

**Cake Division with Minimal Cuts:  
Envy-Free Procedures for 3 Persons, 4 Persons, and Beyond\***

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# **Cake Division with Minimal Cuts: Envy-Free Procedures for 3 Persons, 4 Persons, and Beyond**

Abstract: The minimal number of parallel cuts required to divide a cake into  $n$  pieces is  $n - 1$ . A new 3-person procedure, requiring 2 parallel cuts, is given that produces an envy-free division, whereby each person thinks he or she receives at least a tied-for-largest piece. An extension of this procedure leads to a 4-person division, using 3 parallel cuts, that makes at most one person envious. Finally, a 4-person envy-free procedure is given, but it requires up to 5 parallel cuts, and some pieces may be disconnected. All these procedures improve on extant procedures by using fewer moving knives, making fewer people envious, or using fewer cuts.

## **1. Introduction**

The literature on fair division has burgeoned in recent years, with five academic books (Young, 1994; Brams and Taylor, 1996; Robertson and Webb, 1998; Moulin, 2003; Barbanel, 2004) and one popular book (Brams and Taylor, 1999) providing overviews. There is also a more specific literature on cake-cutting—our focus here—which concerns the fair division of a divisible heterogeneous good over which different people may have different preferences.

Some of the cake-cutting procedures that have been proposed are discrete, whereby players make cuts with a knife—usually in a sequence of steps—but the knife is not allowed to move continuously over the cake. Moving-knife procedures, on the other hand, permit such continuous movement and allow players to call “stop” at any point at which they want to make a cut or mark. While there are now about a dozen such procedures for dividing a cake among three players such that each player is assured of getting a largest or tied-for-largest piece (Brams, Taylor, and Zwicker, 1995)—and so will not envy another player (resulting in an *envy-free division*)—only one procedure

(Stromquist, 1980) makes the envy-free division with only two cuts. This is the minimal number for three players; in general  $n - 1$  cuts is the minimum number of cuts required to divide a cake into  $n$  pieces. A cake so cut ensures that each player gets a single connected piece, which is desirable in certain applications (e.g., for land division).

For two players, the well-known procedure of “I cut, you choose” leads to an envy-free division if the cutter divides the cake 50-50 in terms of his or her preferences. By taking the piece he or she considers larger and leaving the other piece for the cutter (or choosing randomly if the two pieces are tied in his or her view), the chooser ensures that the division is envy-free. However, this procedure does not satisfy certain other desirable properties (Jones, 2002; Brams, Jones, and Klamler, 2004).

The moving-knife equivalent of “I cut, you choose” is for a knife to move continuously across the cake, say from left to right. Assume that the cake is cut when one player calls "stop." If each of the players calls "stop" when he or she perceives the knife to be at a 50-50 point, then the first player to call "stop" will produce an envy-free division if he or she gets the left piece and the other player gets the right piece. (If both players call "stop" at the same time, the pieces can randomly be assigned to the two players.)

Surprisingly, to go from two players making one cut to three players making two cuts cannot be done by a discrete procedure if the division is to be envy-free (Robertson and Webb, 1998, pp. 28-29; additional information on the minimum numbers of cuts required to give envy-freeness is given in Even and Paz, 1984, and Shishido and Zeng, 1999). The 3-person discrete procedure that makes the fewest cuts is one discovered independently by John L. Selfridge and John H. Conway about 1960; it is described in,

among other places, Brams and Taylor (1996) and Robertson and Webb (1998) and requires up to five cuts.

Although there is no known discrete 4-person envy-free procedure that uses a bounded number of cuts, Brams, Taylor, and Zwicker (1997) give a moving-knife 4-person procedure that requires up to 11 cuts. Peterson and Su (2002) give an analogous 4-person envy-free moving-knife procedure for chore division, whereby each player thinks he or she receives the smallest (or tied-for-smallest) piece of an undesirable item.

In this paper, we show that (i) Stromquist's 3-person envy-free moving-knife procedure and (ii) Brams, Taylor, and Zwicker's 4-person envy-free moving-knife procedure can be improved upon, but in two different senses. In the case of (i), its two cuts are already minimal; however, we will give another 2-cut procedure that requires only two *simultaneously* moving knives, not the four that Stromquist's procedure requires. In the case of (ii), we will, like Brams, Taylor, and Zwicker (1997), require more than one simultaneously moving knife (in some cases, we require five) but show that their 11-cut maximum can be reduced to a 5-cut maximum.

Our 3-person, 2-cut procedure is simpler than Stromquist's and will serve to introduce the notion of "squeezing," which will be used repeatedly in our 4-person, 5-cut procedure. This 4-person, 5-cut procedure is arguably no simpler than that of Brams, Taylor, and Zwicker (1997): while it reduces the maximum number of cuts needed to produce an envy-free division by more than half, it requires more stages and finer distinctions to implement than that of Brams, Taylor, and Zwicker (1997).

We pave the way for introducing the 4-person, 5-cut envy-free procedure by describing a simple 4-person, 3-cut procedure that gives each player a *proportional*

*piece*--one that he or she thinks is at least  $1/n$  of the cake if there are  $n$  players. (If all players receive what they believe to be proportional pieces, the division is said to be *proportional*.) But more than giving a proportional division, the 4-person, 3-cut procedure makes at most one player envious, which we characterize as *almost envy-freeness*.

Our 4-person, 5-cut procedure is not as complex as Brams and Taylor's (1995) general  $n$ -person discrete procedure. Their procedure illustrates the price one must pay for a procedure that works for all  $n$ : not only is it more complex than any bounded procedure we know of, but it also places no upper bound on the number of cuts that are required to produce an envy-free division; this is also true of other  $n$ -person envy-free procedures (Robertson and Webb, 1997; Pikhurko, 2000). While the number of cuts needed will depend on the players' preferences over the cake, it is worth noting that Su's (1999) approximate envy-free procedure uses the minimal number of cuts at the expense of little error.

The paper proceeds as follows. In section 2 we give the 3-person, 2-cut envy-free procedure that uses only two simultaneously moving knives. In section 3, we build on this procedure to present the almost envy-free 4-person, 3-cut procedure, which also uses only two simultaneously moving knives.

