Efficient Fair Division: Help the Worst Off or Avoid Envy?

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Abstract. Two or more players rank a set of indivisible items from best to worst. An efficient allocation of items is characterized, which may satisfy such properties as maximin, Borda maximin, and envy-avoidance. Whereas the two maximin properties are in conflict with envy-avoidance, there is always an efficient allocation that does not ensure envy, but it may not be maximin or Borda maximin. Computer calculations show that maximin allocations lead to envy quite often, but Borda maximin allocations do so only rarely. Implications of the theoretical findings for real-world fair-division problems are discussed.

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1. Introduction

Probably the most-discussed trade-off in economics is between efficiency and equity (LeGrand, 1991). While efficiency is arguably best achieved by unfettered competition in the marketplace that maximizes total wealth, the marketplace may be extremely unfair in distributing this wealth, especially to people who lack certain skills or social support (Roemer, 2000).

The efficiency vs. equity trade-off has analogues in other fields. In politics, centralized government may be less democratic, but decentralized government may be more wasteful and corrupt (though some analysts would claim just the opposite). In law, economic justice may best be served by giving people rights to a good education, decent housing, and other economic-enhancing opportunities, but these rights may vitiate their incentives to strive to their fullest, encouraging shirking instead.

In this paper, we postulate that the *sine qua non* of any fair distribution of items is *efficiency* (or *Pareto-optimality*): There is no other distribution that can help some players without hurting others, or at least not making them worse off. Assuming efficiency, we then focus on the trade-offs in the fair *distribution* of items, which we assume are indivisible and which may range from the marital property in a divorce to religious sites in an international dispute.

Our framework for comparing the fairness of efficient distributions is spare. We assume that each player can strictly rank all items that are to be divided from best to worst (ordinal preferences) but not attach numerical values, or cardinal utilities, to the items (cardinal preferences).[1] While this assumption eliminates the practical difficulty of

[1] This is the framework that is used in Brams and Fishburn (2000), Edelman and Fishburn (2001), and Brams, Edelman, and Fishburn (2001a, 2001b), which contain numerous references to the literature on fair

comparing large numbers of subsets of items (or bundles)—over a million, or 2^{20}, if there are 20 items—it also may produce indeterminacy. For example, if there are four items, {1, 2, 3, 4}, and a player ranks them from best to worst in the order 1234, it is impossible to say from the ordinal rankings alone whether this player prefers the bundle {1, 4} to the bundle {2, 3}, or vice versa.[2] Realistically, this is information players may not possess—even of *their own* preferences if the bundles are quite large—so it is reasonable to assume only simple rankings and derive fairness consequences from these.

To facilitate comparisons among efficient distributions of items, we postulate two criteria of fairness (which we will refine shortly):

(1) helping the worst-off player by maximizing the minimum rank of items that any player receives (*maximin*);

(2) avoiding envy by preventing, insofar as possible, any player from receiving a set of items it considers inferior to a set received by another player (*envy-freeness*).

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division of indivisible goods. But these papers do not emphasize, or give general results on, the fundamental trade-off encapsulated in the subtitle of this paper. While the present paper is self-contained, we mention, as background reading, eight recent books by economists, mathematicians, and philosophers relevant to the topics discussed herein: Broome (1991), Young (1994), Moulin (1995), Kolm (1996), Roemer (1996), Brams and Taylor (1996, 1999), and Robertson and Webb (1998). Also worth perusing is a website, "The Equality Exchange: Egalitarian Theories of Justice and Their Applications," organized by Marc Fleurbaey, <marc.fleurbaey@univ-pau.fr>, and viewable at http://aran.univ-pau.fr/ee/index.html. Several of the foregoing references emphasize algorithms for dividing up a set of indivisible items, as does a recent paper that proposes a "descending demand procedure" (Herreiner and Puppe, 2001). By contrast, we focus on properties of fair division, showing which are compatible and which are not; once a choice of compatible properties has been made, then efficient algorithms for finding allocations that satisfy them is certainly an important task.

[2] By contrast, if this player associated cardinal utilities of [5, 3, 2, 1] with items (1, 2, 3, 4), one could say that it prefers {1, 4} (utility: 5 + 1 = 6) to {2, 3} (utility: 3 + 2 = 5), presuming the items are separable and their utilities additive. But if they are not, then a framework in which players can rank both individual items and bundles of items is required; trade-offs in this framework are analyzed in Beviá (1998). Unlike our framework, however, Beviá assumes that there is a divisible good (money), which facilitates "smoothing out" allocations.

