Cutting a Pie Is Not a Piece of Cake

Julius B. Barbanel
Department of Mathematics
Union College
Schenectady, NY 12308
barbanej@union.edu

Steven J. Brams
Department of Politics
New York University
New York, NY 10003
steven.brams@nyu.edu

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Abstract

Gale (1993) posed the question of whether there is necessarily an undominated, envy-free allocation of a pie when it is cut into wedge-shaped pieces or sectors. For two players, we give constructive procedures for obtaining such an allocation, whether the pie is cut into equal-size sectors by a single diameter cut or into two sectors of unequal size. Such an allocation, however may not be equitable—that is, give the two players exactly the same value from their pieces.

For three players, we give a procedure for obtaining an envy-free allocation, but it may be dominated either by another envy-free allocation or by an envy-causing allocation. A counterexample shows that there is not always an undominated, envy-free allocation for four or more players if the players' preferences are not absolutely continuous with respect to each other. If we do make this assumption, the existence question remains open for four or more players; it is open for three players whether their preferences are absolutely continuous or not.

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1. INTRODUCTION. The general problem of fair division, and the specific problem of cutting a cake fairly, have received much attention in recent years (for overviews, see Brams, Taylor, and Zwicker, 1995; Brams and Taylor, 1996; Robertson and Webb, 1998; and Barbanel and Brams, 2004). Cutting a pie into wedge-shaped sectors, by contrast, has received far less attention, though it would seem that the connection between cake-cutting and pie-cutting is close.

Mathematically, if a cake is a line segment, it becomes a pie when its endpoints are connected to form a circle. However, we shall find it more convenient to think of pies as disks rather than circles. All cuts are made between the center and a point on the circumference, as one would cut a real pie, so each cut runs along a radius of the disk.

These cuts divide a pie into sectors, exactly one of which is given to each player. Each player has a countably additive, nonatomic probability measure over the pie. Thus, the value of disjoint pieces of pie can be summed, any piece of pie that has positive value to a player has a subpiece that has smaller positive value to that player, and each player assigns a value of 1 to the whole pie.

Notice that the non-atomic nature of the measures guarantees that every player's measure assigns a value of 0 to each radius of the pie. This, together with the countable additivity of the measures, implies that the preferences of the players, which are based on their measures, are continuous, enabling us to invoke the Intermediate-Value Theorem. For example, imagine two adjacent wedge-shaped pieces, and some player values one piece more than the other. As the radius separating these pieces rotates along the circumference from the less-valued piece to the more-valued piece, there will be some intermediate point where the player values the two pieces equally.

We assume that players do not know the preferences of other players. At various points in the arguments that follow, we will claim that a certain function has a maximum value. In all cases we consider, this is always justified by the Extreme-Value Theorem: A continuous function on a closed and bounded subset of a Euclidean space achieves a maximum.

More than ten years ago, Gale (1993) asked an intriguing question: Is there necessarily an allocation of the pie sectors that is envy-free and undominated. An *envy-free allocation* is one in which each player receives a sector that it believes is at least as valuable as that which any other player receives. An *undominated allocation* is one for which there is no other allocation in which at least one player receives a sector it strictly prefers, and the other players receive sectors they value at least as much.

We answer Gale's question affirmatively for two players by specifying constructive procedures that yield envy-free and undominated allocations. We do this for both wedge cuts and diameter cuts, when the wedges are 180-degree sectors that divide the pie exactly in half. (Unless otherwise stated, when we discuss wedge cuts and say that an allocation is undominated, we mean that it is undominated with respect to wedge-cut allocations. A similar statement holds for diameter cuts.)

For three players, we give a procedure for cutting a pie into three sectors such that the resulting allocation is envy-free but not necessarily undominated. In fact, we do not know whether there always exists a three-player undominated, envy-free allocation.

For four players, surprisingly, we do have an answer: There need not exist such an allocation. While the players' preferences are continuous in our counterexample, they are not absolutely continuous with respect to each other. We will say more about this later.