In section 4, we give the 4-person envy-free procedure that uses at most 5 cuts. Unlike the preceding procedures, in which the pieces assigned to the players are connected, some of the four pieces that constitute the envy-free division produced by this procedure may be the union of two or three non-adjacent pieces. Moreover, the 4-person,

5-cut procedure is far more complicated than either the 3-person, 2-cut envy-free procedure or the 4-person, 3-cut almost envy-free procedure.

Curiously, while we know that there exists a 4-person, 3-cut envy-free division (more on the existence question later), we know of no procedure that implements it. In section 5 we speculate on how such a procedure might work. We also discuss the possibility of finding bounded procedures that yield envy-free divisions for more than four persons. We conclude that if they exist, they may be of mathematical interest but are likely to be very complicated and of little or no practical value. Accordingly, we suggest new directions in cake-cutting research.

## **2. A 3-Person, 2-Cut Envy-Free Procedure**

To begin the analysis, we make the following assumptions that will be used throughout the paper:

1. The goal of each player is to maximize the minimum-size piece (*maximin piece*) he or she can guarantee for himself or herself, regardless of what the other players do. To be sure, a player might do better by not following such a *maximin strategy*; this will depend on the strategy choices of the other players. In the subsequent analysis, however, we assume that all players are *risk-averse*: they never choose strategies that might yield them larger pieces if they entail the possibility of giving them less than their maximin pieces.

2. The preferences of the players over the cake are continuous, enabling us to invoke the intermediate-value theorem. Suppose, for example, that a knife moves across a cake from left to right and, at any moment, the piece of the cake to the left of the knife is A and the piece to the right is B. If, for some position of the knife, a player views

piece A as being larger than piece B, and for some other position he or she views piece B as being larger than piece A, then there must be some intermediate position such that the player values the two pieces the same.

3. We want the cake between two cuts to be connected. (In general, this need not be so. Imagine a two-dimensional circular cake with a hollow center.) In order to guarantee that such a piece is connected, we can make either of the following two equivalent assumptions: the cake is convex and all cuts are parallel to each other, or the cake is a line segment. We make the latter assumption, which will simplify our subsequent discussion.

4. Let A be the piece of a cake between two given knives, and suppose that the left knife is moved rightward while the right knife is kept stationary. Then we want the movement of the left knife to be such that every player sees piece A as converging to size 0 as this process continues. To ensure convergence, we assume that the knife is moved at a constant speed by a neutral party, whom we call a *referee*. We will also allow players to move knives—sometimes two at once—to change the sizes of pieces. In this case, the speeds of these knives may vary in a manner that will depend on the situation.

The notion of “speed” is sensible if we imagine that the cake is located on the real line. By assuming that a knife moves at a constant speed, we avoid the situation in which the piece is seen as decreasing in size but not converging to 0. To show how the latter situation can arise, fix some point  $x$  strictly between the position of the left and right knives. If the left knife is moving in such a way that, in each second that passes, its distance to point  $x$  is halved (and thus the speed of the knife is decreasing), the players

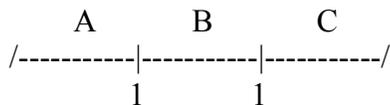
will not view the size of piece A as converging to 0. The assumption of constant speed, however, ensures convergence to 0.

Throughout the paper, we will refer to players by number, i.e., player 1, player 2, etc. We will call odd-numbered players “she” and even-numbered players “he.”

We next describe the 3-person, 2-cut envy-free procedure and show that it gives an envy-free solution. While the cuts are made by two knives in the end, initially one player makes “marks,” or virtual cuts, on the line segment defining the cake; these marks may subsequently be changed by another player before the real cuts are made.

**Theorem 1.** *There is a moving-knife procedure for three players that yields an envy-free division of a cake using two cuts.*

**Proof.** Assume a referee moves a knife from left to right across a cake. The players are instructed to call "stop" when the knife reaches the  $1/3$  point for each. Let the *first* player to call "stop" be player 1. (If two or three players call "stop" at the same time, randomly choose one.) Have player 1 place a mark at the point where she calls "stop" (the right boundary of piece A in the diagram below), and a second mark to the right that bisects the remainder of the cake (the right boundary of piece B below). Thereby player 1 indicates the two points that, for her, trisect the cake into pieces A, B, and C:



Because neither player 2 nor player 3 called "stop" before player 1 did, each of players 2 and 3 thinks that piece A is at most  $1/3$ . They are then asked whether they prefer piece B or piece C. There are three cases to consider:

1. If players 2 and 3 each prefer a different piece—one player prefers piece B and the other piece C—we are done: players 1, 2, and 3 can each be assigned a piece that they consider to be at least tied-for-largest.

2. Assume players 2 and 3 both prefer piece B. A referee places a knife at the right boundary of B and moves it to the left. Meanwhile, player 1 places a knife at the left boundary of B and moves it to the right in such a way that the amounts of cake traversed on the left and right are equal for player 1. Thereby pieces A and C increase equally in player 1's eyes. At some point, piece B will be diminished sufficiently to B'—in either player 2 or player 3's eyes—to tie with either piece A' or C', the enlarged A and C pieces. Assume player 2 is the first, or tied for the first, to call "stop" when this happens; then give player 3 piece B', which she still thinks is the largest or the tied-for-largest piece. Give player 2 the piece he thinks ties for largest with piece B' (say, piece A'), and give player 1 the remaining piece (piece C'), which she thinks ties for largest with the other enlarged piece (A'). Clearly, each player will think he or she got at least a tied-for-largest piece.

3. Assume players 2 and 3 both prefer piece C. A referee places a knife at the right boundary of B and moves it to the right. Meanwhile, player 1 places a knife at the left boundary of B and moves it to the right in such a way as to maintain the equality, in her view, of pieces A and B, as they increase. At some point, piece C will be diminished sufficiently to C'—in either player 2 or player 3's eyes—to tie with either piece A' or B', the enlarged A and B pieces. Assume player 2 is the first, or the tied for first, to call "stop" when this happens; then give player 3 piece C', which she still thinks is the largest or the tied-for-largest piece. Give player 2 the piece he thinks ties for largest with piece



Since players 2, 3, and 4 did not call “stop,” each of these players thinks that A is at most  $1/4$ , and hence that B is at least  $3/4$ .