Because the maximin criterion takes into account only the lowest-ranked items of players, we offer a second maximin criterion, called *Borda maximin*, that better reflects the overall satisfaction of the players (this concept is defined in section 2). Also, because efficient, envy-free allocations may not exist, we will concentrate on allocations that do not *ensure* envy, based on ordinal preferences, rather than those that are definitely envy-free (this distinction will be illustrated in section 4).

The plan of the paper is as follows. In section 2 we define maximin and Borda maximin allocations and illustrate differences between these concepts. In section 3 we characterize efficient allocations and show that at least one maximin and one Borda maximin allocation must be efficient. At the same time, we demonstrate there may be maximin and Borda maximin allocations that are inefficient.

In section 4 we show that maximin and Borda maximin allocations may be disjoint. Then we prove our main result—that maximin and Borda maximin allocations may be envy-ensuring. Indeed, if the preferences of players for items are unrestricted, we prove that the conflict between the two maximin criteria, on the one hand, and the avoidance of envy on the other, is inescapable, with one exception. This exception occurs when there are just two players; if each receives at least two items, maximin and Borda maximin allocations never ensure envy, but they do allow for it, which we call an *envy-possible* allocation and illustrate in section 4.

In section 5 we provide a sufficient condition for an efficient allocation not to ensure envy, but it is not necessary. Nevertheless, we show that there always exists at least one efficient envy-unensuring allocation if each player receives at least two items.

On the other hand, there may be no envy-free allocation; moreover, even if one exists, it may be inefficient.

In section 6 we show that maximin and Borda maximin allocations may require allocating unequal numbers of items to the players. Thereby "inequality" may be a virtue rather than a vice. Furthermore, equal allocations do not necessarily maximize the sum of Borda scores of all players, which may be viewed as an indicator of overall well-being.

To understand better the quantitative dimensions of the conflict between the two maximin criteria and envy-ensuringness, we briefly report in section 6 on a computer analysis of the relative frequencies of this conflict for different populations. Our principal finding is that maximin allocations ensure envy quite often, but Borda maximin allocations do so only rarely.

In section 7 we conclude with a discussion of both the obstacles and the opportunities for efficient fair division of a set of indivisible items. In theory, the obstacles are large: Not only is the envy-ensuringness of maximin and Borda maximin allocations unavoidable, except in the case of two players receiving at least two items, but even in this case there is no guarantee of envy-freeness—envy depends on the players' preferences for subsets of items. On the brighter side, while maximin and Borda maximin allocations cannot expunge envy, there is always an efficient allocation that is not envy-ensuring.

2. Maximin and Borda Maximin Allocations

We begin by defining and illustrating the concepts of maximin and Borda maximin allocations. A *maximin allocation* of items to players is an allocation that renders the lowest-ranked item that any player receives as high as possible. To illustrate, consider two players, A and B, with the following preferences for four items {1, 2, 3, 4}, ranked from best (on the left) to worst (on the right):

Example 1

A: 1 2 3 4

B: 2 3 4 1

If A receives items {1, 3} and B receives items {2, 4}, the lowest-ranked item that either player receives is its 3rd-best; this is also true if A receives items {1, 2} and B receives items {3, 4}. Because every other allocation in which A and B receive two items each involves A's receiving item 4 or B's receiving item 1 (their 4th-best items), the two aforementioned allocations are the only maximin ones.

In addition, there are two unequal allocations in which no player receives a 4th-best item: A - {1}, B - {2, 3, 4}; and A - {1, 2, 3}, B - {4}. To include these allocations in the maximin set seems highly questionable, however, because the player receiving its top three items can hardly be considered worse off (because it receives a 3rd-best item) than the player receiving only its best item.[3]

A *Borda maximin allocation* of items to players is an allocation that maximizes the minimum Borda score that any player receives. The *Borda score* of a player is the sum of the individual scores it receives for each item it is allocated, wherein the score for a

[3] To avoid this problem, we could restrict maximin comparisons to those that provide as "even" a distribution of the set of items as possible. For example, if there were 3 players and 8 items, this would be a distribution that gives one player 2 items and the other two players 3 items each. Among all such distributions, a maximin distribution would be one that maximizes the lowest-ranked item received by the player who receives only two items. Because even this comparison is dubious, we restrict future maximin comparisons to allocations in which players receive exactly the same number of items.

player's lowest-ranked item is 1 point, the score for its next-lowest-ranked item is 2 points, and so on up to its top-ranked item.[4]

To illustrate the calculation of Borda maximin, consider the six equal allocations in Example 1. In order to simplify notation, let the allocation of {1, 3} to A and {2, 4} to be B be represented by (13, 24); and let the resulting Borda scores of 4 + 2 = 6 to A, and 4 + 2 = 6 to B, be represented by [6, 6]. This representation is given as allocation (1) below; the five other allocations to (A, B), and their Borda scores, are also shown:

1. (13, 24) [6, 6]
2. (12, 34) [7, 5]
3. (14, 23) [5, 7]
4. (23, 14) [5, 3]
5. (24, 13) [4, 4]
6. (34, 12) [3, 5]

Allocation (1) is the Borda maximin allocation, because it gives each player 6 points, which is greater than the minimum number of points that each of the five other equal allocations gives the players. Moreover, every unequal allocation, which gives one player either one or zero items, necessarily gives that player a lower Borda score than 6 (indeed, 4 or less). Hence, allocation (1) is the unique Borda maximin allocation among *all* allocations (equal or unequal) in Example 1.