To summarize, for two players we have a procedure that yields an undominated and envy-free allocation of a pie, and for four players we know that such an allocation may not exist when preferences are not absolutely continuous with respect to each other. For three players, Gale's question remains open, with or without absolute continuity.

2. TWO-PLAYER DIAMETER PROCEDURES. We next specify two procedures that lead to an envy-free allocation of a pie using a single diameter cut, under the assumption that a player seeks to maximize the minimum-value piece that it can guarantee for itself, whatever the other player does. One procedure yields an *equitable allocation*—in which each player gets exactly the same value from its piece (according to its measure) as the other player gets from its piece—whereas the other yields an undominated allocation. As we will show in the next section, the undominated allocation may be dominated by an allocation that uses wedge cuts.

We give below two rules (D1 and D2) that give an envy-free and equitable allocation, but this allocation need not be undominated. When we substitute revised rules D1' and D2' for D1 and D2, respectively, we obtain an envy-free and undominated allocation, but this allocation need not be equitable.

D1. *Randomly choose a diameter of the pie and randomly assign an "up" and "down" orientation to this diameter (so that the notions of "left piece" and "right piece" determined by this diameter make sense). Assign the left piece determined by this diameter to Player A and the right piece to Player B. Rotate the diameter 360 degrees.*

As it rotates, draw two graphs. At each point in the rotation, one graph, denoted by f_A*, indicates the value that Player A assigns to its piece, and another graph, denoted by* f_B*, indicates the value that Player B assigns to its piece.*

Theorem 1. f_A *and* f_B *have at least one point of intersection.*

Proof. Suppose, by way of contradiction, that this is not so. Since the measures are countably additive and non-atomic, f_A and f_B are graphs of continuous functions. If two such graphs do not intersect, then one of the graphs must always be above the other. Assume, without loss of generality, that f_A is always above f_B. Since this is true at the beginning and at the 180-degree point, additivity tells us that Player A assigns greater value to the whole pie than Player B does. But this contradicts our assumption that all measures assign value 1 to the whole pie. □

Theorem 2. *For at least one point of intersection, both players receive a common value of at least 50%.*

Proof. Any point of intersection corresponds to an allocation in which both players get a common value. It is easy to see that intersection points come in pairs, separated by 180 degrees. If both players get less than 50% at some intersection point, then both players must get more than 50% at this intersection's 180-degree pair. □

D2. *Choose an intersection point that maximizes the common value of the players. Make the diameter cut at this point, allocating to each player its preferred half, or either half if the maximum is 50% for both players.*

Theorem 3. *The resulting allocation is envy-free and equitable.*

Proof. Immediate from the construction. □

We next extend D1, and revise D2, to give rules that ensure that the resulting allocation is envy-free and undominated.

D1'. *Randomly choose a diameter of the pie and randomly assign an "up" and "down" orientation to this diameter. Assign the left piece determined by this diameter to Player A and the right piece to Player B. Rotate the diameter 360 degrees. As it rotates, draw two graphs. At each point in the rotation,* f_A *indicates the value that the Player A assigns to its piece, and* f_B *indicates the value that Player B assigns to its piece. Use these two graphs to draw the two new graphs,* $max(f_A, f_B)$ *and* $min(f_A, f_B)$*, the maximum and the minimum of* f_A *and* f_B*, respectively.*

D2'. *Make the diameter cut that corresponds to the maximum value of* $min(f_A, f_B)$*. If the maximum value of* $min(f_A, f_B)$ *occurs at more than one point, choose the point that has the largest (or a tied-for-largest)* $max(f_A, f_B)$ *value. Make the diameter cut at this point, and allocate to each player its preferred half.*

We note that because the measures are countably additive and non-atomic, f_A and f_B, and therefore $\max(f_A, f_B)$ and $\min(f_A, f_B)$, are graphs of continuous functions. Hence, the Extreme-Value Theorem applies.