Next, use Theorem 1, applied to just piece B, to obtain an envy-free division of B among players 2, 3, and 4. Now consider the division of the entire cake that gives the pieces so obtained to players 2, 3, and 4, and gives piece A to player 1. Player 1 thinks that piece A is exactly  $1/4$ . Players 2, 3, and 4 each think that their piece is at least  $1/3$  of B, and that B is at least  $3/4$ . Hence, each thinks that their piece is at least  $1/4$ . This establishes that the division is proportional.

Players 2, 3, and 4 do not envy each other. Neither do they envy player 1, since they all believe that piece A is at most  $1/4$  of the cake. However, player 1, even though she gets a proportional piece (i.e., piece A is at least  $1/4$  for her), may still envy either one or two of the other players. (Player 1 cannot envy all three of the other players, because her proportional piece A rules out the possibility that all three remaining pieces are greater than  $1/4$ .) Thus, the division is proportional and at most one of the four players (i.e., player 1) is envious, making the procedure almost envy-free. Q.E.D.

In general, a different almost envy-free division of the cake will result if the knife of the referee moves from right to left instead of from left to right. In this case, the possibly envious player will be the one that is the first to call "stop" from the right and who therefore gets the piece defined by the right boundary of the cake and the mark placed by the second player to call "stop" from the right.

Although we have not succeeded in finding a 4-person envy-free procedure that uses only 3 cuts, the almost envy-free 3-person procedure just described is better at

reducing envy than the well-known moving-knife procedure of Dubins and Spanier (1961). Under the Dubins-Spanier procedure, a referee moves a knife from left to right across a cake. The first player to call "stop" gets the piece to his or her left of the point where the knife stops, the next player to call "stop" gets the next piece to his or her left, and so on.

A maximin strategy for this procedure is for each player to call "stop" when he or she perceives the knife to have traversed  $1/m$  of the cake not already allocated, where  $m$  is the number of players that have not yet called "stop". Thereby each player ends up with a proportional piece. In particular, the first player to call "stop" will get what he or she believes to be  $1/n$ . But, if no other player called "stop" at the same time as this player, all the other players will obtain pieces they believe to be greater than  $1/n$ , because they perceive the first piece to be less than  $1/n$  of the cake and therefore have more than  $(n-1)/n$  of the cake to divide.

Suppose the Dubins-Spanier procedure is used by four players, and player 1 is the first to call "stop" when she perceives  $1/4$  of the cake to have been traversed. She is now out of the picture, so to speak, and will envy at least one of the other players unless she thinks player 2 and player 3, the second and third players to call "stop," did so exactly at the two points where she (player 1) would have trisected the remainder of the cake. Likewise, player 2 will be envious unless he thinks player 3 called "stop" exactly at the point where he (player 2) would have bisected the remainder.

Note that while player 3 thinks she creates a two-way tie for largest in making the last cut, player 4, who never called "stop" before any other player (but perhaps called "stop" at the same time as another player), will get what he thinks is the single largest

piece (on the right), unless he called "stop" at the same times as players 1, 2, and 3.

Although neither player 3 nor player 4 will be envious, players 1 and 2 may be. Thus, the Dubins-Spanier procedure, because it can make as many as two players envious, is not almost envy-free.

#### 4. A 4-Person, 5-Cut Envy-Free Procedure

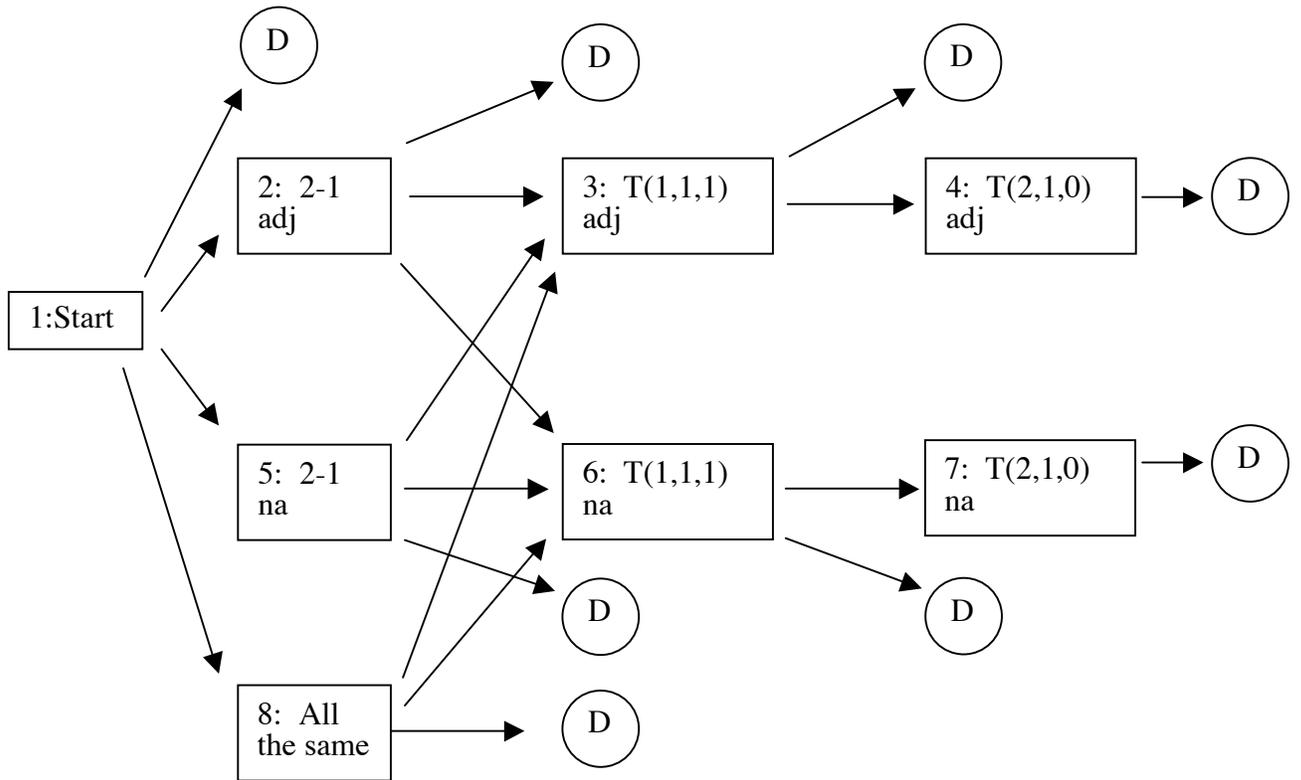
In this section, we show how to use the notion of squeezing to produce an envy-free division among four people using at most five cuts. It will be convenient to do most of the analysis in the context of pie division, rather than cake division. We will prove a theorem on envy-free pie division from which the cake-division theorem will easily follow.