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[4] The 1-unit difference of cardinal scores is a simplification: Preferences are rarely linear and equally spaced. While this cardinalization of ranks reflects the scoring method of the Borda voting system, the reflection is not exact: Borda vote-scoring gives the lowest-ranked alternative 0 points, the next lowest-ranked alternative 1 point, and so on up to an award of n - 1 points to the highest-ranked alternative if there are n alternatives. We start with 1 point for the lowest-ranked item, rather than 0, to ensure that every item adds positive value to a player's set of items. While Borda scores are but one of an infinite number of

Allocation (1) is also a maximin allocation—no player gets lower than a 3rd-best item—but it shares this status with allocation (2). Sometimes, however, the set of Borda maximin and maximin allocations may not overlap, as we will illustrate in section 4.

It is worth noting that unequal allocation (1, 234), with Borda scores of [4, 9], maximizes the *sum* of Borda scores, giving a total of 13 points to both players. Thus, Borda maximin allocations do not necessarily lead to what we will call *Borda maxsum* allocations. This is a point we will return to in sections 5 and 6, where we will show that Borda maximin allocations also may give players unequal numbers of items.

To recapitulate, maximin selects allocations (1) and (2); Borda maximin singles out allocation (1). Sometimes these two maximin criteria may give entirely different allocations, and they need not be Borda maxsum allocations.

3. Characterization of Efficient Allocations

In this section we characterize all efficient allocations. Then we show that there is at least one maximin and one Borda maximin allocation that are efficient, though both maximin and Borda maximin allocations may also be inefficient, as we will illustrate.

To make precise a notion of efficiency in our ordinal framework, we need the following definitions. Assume two sets of items, *S* and *S*′, have the same cardinality but not the same items. For an individual player, a set *S* of items *dominates* set *S*′ if, for every item in *S* but not in *S*′, there is a different item in *S*′ but not in *S* that that player ranks lower.

Consider two different allocations, ***A*** and ***A*′**, to *all* players. ***A*** *Pareto-dominates* ***A*′** if, for every player, its allocation in ***A*** dominates its allocation in ***A*′**. In this situation,

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possible cardinalizations of ordinal preferences, all seem vulnerable to the problems we discuss (see note we say ***A′*** is *Pareto-dominated* by ***A***. An allocation is *efficient* if it is not Pareto-dominated.[5]

Thus in Example 1, allocation (1) Pareto-dominates allocation (4), because both A and B prefer their allocations in the former to those in the latter, rendering allocation (4) Pareto-dominated. Allocations (2) and (3), as well as allocation (1), are efficient, because they are not Pareto-dominated by any other allocations.

By contrast, allocations (5) and (6), as well as allocation (4), are Pareto-dominated, because both A and B prefer their allocations in (1), which gives each player 1st and 3rd-best choices, to their allocations in (4), (5), and (6). It is easy to check that allocation (2) Pareto-dominates allocations (4) and (5), and allocation (3) Pareto-dominates allocations (5) and (6).

We next characterize efficient allocations, offering a simple test for efficiency. When players choose items one at a time in some sequence, a *sincere choice* by a player is its choice of its top-ranked item from those not yet chosen.

Proposition 1. *An allocation of items is efficient if and only if it is the product of sincere choices by the players in some sequence.*

Proof. *Sufficiency:* Consider a set of items ***I***, containing *p* items, and an arbitrary sequence of players of length *p*. Let ***A*** be the allocation resulting from sincere choices by all the players, selecting in the order of the sequence. Suppose ***A′*** is any other allocation. We claim that ***A*** is not Pareto-dominated by ***A′***.

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8).

[5] It is worth emphasizing that efficiency, as defined here, does not take into account information about preferences over bundles of items. Thus, for instance, if two players, A and B, rank four items in the order 1234, the two allocations, (14, 23) and (23, 14), to (A, B) are efficient. But if both A and B prefer the first

To show this, let I^* be the nonempty subset of items in I that are allocated to different players in $\boldsymbol{A}$ and $\boldsymbol{A}'$. Let i be the item in I^* that was selected first in $\boldsymbol{A}$, and let X be the player that selected i. Because X selected i over all other items in I^*, X must prefer i to all other items in I^*. Thus, there is no item that X receives in $\boldsymbol{A}'$, but not in $\boldsymbol{A}$, that it prefers to i.