Theorem 4. *The resulting allocation is envy-free and undominated.*

Proof. Assume, without loss of generality, that the resulting allocation occurs at a point where $\min(f_A, f_B)$ agrees with f_A, and that the value of the piece of pie corresponding to this point is $x\%$. Thus, Player A obtains $x\%$, which must be at least 50% by Theorem 2. Then Player B must obtain $y\% \geq x\%$. Hence, the resulting allocation is envy-free. By D2', whichever player receives the less-valued piece in any other allocation cannot obtain more than x% in that allocation; and if one player receives x%, the other player cannot obtain more than y%. Hence, the resulting allocation is undominated. □

Next we present an example to show that an envy-free allocation may be (i) equitable but dominated or (ii) undominated but inequitable.

Example 1. *An envy-free and equitable allocation that is dominated, and an envy-free and undominated allocation that is inequitable*.

We associate points on the circumference of the pie with the numbers on a clock, indicating the degrees of each sector in parentheses. Players A and B associate the following values with two different sectors of the pie, where each player's two sectors together comprise 12 hours:

Player A: 11-1 o'clock (60°)—90%; 1-11 o'clock (300°)—10%.

Player B: 3-9 o'clock (180°)—60%; 9-3 o'clock (180°)—40%.

We assume that each player's valuation is uniformly distributed within each sector. The initial assignments, as given by D1 or D1', are the diameter from 12 to 6 o'clock, with Player A taking the 12-6 o'clock piece and Player B taking the 6-12 o'clock piece. In addition, we assume the rotation is clockwise. The corresponding graphs are as in Figure 1.

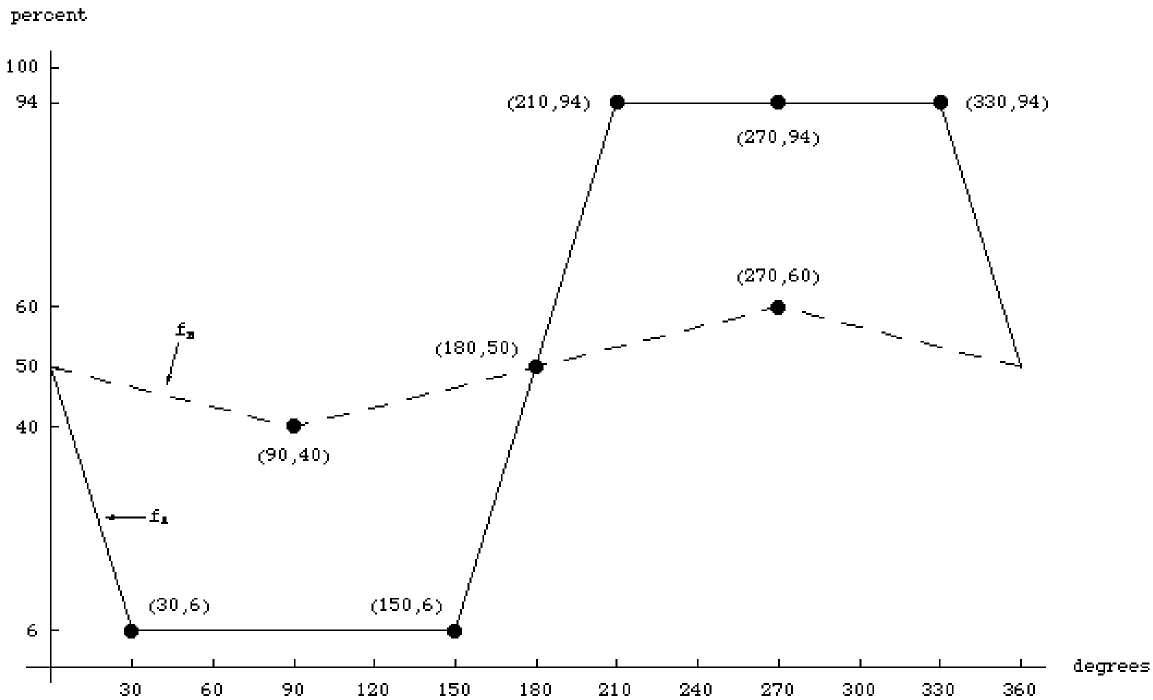

Figure 1:

Percent Allocations of Pie to Players A and B in Example 1 as

Diameter Cut Rotates Clockwise 360 Degrees from 12-6 O'Clock Position

The two graphs intersect at three points: (0, 50), (180, 50), and (360, 50). Of course, the first and third of these points correspond to the same allocation, whereas the second is different. Because the two different allocations, which are the result of applying rules D1 and D2, give both players exactly 50% of the pie, they are envy-free and equitable. But it is clear from Figure 1 that each is dominated: Any rotation greater than 180 degrees and less than 360 degrees results in an allocation that is better for both players.

In particular, note that the point (270, 60) is the maximum point on $\min(f_A, f_B)$. This is the only point at which the maximum is attained. Thus, the application of rules

D1' and D2' results in a rotation of 270 degrees, giving an allocation in which Player A receives 60% of the pie and Player B receives 94% of the pie, each according to its own valuation. While this allocation is envy-free and undominated, it is clearly inequitable.

Of course, there are examples in which an envy-free allocation is both undominated and equitable. For example, if Players A and B both value a pie uniformly, giving them each a different half yields an envy-free, equitable, and undominated allocation. There are also non-trivial situations in which rules D1 and D2 yield the same allocation as do rules D1' and D2', and hence the resulting allocation is envy-free, equitable, and undominated. We conclude this section by presenting such an example.

Example 2. *Rules D1 and D2 may give the same result as rules D1' and D2'.*

Players A and B associate the following values with two different sectors that comprise 12 hours:

Player A: 12-6 o'clock (180°)—40%; 6-12 o'clock (180°)—60%.

Player B: 4-8 o'clock (120°)—60%; 8-4 o'clock (240°)—40%.

We assume that each player's valuation is uniformly distributed within each sector. As in Example 1, the initial assignments, as given by D1 and D1', are the diameter from 12 to 6 o'clock, with Player A taking the 12-6 o'clock piece and Player B taking the 6-12 o'clock piece. The rotation is clockwise. The corresponding graphs are as in Figure 2.