What is the difference between pie division and cake division? When cutting a cake, our convention is that any two cuts are parallel, and this justified our perspective that our cake can be viewed as a line segment. When cutting a pie, by contrast, we assume that the pie is a disk and that all cuts are between the center and a point on the circumference (as we would cut a real pie). Then, just as our parallel-cut assumption for a cake justified our viewing the cake as a line segment, our present assumption justifies our viewing the pie as a circle. Finally, we randomly choose a point on the circle, break the circle at this point, and view the pie as a line segment with the endpoints identified.

**Theorem 3.** *There is a moving knife procedure for four players that yields an envy-free division of a pie using at most five cuts. Three of the four players will each receive a connected piece and the other player will receive either a connected piece or else a union of two such pieces.*

In the proof, we shall frequently refer to Figure 1, so we first discuss this figure and then begin giving the details of the proof. The figure provides a kind of flow chart for the procedure we are about to describe.

**Figure 1**



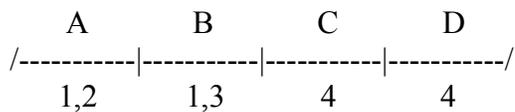
In the figure, each box or circle represents a state in the process. An arrow from state  $i$  to state  $j$  indicates that, in following the procedure to be described, moving from state  $i$  to state  $j$  is a possibility. If there is only one arrow leaving state  $i$ , and that arrow goes to state  $j$ , then going to state  $j$  is the only possibility upon leaving state  $i$ .

The D's in circles stand for "done." When we arrive at such a state, we will have produced an envy-free division of the pie using the required number of cuts.

We must explain the  $T(p,q,r)$  notation in the figure. At each stage in the process after the Start state, there will be a temporary assignment of pieces of pie to each of the four players. Thus, at any point in the process, we may ask questions such as, "Which piece does player 1 think is the largest piece," or "Does player 2 think that there are two pieces that are tied for largest?" We define  $T(p,q,r)$ , where  $T$  denotes "tie," as follows:

1. There are  $p$  players that believe there is a (two-way) tie for largest piece. Let this tie be between pieces A and B.
2. Besides these  $p$  players,  $q$  players believe piece A is largest.
3. Besides these  $p$  players,  $r$  players believe piece B is largest.
4. Any players not among these  $p + q + r$  players believe that the other two pieces (i.e., C and D) are tied for largest. (In every case we consider, it will turn out that  $p + q + r = 3$ , so there is only one "other" player.)

An example of  $T(1,1,1)$  is given by

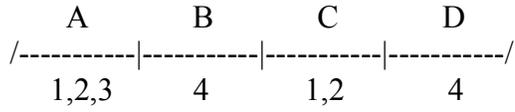


where a player's number under a piece of pie indicates that that player views the piece as at least tied-for-largest.

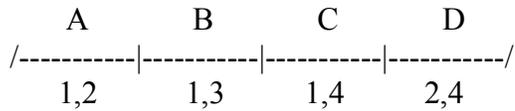
Notice that this type of diagram is similar, but not identical, to that used earlier. In sections 2 and 3, the numbers indicated marks put by players on the cake, whereas in this

section the numbers are used to keep track of the largest and tied-for-largest pieces of the players.

An example of  $T(2,1,0)$  is given by



When we write “ $T(p,q,r)$ ,” we do *not* exclude the possibility of other ties. So, for example,



is still an illustration of  $T(1,1,1)$ .

Observe that this example is  $T(1,1,1)$  in three different ways. One involves the tie between pieces A and B for player 1, one involves the tie between pieces A and D for player 2, and one involves the tie between pieces C and D for player 4.

In the situations just considered, there is a natural distinction to be made, depending on whether the two tied pieces are adjacent or non-adjacent. In the Figure 1, “adj” denotes “adjacent” and “na” denotes “non-adjacent.” So, in our two examples above, the first is “ $T(1,1,1)$  adj” and the second is “ $T(2,1,0)$  na.”

Notation for states 2, 5, and 8 in Figure 1 will be explained in the proof.

Throughout the proof, we shall refer to pieces A, B, C, and D of pie. These are the pieces of the division shown in our diagrams above; when the process is complete, each player will be given exactly one of these pieces. But in contrast with our usage in sections 2 and 3 - where A, B, C, and D denoted the initial pieces and A', B', C', and D'

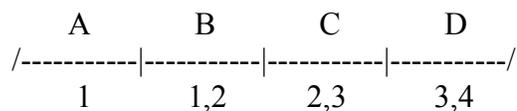
denoted the pieces at the end of the process - the pieces are not fixed but change throughout the procedure.

We will refer to the knife between pieces A and B as knife A/B, the knife between pieces B and C as knife B/C, and so on. We remind the reader that, because we are presently considering a pie rather than a cake, the left and right endpoints are identified. Hence, pieces A and D are adjacent and separated by knife D/A.

