Figure 8; players B and C distribute $\frac{1}{5}$ uniformly over each of their corresponding five sectors.

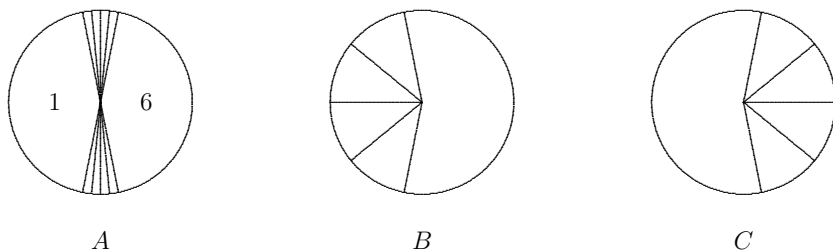


Fig. 8 An allocation of pie in the proportion $3 : 1 : 1$ does not exist for players A , B , and C .

Number player A 's sectors clockwise from 1 to 10, as shown in Figure 8 (we omit numbers 2-5 and 7-10 inclusive). To give A the $\frac{3}{5}$ of the pie to which it is entitled, A must receive a piece that contains five consecutive sectors and a portion of at least one other sector. Any five consecutive sectors must contain sector 1 or sector 6. Because player B values A 's sector 1 at $\frac{4}{5}$ and

player C values A 's sector 6 at $\frac{4}{5}$, at least one of players B or C cannot receive a piece that it values at $\frac{1}{5}$.

Lest one think that one player must be entitled to more than $\frac{1}{2}$ to preclude a proportional solution in the three-player case, consider the following example.

Example 5 (Measures and entitlements for which no 3-player proportional allocation exists in which each player is entitled to less than $\frac{1}{2}$.)

Players A and B are each entitled to $\frac{1}{3} + x$, where $0 < x < \frac{1}{24}$, and player C is entitled to $\frac{1}{3} - 2x$. Assume that the pie consists of twelve sectors, each player has a uniform distribution over each sector, and the sectors are valued as shown in Figure 9. Player C has a uniform distribution over the whole pie.

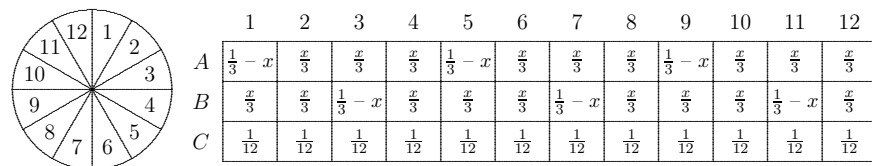


Fig. 9 An allocation of pie in the proportion $(\frac{1}{3} + x) : (\frac{1}{3} + x) : (\frac{1}{3} - 2x)$ does not exist for players A , B , and C .

For any division of the pie into equal thirds, not necessarily restricted to consist of the whole sectors shown in Figure 9, players A and B value each one-third at $\frac{1}{3}$. Hence, to receive pieces valued at their entitlements, players A and B must receive pieces that represent more than two-thirds

of the pie. Player C must receive more than one-quarter of the pie because $\frac{1}{3} - 2x > \frac{1}{4}$ when $x < \frac{1}{24}$.

Assume that A and B have each received $\frac{1}{3}$ of the pie and C has received $\frac{1}{4}$, leaving $\frac{1}{12}$ to be shared among A , B , and C . Now A and B each must get x of the pie, and C must get $\frac{1}{12} - 2x$, according to their respective measures. Because sectors valued at $\frac{1}{3} - x$ by player A and B are always between sectors valued at $\frac{x}{3}$ by both players, there is no one-twelfth of the pie that is valued at more than x by both players A and B ; at least one of these players must receive a piece valued at no more than $\frac{x}{3}$. This guarantees that no proportional allocation exists whereby each player receives a piece at least equal to its entitlement.

Just as we showed that proportional *cake* division breaks down if there are two or more players (Example 1), proportional *pie* division breaks down if there are three or more players. Still, it is pleasing that for two players, not only can a pie always be divided between them so that each, by getting at least a proportional share, is envy-free, but there is also a procedure that induces risk-averse players to be truthful, allowing such a fair division actually to be implemented.

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