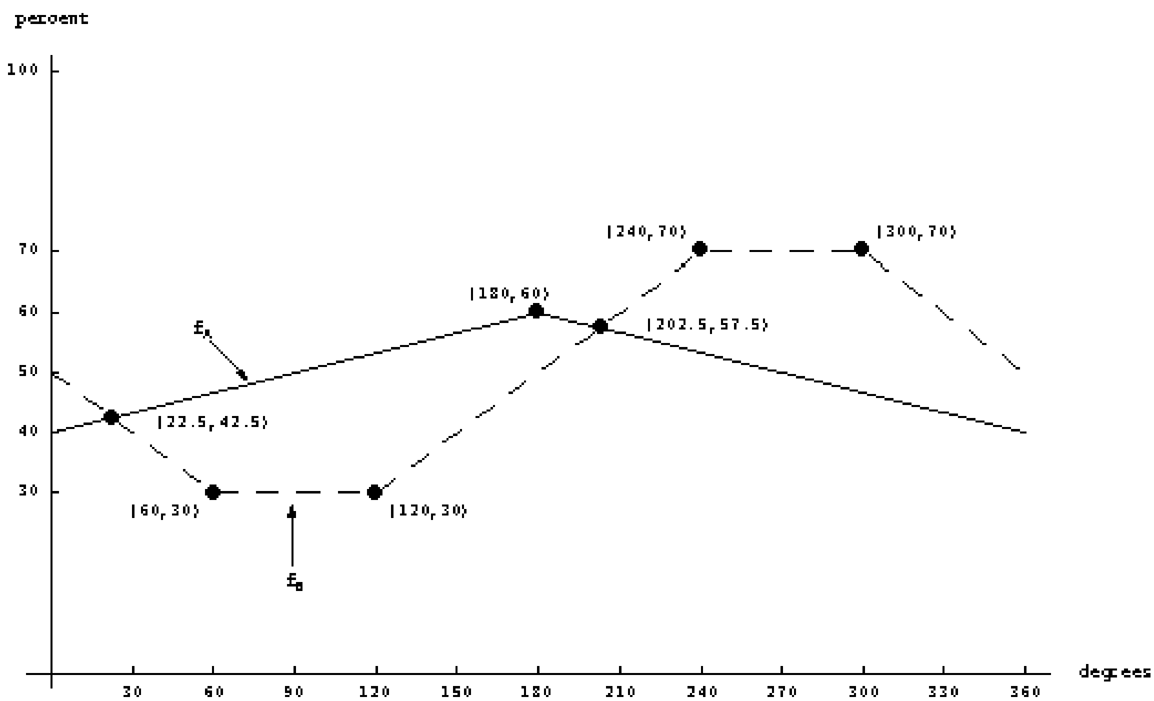

Figure 2:

Percent Allocations of Pie to Players A and B in Example 1 as

Diameter Cut Rotates Clockwise 360 Degrees from 12-6 O'Clock Position

The two graphs intersect at the points (22.5, 42.5) and (202.5, 57.5). (The allocations corresponding to these points can be viewed as being obtained from each other by having the players switch pieces: They are 180 degrees apart, and the sum of each player's valuation of its piece in these two allocations is 100%.) Clearly, (202.5, 57.5) corresponds to an envy-free allocation, and (22.5, 42.5) does not; the former is the envy-free and equitable allocation obtained from rules D1 and D2. In this allocation, the two players receive pieces of pie that they value equally at 57.5%, which occurs at a rotation of 202.5 degrees.

Now we apply rules D1' and D2'. Something very curious—and different from Example 1—occurs. The point (202.5, 57.5) is the maximum point on the graph of min(f_A, f_B), showing that at a rotation of 202.5 degrees, each player receives 57.5% of the pie, according to each's own valuation. Consequently, rules D1 and D2 result in the same allocation as do rules D1' and D2' Thus, this allocation is envy-free, equitable, and undominated, whereas one of the envy-free allocations in Example 1 was equitable but dominated, and the other was undominated but inequitable.

3. TWO-PLAYER WEDGE PROCEDURE. In this section, we give four rules, W1, W2, W3, and W4, for obtaining a wedge allocation that is undominated and envy-free. After doing so, we use Example 2 to show that such an allocation can dominate an undominated diameter allocation. (Recall that "undominated diameter allocation" means "undominated with respect to other diameter allocations.")

W1. *Player A places two knives at radii such that the piece of pie so determined is 50% of the pie, according to Player A. Player A rotates the knives 360 degrees around the pie, maintaining the 50% size of its piece.*

W2. *Player B chooses the position of the knives such that one of the pieces so determined by A's rotation is of maximal size in its view.*

It is straightforward to show that the aforementioned piece will have a value of at least 50% to Player B. Thus, if we cut the pie at these knife locations and give Player B its preferred piece and Player A the other piece, then the allocation is envy-free. However, it may be necessary to perform an additional operation to ensure that the allocation is undominated.

W3. *Player B places the knives at the boundary of its maximal piece obtained in W2. Suppose that, in Player B's view, this piece is y% of the pie. Player B rotates the knives 360 degree around the pie, maintaining this y% size in its view.*

W4. *Player A chooses the position of the knives such that the other piece of the pie (i.e., not the one whose size, according to Player B, is being maintained at y%) is of maximal size in its view. At this maximum value for Player A in the rotation, give Player B the sector whose size it was maintaining at y%, and give Player A the other sector.*

Theorem 5. *The resulting allocation is envy-free and undominated.*

Proof. Suppose that Player A views its piece as x% of the pie. By construction, $x \geq 50$ and, as noted above, $y \geq 50$. Thus, the allocation is envy-free. Also, by construction, if Player B obtains y% in any other allocation, then Player A cannot obtain more than x%; and if Player A obtains 50%, then Player B cannot obtain more than y%. It follows that if Player A obtains x%, then Player B cannot obtain more than y% (because $x \geq 50$). Hence, the allocation is undominated. □