**Proof of Theorem 3.** We first note that in order to divide a pie into four pieces, one for each player, at least four cuts are required. If we divide the pie using four cuts, each player will receive a connected piece. If we use five cuts, then three players receive a connected piece, and the fourth player receives either a connected piece, or else a union of two such pieces. Hence, the second sentence of the theorem follows easily from the first.

To prove the first sentence, refer to Figure 1. It is clear from the figure that all paths lead to state D. Thus, we must show that the figure is correct and the number of cuts is as claimed. We examine each state in the figure.

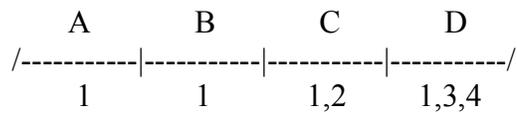
Assume that our players have been arbitrarily named players 1, 2, 3, and 4. Given a division of the pie, we say that a player *prefers* a given piece of pie if that piece is largest or tied for largest in that player's view. A player may prefer more than one piece of pie; we will not mention this fact if it is not needed. Thus, for example, for the division



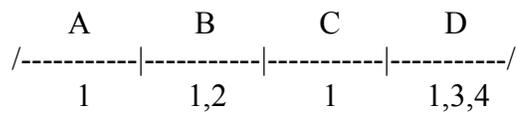
we say that each player prefers a different piece because, if we give piece A to player 1, piece B to player 2, piece C to player 3, and piece D to player 4, then each player receives a piece that he or she believes is largest or tied for largest.

Notice that whenever we arrive at a point in our procedure where each player prefers a different piece of pie (as in the previous example), we are done.

State 1: This is where we begin. Player 1 positions knives so as to divide the pie into four equal pieces, in her view. Each of players 2, 3, 4 picks a piece that he or she prefers. (Although ties will be central later in the proof, we ignore ties at this stage and have players select just one piece, breaking a tie randomly if necessary.) If all three players select just one piece, then we are done. If two prefer the same piece and one prefers a different piece then, without loss of generality, we may assume that one of the following situations occur:

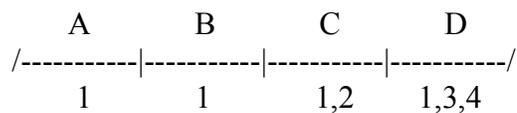


or



These are states 2 and 5, respectively. If all three prefer the same piece, then we are in state 8.

State 2: Assume, without loss of generality, that the situation is as follows:



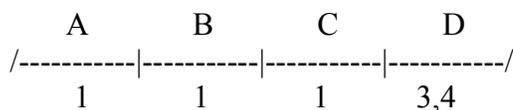
With knife C/D kept stationary, squeeze piece D by having a referee move knife D/A to the left. (There will now be some of piece A on the right end of the diagram.) Player 1 controls knives A/B and B/C and moves these knives in such a way so as to maintain, in her view, the equality of pieces A, B, and C.

Only player 3 or 4 can call “stop.” One of them will do so when she or he believes that piece D shrinks, and piece A, B, or C expands, to the point that piece A, B, or C is now tied for largest with piece D.

We make the following observations:

- a. From player 1’s perspective, piece D is shrinking and pieces A, B, and C are getting larger. Hence, she will not think that (the new) piece D is the largest piece.
- b. Since knife C/D is not moving, knife B/C is moving to the left. Hence, piece C is going through superset changes and piece D is going through subset changes. Hence, player 2 will not think that (the new) piece D is largest or tied for largest.
- c. Player 2 may think that (the new) piece A or (the new) piece B is now the largest piece.
- d. Player 3 or 4 must eventually call “stop,” because each believes that piece D is tending toward size 0.

Assume, without loss of generality, that player 3 says “stop.” Then the situation is



where, in addition to the preferences shown, players 2 and 3 each prefers piece A, B, or C. If they prefer the same piece, and this is piece A or C, then we are in state 3. If they prefer the same piece, and this is piece B, then we are in state 6. If they prefer different pieces, then all four players prefer different pieces, and we are done.

State 3: Assume, without loss of generality, that the situation is as follows:

A	B	C	D
1	1	2,3	3,4

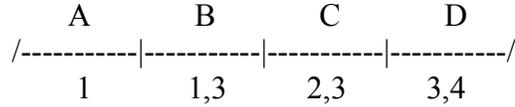
We wish to squeeze pieces C and D. With knife C/D stationary, a referee moves knife D/A to the left. Player 3 controls knife B/C and moves it to the right so as to maintain, in her view, the equality of pieces C and D. Player 1 controls knife A/B and moves it so as to maintain, in her view, the equality of pieces A and B.

Only player 2, 3, or 4 can call “stop.” Player 3 calls “stop” if piece A or B becomes tied for largest. Player 2 calls “stop” if piece A, B, or D becomes tied for largest. Player 4 calls “stop” if piece A, B, or C becomes tied for largest.

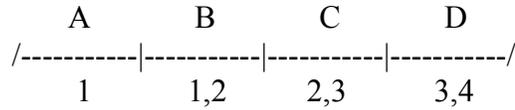
We make the following observations:

- a. From player 1’s perspective, pieces C and D are shrinking and pieces A and B are getting larger. Hence, she will not think that (the new) piece C or (the new) piece D is the largest piece.
- b. Player 2, 3, or 4 must eventually call “stop,” because each believes that pieces C and D are tending toward size 0.

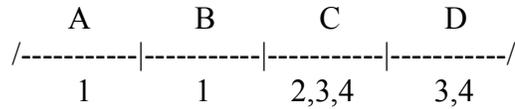
Without loss of generality, we may assume that this procedure leads to one of the following situations:



or

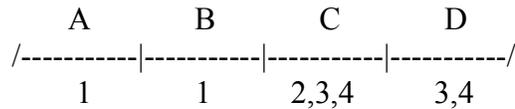


or



In the first two cases, the players prefer different pieces, and we are done. In the third case, we are in state 4.

State 4: Assume, without loss of generality, that the situation is as follows:



We wish to squeeze pieces C and D. With knife C/D kept stationary, a referee moves knife D/A to the left. Players 2, 3, and 4 each has a knife. Call these knives B/C-2, B/C-3, and B/C-4, respectively, because these knives will be taking the place of knife B/C. Each of these three players moves his or her knife so as to maintain (in each's view) the equality of pieces C and D. Then knives B/C-3 and B/C-4 begin where knife B/C was. Since player 2 initially thinks that piece C is larger than piece D, he will begin by placing his knife (i.e., B/C-2) to the right of where knife B/C was.

We now rename these three knives. At any point in the process, let B/C-x, B/C-y, and B/C-z, denote these knives in left-to-right order. It will also be convenient to rename players 2, 3, and 4 in this way. Thus, we let player x be the player that controls knife

B/C- $x$ , player  $y$  be the player that controls knife B/C- $y$ , and player  $z$  be the player that controls knife B/C- $z$ . Since the order of B/C-2, B/C-3, B/C-4 can change during the process, the meanings of “knife B/C- $x$ ,” “knife B/C- $y$ ,” and “knife B/C- $z$ ,” and the meaning of “player  $x$ ,” “player  $y$ ” and “player  $z$ ” can change.

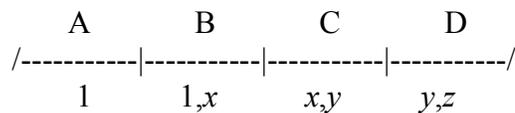
Player 1 controls knife A/B. She moves knife A/B so as to maintain, in her view, the equality of pieces A and B, where “B” refers to the piece between knife A/B and knife B/C- $y$ .

Only player 2, 3 or 4 can call “stop.” In determining when to call “stop,” each player looks only at knife B/C- $y$ . Player 2, 3, or 4 calls “stop” when piece A or B becomes tied for largest, in his, her, or his own view. The cut that separates pieces B and C is made by knife B/C- $y$ .

We make the following observations:

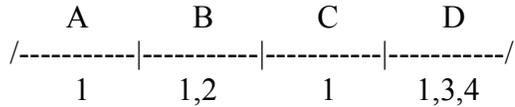
- a. From player 1’s perspective, pieces C and D are shrinking and pieces A and B are getting larger. Hence, she will not think that (the new) piece C or (the new) piece D is the largest piece.
- b. Player 2, 3, or 4 must eventually call “stop,” because each believes that pieces C and D are tending toward size 0.

Player  $y$  prefers pieces C and D, and players  $x$  and  $z$  prefer different pieces C or D. In addition, one of players  $x$ ,  $y$ , or  $z$  prefers piece A or B, and player 1 prefers pieces A and B. Without loss of generality, we may assume that this procedure leads to the following situation:



The players prefer different pieces, and we are done.

State 5: This is similar to state 2. Assume, without loss of generality, that the situation is as follows:



With knives B/C and C/D kept stationary, squeeze piece D by having a referee move knife D/A to the left. Player 1 controls knife A/B and an additional knife that we will call knife X. Knife X starts at the same place as knife A/B. As we proceed, knife X will be to the left of knife A/B. The piece between knives X and A/B is now a part of piece C.

Notice that because knives B/C and C/D do not move, the piece between these two knives obviously does not change in size. Player 1 moves knives A/B and X so as to maintain, in her view, the equality of pieces A, B, and C, where piece A is the piece between knives D/A and X, and piece C now consists of two parts, the old part and the new part, which is the piece between knives X and A/B.

Only player 3 or 4 can call “stop.” One of these players will do this when she or he believes that piece D shrinks, and piece A, B, or C expands, to the point that piece A, B, or C is now tied for largest with piece D.

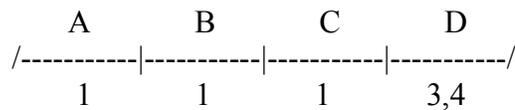
We make the following observations:

- a. From player 1’s perspective, piece D is shrinking and pieces A, B, and C are each getting larger. Hence, she will not think that (the new) piece D is the largest piece.
- b. Since knife B/C is stationary and player 1 sees pieces A, B, and C as all getting larger, it follows that knife A/B is moving to the left. Thus,

piece B is going through superset changes. Since piece D is going through subset changes, player 2 will not think that (the new) piece D is the largest piece.

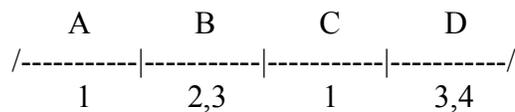
- c. Player 2 may think that (the new) piece A or (the new) piece C is now the largest piece.
- d. Player 3 or 4 must eventually call “stop,” because both believe that piece D is tending toward size 0.

Then, the situation is



where, in addition to the preferences shown, player 2 and either player 3 or 4 prefers piece A, B, or C. (For clarity, we have not shown knife X or the new part of piece C in the diagram above.) If they prefer different pieces, then all four players prefer different pieces, and we are done. If they prefer the same piece, and this is piece A or C, then we are in state 3. If they prefer the same piece, and this is piece B, then we are in state 6.

State 6: This is similar to state 3 and also includes ideas introduced in our study of state 5. Assume, without loss of generality, that the situation is as follows:



We wish to squeeze pieces B and D. With knives B/C and C/D stationary, a referee moves knife D/A to the left. Player 3 controls knife A/B and moves it to the right so as to maintain, in her view, the equality of pieces B and D.

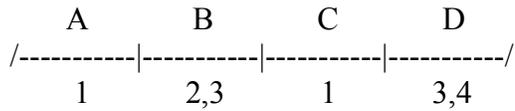
Player 1 controls a new knife, knife X. As in state 5, knife X starts at the same place as knife A/B. As we proceed, knife X will be to the left of knife A/B. The piece between knives X and A/B is now a part of piece C. Player 1 moves knife X so as to maintain, in her view, the equality of pieces A and C, where piece A is the piece between knives D/A and X, and piece C now consists of two parts, the old part and the new part, which is the piece between knives X and A/B.

Only player 2, 3 or 4 can call “stop.” Player 2 calls “stop” if piece A, C, or D becomes tied for largest in his view. Player 3 calls “stop” if piece A or C becomes tied for largest in her view. Player 4 calls “stop” if piece A, B, or C becomes tied for largest in his view.