It may be that a wedge allocation dominates an allocation obtained using rules D1' and D2' (which need only be undominated with respect to diameter cuts). To illustrate, we return to Example 2. As we saw, rules D1' and D2' yield a diameter cut after a rotation of 202.5 degrees, and this allocation gives Players A and B a common value of 57.5%, according to each's own valuation. This allocation is dominated by the wedge allocation that gives the 8-4 o'clock sector to Player A and the 4-8 o'clock sector to Player B, which yields Player A a value of (2/3)(60%) + (2/3)(40%) = 66 2/3% and Player B a value of 60% of the pie.

4. THREE-PLAYER WEDGE PROCEDURE. We next give a procedure, whose rules are stated in the next paragraph, for dividing a pie into three sectors that results in an envy-free allocation. Unlike the two-player procedures, this procedure does not ensure either an equitable or an undominated allocation.

Rules of three-player wedge procedure. Player A rotates three knives around a pie, each along a radius, maintaining a 1/3-1/3-1/3 allocation for itself. Player B calls "stop" when it thinks two of the pieces are tied for largest, which must occur for at least one set of positions in the rotation (see below). The players then choose pieces in the order C first, B second, and A third.

Theorem 6. *The three-player wedge procedure yields an envy-free allocation.*

Proof. To show that there must be at least one set of knife positions in the rotation such that Player B thinks there are two pieces that tie for most-valued, let us call the three pieces determined by the beginning positions of the knives piece *i*, piece *ii*, and piece *iii*. (These pieces will change as Player A rotates the knives.) Let Player B specify its most-valued piece at the start of the rotation. If there is a tie, then we are done. If not, then Player A begins rotating the three radial knives. We assume, without loss of generality, that Player B's most-valued piece at the start of the rotation is piece *i*, and that Player A rotates the three knives in such a way that piece *i* moves toward the original position of piece *ii*. Because, in Player A's view, each of the three pieces is 1/3 of the pie, piece *i* will eventually occupy the position of the original piece *ii*. At this point, piece *iii* occupies the position of the original piece *i*, and hence Player B must think that this new piece *iii* is the largest piece. Because, in Player B's view, piece *i* starts out largest and another piece becomes largest as the rotation proceeds, it follows that there must be a position in the rotation when Player B views two pieces as tied for largest. (This uses the continuity that follows from the fact that the measures are countably additive and non-atomic.)

To see that the procedure gives an envy-free allocation, note that the first player to choose, Player C, can take a most-valued piece, so it will not be envious. If Player C takes one of Player B's tied-for-most-valued pieces, Player B can take the other one; otherwise, Player B can choose either of its two tied-for-most-valued pieces. Because Player A values all three pieces equally, it does not matter which piece it gets. □

An allocation given by the three-player wedge procedure need not be undominated. For example, a rotation of the three knives by Player A could break Player B's tie of two largest pieces in a way that gives some players more-valued pieces and no player a less-valued piece. Also, the three-player wedge procedure does not, in general, give equitability.

We next give an example to illustrate the three-player wedge procedure. Not only does it show that that an envy-free allocation may be dominated by another envy-free allocation, but it also shows that it may be dominated by an *envy-causing allocation*, in which at least one player thinks that another player receives a larger portion than it does.

Example 3. *An envy-free allocation among three players that is dominated by another envy-free allocation and by an envy-causing allocation.*

Players A, B, and C associate the following values with two different sectors each:

Player A: 12-4 o'clock (120°)—2/3; 4-12 o'clock (240°)—1/3.

Player B: 12-4 o'clock (120º)—1/2; 4-12 o'clock (240º)—1/2.

Player C: 12-2 o'clock (60º)—1/2; 2-12 o'clock (300º)—1/2.