We make the following observations:

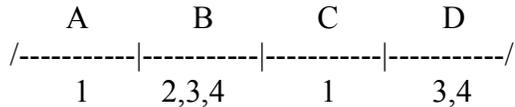
- a. From player 1’s perspective, pieces B and D are shrinking and pieces A and C are getting larger. Hence, she will not think that (the new) piece B or (the new) piece D is the largest piece.
- b. Player 2, 3, or 4 must eventually call “stop,” because each believes that pieces B and D are tending toward size 0.
- c. Player 4 views pieces B and D as getting smaller (as does everyone), because each is going through subset changes. However, player 4 may think that piece B is getting smaller at a slower rate than is piece D and so, at some point, he may think that piece B is tied for largest. The same is true for player 2, with the roles of B and D reversed.

Then the situation is



where, in addition to the preferences shown, player 2, 3, or 4 prefers a new piece. (As in our study of state 5, we have not shown knife X or the new part of piece C in the diagrams above.) If one of these players prefers piece A or C, then all four players prefer different pieces, and we are done. If player 2 prefers piece D, or player 4 prefers piece B, then we are in state 7.

State 7: This is similar to state 4 and also includes ideas introduced in our study of state 5. We assume, without loss of generality, that the situation is as follows:



We wish to squeeze pieces B and D. With knives B/C and C/D kept stationary, a referee moves knife D/A to the left. Players 2, 3, and 4 each has a knife. Call these knives A/B-2, A/B-3, and A/B-4, respectively. These knives take the place of knife A/B. Each of these three players moves his or her or his knife so as to maintain (in each's own view) the equality of pieces B and D. Note that knives A/B-3 and A/B-4 begin where knife A/B was. Since player 2 initially thinks that piece B is larger than piece D, he will begin by placing his knife (i.e., A/B-2) to the right of where knife A/B was.

Next, we rename these knives, and the corresponding players, as we did for state 4. At any point in the process, let A/B-x, A/B-y, and A/B-z, denote these knives in left-to-right order, and rename players 2, 3, and 4 in this way. In other words, player x is the

player that controls knife  $A/B-x$ , player  $y$  is the player that controls knife  $A/B-y$ , and player  $z$  is the player that controls knife  $A/B-z$ . As in state 4, we note that the meanings of “knife  $A/B-x$ ,” “knife  $A/B-y$ ,” and “knife  $A/B-z$ ,” and the meaning of “player  $x$ ,” “player  $y$ ” and “player  $z$ ” can change.

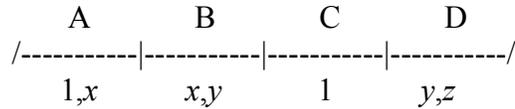
Player 1 controls a new knife, knife  $X$ . As in states 5 and 6, knife  $X$  starts at the same place as (the original) knife  $A/B$ . The piece between knives  $X$  and  $A/B-y$  is now a part of piece  $C$ . Player 1 moves knife  $X$  so as to maintain, in her view, the equality of pieces  $A$  and  $C$ , where piece  $A$  is the piece between knives  $D/A$  and  $X$ , and piece  $C$  is the old piece  $C$  together with the piece between knives  $X$  and  $A/B-y$ .

Only player 2, 3, or 4 can call “stop.” In determining when to call “stop,” each player looks only at knife  $A/B-y$ . Player 2, 3, or 4 calls “stop” when piece  $A$  or  $C$  becomes tied for largest, in his, her, or his own view. The cut that separates pieces  $A$  and  $B$  is made by knife  $A/B-y$ .

We make the following observations:

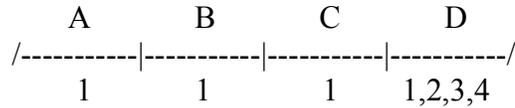
- a. From player 1’s perspective, pieces  $B$  and  $D$  are shrinking and pieces  $A$  and  $C$  are getting larger. Hence, she will not think that (the new) piece  $B$  or (the new) piece  $D$  is the largest piece.
- b. Player 2, 3, or 4 must eventually call “stop,” because each believes that pieces  $B$  and  $D$  are tending toward size 0.

Player  $y$  prefers pieces  $B$  and  $D$ , and players  $x$  and  $z$  prefer different pieces  $B$  or  $D$ . In addition, one of players  $x$ ,  $y$ , or  $z$  prefers piece  $A$  or  $C$ , and player 1 prefers pieces  $A$  and  $C$ . Without loss of generality, we may assume that this procedure leads to the following situation:



The players prefer different pieces, and we are done.

State 8: We assume, without loss of generality, that the situation is as follows:



We squeeze piece D by keeping knife C/D fixed and having a referee move knife D/A to the left. Player 1 controls knives A/B and B/C and moves them so as to maintain, in her view, the equality of pieces A, B, and C.

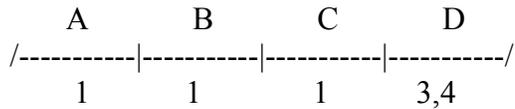
Only player 2, 3, or 4 can call “stop.” We do not actually stop the process until the moment when the second player has called “stop.” Each player calls "stop" when piece A, B, or C is, in his or her view, tied for largest.

An issue arises in our analysis of this state that did not arise in any other state. Notice that it need not be the case that players 2, 3, and 4 all view pieces A and B as increasing. (They will view piece C as increasing, because it is going through superset changes.) Therefore, as the process goes on, we must allow a player who has called “stop” to take it back. For example, assume that player 2 decides, at some point, that piece A is tied for largest with piece D and calls “stop.” But, before a second player calls “stop,” player 2 might decide that piece D is now the largest (not tied with any other). In this case, we allow player 2 to take back his “stop.”

We make the following observations:

- a. From player 1's perspective, piece D is shrinking and pieces A, B, and C are getting larger. Hence, she will not think that (the new) piece D is the largest piece.
- b. We will eventually have a second player that calls "stop," because players 2, 3, and 4 each believes that piece D is tending toward size 0.

Without loss of generality, we may assume that the situation is



where, in addition to the preferences shown, players 2 and 3 each prefer piece A, B, or C. If they prefer different pieces, then the four players prefer different pieces, and we are done. If they prefer the same piece, and this is piece A or C, then we are in state 3. If they prefer the same piece, and this is piece B, then we are in state 6.

This concludes our analysis of the eight states. We have shown that we always complete the procedure and arrive at an envy-free division. Concerning the number of cuts, we need only observe that in every case, we made cuts using knives A/B, B/C, C/D, D/A, and sometimes X. Hence, we have used at most 5 cuts. Q.E.D.

**Theorem 4.** *There is a moving knife procedure for four players that yields an envy-free division of a cake using at most five cuts. Also, either*

- a. *two of the four players each receives a connected piece, and each of the other two players receives either a connected piece or else a union of two such pieces, or*
- b. *three of the four players each receives a connected piece, and the other player either receives a connected piece, a union of two such pieces, or a union of three such pieces.*

**Proof.** Theorem 4 follows easily from Theorem 3. Given a cake, we temporarily pretend that it is a pie by identifying the endpoints. We then apply Theorem 3 to obtain an envy-free division of the pie, using at most 5 cuts, such that three of the four players each has a connected piece and the other player either has a connected piece or else a union of two such pieces. Then we return to our original cake by breaking the identification of the endpoints. Clearly, the number of cuts is still at most five. The various possibilities listed in the theorem correspond to whether breaking the identification of the endpoints causes no new disconnection, or causes a disconnection in a previously connected piece and, if it does cause a new disconnection, whether this new disconnection occurs in a piece that was already disconnected (and so now is the union of three pieces). Q.E.D.

## 5. Conclusions

It would be wonderful if we could somehow eradicate envy entirely for four or more players with a procedure that requires only the minimal  $n - 1$  parallel cuts. In principle, this is possible. Stromquist (1980) and Woodall (1980) proved that there exists an  $n$ -person envy-free division of a cake, using only  $n - 1$  parallel cuts (for recent extensions, see Ichiishi and Idzik (1999)). But how to achieve such a lovely division is by no means evident.

The squeezing operation that we successfully used for three persons seems only capable of giving almost envy-freeness for four persons if we insist on only three cuts. To guarantee envy-freeness, we showed that two additional cuts beyond the minimal three suffice for four persons, which implies that the pieces some players receive may be disconnected. This is not appealing if it is land that is being divided and all the players want connected pieces.

The problem with finding an envy-free solution, using only  $n - 1$  cuts, seems to be that the operations for moving knives that we allow, as well as the information that the players have, is insufficient to give such a solution. While the procedures put a great deal of weight on creating ties, it seems that the players need to be able to make cuts that take into account more information about the valuations of the other players to effect an envy-free division with  $n - 1$  cuts. Just as trisecting an angle with only a straightedge and a compass is impossible, we suspect that a 4-person, 3-cut procedure is also impossible unless new operations are allowed or new information about the relative valuations of the pieces by different players is introduced.

Consider the possibility of giving the players more information. Assume they know not only their own valuations of the cake but also are told the other players' valuations. Then it should be possible for them to calculate an envy-free solution that uses only  $n - 1$  cuts, because we know such a solution exists.

But this calculation introduces two problems. First, there may be many solutions. Indeed, because such an envy-free solution is efficient (or Pareto-optimal) among the set of solutions using  $n - 1$  parallel cuts (Brams and Taylor, 1996, pp. 149-151), different solutions will favor different players. (It is not known whether envy-free pie division with radial cuts is efficient; see Gale, 1993, p. 51.) Which of a possible infinity of solutions is fairest?

Even if a unique solution is agreed upon, the second problem is finding rules of a game that would enable the players to implement such a solution as an equilibrium outcome. It should be an equilibrium so that the players, once they reach it, will have no reason to depart from it. But the rules should also give the players an incentive to choose it, especially if there are other equilibria, by making the desired equilibrium dynamically stable in the sense that the players' optimal strategies in a multi-stage game would lead them to select it.

Alternatively, an arbitrator might be asked to calculate such a solution from the players' preferences. In that case, however, the players may not have an incentive to be truthful in revealing their preferences. Creating "incentive compatibility"—by making it in the interest of the players to be truthful—is also a problem in designing the rules of a game without an arbitrator if the players can indeed benefit from not being truthful.

The procedures we have described are not incentive compatible—they can be manipulated by wily players. As we indicated earlier, however, any attempt by a player to gain a larger piece of cake (e.g., by not calling "stop" when there is a tie but waiting a bit longer) carries the risk of that player's getting less. In effect, the strategies our procedures prescribe ensure the maximin outcomes of envy-freeness and almost envy-freeness, but players willing to take chances may, on occasion, do better by departing from these strategies.

Patently, challenges remain for finding better cake-cutting procedures. Our 4-person, 5-cut envy-free procedure is hardly one we would expect players to use; the situation surely gets worse for five or more players if one makes envy-freeness the *sine quo non* of a cake-cutting solution. Almost envy-free procedures, or those that give approximate envy-free solutions (Brams and Kilgour, 1996, pp. 130-133; Su, 1999; Zeng, 2000) or invoke other criteria of fairness like the amount of competition for a good (Brams and Kilgour, 2001), seem fruitful ways to go. Another promising direction is to change the rules of the cake-cutting game along the lines mentioned earlier by (i) putting more information at the disposal of the players and (ii) giving them more opportunities to make adjustments in boundaries in a manner that facilitates the selection of fair outcomes. Still another approach that has proved fruitful (Brams, Jones, and Klamler, 2004) is to introduce another divisible good, money, that enables players, aided by a referee, to make a division that is not only envy-free but also "equitable" (each player thinks that he or she receives the same-size portion).

We encourage thinking hard about these alternatives to expand the storehouse of simple and practicable procedures. Ultimately, we hope, they would be applicable to the settlement of real-life disputes of the kind discussed in Brams and Taylor (1999).

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