We assume that each player's valuation is uniformly distributed within each sector. We start by applying the 3-person wedge procedure. One set of positions of Player A's three knives that cuts the pie into three equal-valued sectors for A is (1, 3, 8) o'clock, because A obtains the following values from each sector:

1-3 o'clock: (1/2)(2/3) = 1/3;

3-8 and 8-1 o'clock: (1/4)(2/3) + (1/2)(1/3) = 1/3 each.

If (1, 3, 8) o'clock are, in fact, the *initial* positions of Player A's three knives, then Player B will call "stop" immediately, because two of the sectors (3-8 and 8-1 o'clock) tie for largest for B [value: (1/4)(1/2) + (1/2)(1/2) = 3/8 each]. By contrast, the 1-3 o'clock sector is worth (1/2)(1/2) = 1/4 to B.

Now Player C will choose the 8-1 o'clock sector, whose value to it is (2/5)(1/2) + (1/2)(1/2) = 9/20, which is more than the 1-3 o'clock sector [value: (1/2)(1/2) + (1/10)(1/2) = 3/10] or the 3-8 o'clock sector [value: (1/2)(1/2) = 1/4]. Next, Player B will choose the 3-8 o'clock section (3/8), which is more than the 1-3 o'clock sector (1/4); and Player A will be left with the 1-3 sector (1/3). In sum, the 3-person wedge procedure yields values of

(1/3, 3/8, 9/20) = (.333…, .375, .450) to (A, B, C) in sectors (1-3, 3-8, 8-1). (i)

This is an envy-free allocation, as we have shown. However, it is dominated by cutting the pie at (2, 6, 12), which yields values of

(5/12, 3/8, 1/2) = (.416…, .375, .500) to (A, B, C) in sectors (2-6, 6-12, 12-2). (ii)

This is not only an envy-free allocation but also an undominated one. (We leave the proof of the latter to the reader, because our main purpose is to illustrate that one envy-free allocation may dominate another one.)

We next show that (i) may be dominated by an envy-causing allocation. Consider the undominated, envy-free allocation (ii), and switch the 2 o'clock cutpoint to 2:30 o'clock. The resulting allocation yields values of

(1/3, 3/8, 21/40) = (.333…, .375, .525) to (A, B, C) in sectors (2:30-6, 6-12, 12-2:30). (iii)

This allocation is not envy-free: A envies C for getting what it thinks is 5/12 in the 12-2:30 o'clock sector, which is more than its 1/3 allocation in the 2:30-6 o'clock sector. But (iii) dominates wedge-procedure allocation (i), which is envy-free.

Although envy-causing allocation (iii) dominates envy-free allocation (i), it does not dominate envy-free allocation (ii). Indeed, (iii), like (ii), is undominated, which we leave for the reader to show.

We leave open the question of whether there is always a three-player undominated, envy-free allocation of a pie. For cake, the answer is "yes," and there are two procedures, using the minimal two cuts, for finding such an allocation (Stromquist, 1980; Barbanel and Brams, 2004). In fact, *every* envy-free allocation of a cake using the minimal number of cuts (n-1 if there are n players) is undominated (Gale, 1993; see also Brams and Taylor, 1996, pp. 150-151), but there is no known procedure for finding such an allocation if $n > 3$.

5. FOUR PLAYERS: THERE MAY BE NO UNDOMINATED, ENVY-FREE ALLOCATION

Theorem 7. *If there are four players, there exists a pie and corresponding measures for which there is no allocation that is both envy-free and undominated.*

Proof. The players associate the following values with two different sectors each, but these sectors do not comprise all 12 hours:

- **Players A & B**: 12-3 o'clock (90º)—50%; 6-9 o'clock (90º)—50%;
- **Players C & D**: 3-6 o'clock (90º)—51%; 9-12 o'clock (90º)—49%.

We assume that each player's valuation is uniformly distributed within each sector.

Claim 1. No envy-free allocation can give Player A or Player B a wedge that contains the entire 3-6 o'clock sector or a wedge that contains the entire 9-12 o'clock sector.

Proof. Suppose this is not so and assume, without loss of generality, that Player A receives all the 9-12 o'clock sector. Then envy-freeness demands that Players C and D receive equal portions of the 3-6 o'clock sector. But then Players C and D each view their pieces as at most 25 1/2% of the pie and are, therefore, envious of Player A, which receives 49% of the pie in their view. This contradicts our assumption that the allocation is envy-free and thus establishes the claim.

Claim 2. No envy-free and undominated allocation can give both Player A and Player B some of the 12-3 o'clock sector, and no envy-free and undominated allocation can give both Player A and Player B some of the 6-9 o'clock sector.

Proof. Assume this is not so and suppose, without loss of generality, that some envy-free and undominated allocation gives both Player A and Player B some of the 12-3 o'clock sector, and that Player B's portion of this sector is clockwise of Player A's portion. By Claim 1, neither Player A nor Player B can receive any of the 6-9 o'clock sector.

Assume, without loss of generality, that Player C's piece is the next piece clockwise from Player B's piece. Since the allocation is undominated, the radius separating Player B's and Player C's pieces must be at 3 o'clock (else a movement of this radius would produce an allocation that dominates the given allocation). Likewise, the radius separating Player D's and Player A's pieces must be at 12 o'clock.

Envy-freeness implies that the radius separating Player A's and Player B's pieces must be at 1:30, splitting the 12-3 o'clock sector equally. Thereby, each of these players gets a piece that it views as 25% of the pie.

Where is the radius separating Player C's and Player D's pieces? This radius must be at 7:30, thereby splitting the 6-9 o'clock sector into equal pieces of size 25% of the pie from the perspectives of Players A and B. Any other position of this radius would make Players A and B envy either Player C or Player D. But then Player D will be envious of Player C, because Player D views its piece as 49% of the pie, but it views Player C's piece as 51% of the pie. This is a contradiction and thus establishes the claim.

Continuing with the proof of the theorem, it follows from the two claims that we may assume, without loss of generality, that in any envy-free and undominated allocation, Player A receives some of the 12-3 o'clock sector, that Player B receives some of the 6-9 o'clock sector, that Player C's piece is the next piece clockwise from Player

A's piece, and that Player D's piece is the next piece clockwise from Player B's piece. An argument similar to that used in the proof of Claim 2 shows that because the allocation is undominated, the radii separating the players' pieces must be at 3, 6, 9, and 12 o'clock. It follows that the allocation gives the players wedges that they value as follows:

Players A & B: 50% each; **Player C**: 51%; **Player D**: 49%.

Because Player D views Player C's wedge as being 51% of the pie, Player D envies Player C. Thus, there is no allocation that is envy-free and undominated. □

It is trivial to extend the proof of Theorem 7 to more than four players. For example, for five players, simply add a new sector that Player E views as 100% of the value of the pie and the other four players see as valueless.

Measures on a pie are *absolutely continuous with respect to each other* if, whenever a piece of pie has positive measure to one player, it has positive measure to all players. The measures that underlie the players' preferences in Theorem 7 are not absolutely continuous with respect to each other. By contrast, all our earlier theorems required no assumption about absolute continuity—they held with or without this assumption.

We summarize our results, with "yes" and "no" answers, in the table below.

	Does there always exists an allocation that is envy-free and undominated	
	—if the measures are absolutely continuous with respect to each other?	—with no assumption about absolute continuity?

Two Players	Yes	Yes
Three Players	Unknown	Unknown
Four or More Players	Unknown	No

The “unknowns” in the table leave us with two open questions:

Open Question 1. For three players, does there always exist an undominated, envy-free allocation of a pie (with or without the assumption that players’ measures are absolutely continuous with respect to each other)?

Open Question 2. For four or more players, does there always exist an undominated, envy-free allocation of a pie if the players’ measures are absolutely continuous with respect to each other?

Clearly, answers to these questions—as well as others that have been recently discussed by Brams, Jones, and Klamler (2005) and Thomson (2005)—would fill gaps in our understanding of pie-cutting, which is certainly not a piece of cake.

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