Player X's allocation in $\boldsymbol{A}'$, therefore, does not dominate its allocation in $\boldsymbol{A}$. Because $\boldsymbol{A}'$ was arbitrary, $\boldsymbol{A}$ is not Pareto-dominated by any other allocation.

Necessity: Now consider a situation in which there are m players and a set I of p items that must be allocated. Let $\boldsymbol{A}$ be an arbitrary efficient allocation of the items to the players. We next present an algorithm that is well-defined and produces a sequence of players, called O (for order), for which sincere choices produce $\boldsymbol{A}$.

Step 1. Select any player from among those players that received a top-ranked item in $\boldsymbol{A}$.

Step 2. Place this player first in the sequence O.

Step 3. Remove the selected player's top-ranked item from the rankings of all players.

Step 4. Repeat Steps 1-3, each time placing the selected player in the next position of the sequence, until all items are removed and a sequence O of length p is completed.

We will illustrate this algorithm with an example following the completion of this proof.

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allocation to the second, then the second allocation is inefficient, based on this bundle information, which we do not assume.

We claim the algorithm is well-defined: It will continue uninterrupted until a sequence of players of length p is generated. Suppose, for the sake of contradiction, that this is not the case. Specifically, suppose that the process stops before O is fully generated to a length of p players. This can occur only if, at some point, we reach a situation in which no player receives its top-ranked item (among those items remaining).

Consider an arbitrary player X_1, and denote by x_1 its top-ranked item at the current moment. By the supposition in the preceding paragraph, we know X_1 does not receive item x_1 in ***A***. Let X_2 be the player that does receive this item in ***A***. We know from our supposition that x_1 is not player X_2's currently top-ranked item. Denote by x_2 the top-ranked item in X_2's current ranking. We know that X_2 did not receive item x_2 in ***A***. Denote by X_3 the player that did receive item x_2. Continue this process, identifying players X_1, X_2, X_3, … and items x_1, x_2, x_3, ….

Because the number of players is finite, we eventually arrive at a duplication in the listing X_i of players. Suppose the first duplication occurs in the naming of player X_n, and suppose X_n is X_j for some $j < n$. Now consider the set of players X_j, X_{j+1}, X_{j+2}, …, X_{n-1}, and a trade between these players in which X_j receives the item x_j from X_{j+1}, X_{j+1} receives the item x_{j+1} from X_{j+2}, …, X_{n-1} receives item x_{n-1} from X_j.

This trade will be beneficial to all players involved, because each player will receive a higher-ranked item in exchange for a lower-ranked item. Thus, ***A*** is Pareto-dominated by the allocation resulting from this trade. This contradicts our assumption that ***A*** is efficient. Hence, the algorithm always produces a sequence of players of length p, without stopping, and so is well-defined.

Finally, we claim that efficient allocation ***A*** does in fact result from sincere choices with respect to the sequence *O* produced by the algorithm. Suppose, for the sake of contradiction, that some allocation other than ***A*** results from sincere choices involving sequence *O*. Let X be the first player appearing in the sequence that sincerely selects an item, say *x*, not allocated to it in ***A***. At that moment, *x* would be the top-ranked item of X; otherwise, X would not select it sincerely. But, under the rules of the algorithm, X would have appeared at that position in the sequence *O* only if its top-ranked item at that moment, namely *x,* had been among the set of items that X receives under ***A***. This contradicts the supposition that *x* is not among the set of items that player X receives in ***A***. Thus, ***A*** must result from sincere choices using sequence *O*.

Because ***A*** was chosen arbitrarily, every efficient allocation results from sincere choices with respect to some player selection order *O*. Q.E.D.

To illustrate the algorithm given in the proof of Proposition 1, consider

Example 2

A: 1 2 3 4 5 6

B: 5 6 2 1 4 3

C: 3 6 5 4 1 2

Consider the allocation (12, 56, 34) to (A, B, C). Clearly, it is not Pareto-dominated by another allocation of two items each to the players, because either A or B, which each get their top two choices, would be hurt by such an allocation. Furthermore, no unequal allocation Pareto-dominates allocation (12, 56, 34), because any player receiving one or zero items would be worse off, even if the one item was its top choice. Thus, allocation (12, 56, 34) is efficient.

We now apply the algorithm to Example 2. Observe that all three players receive their top-ranked items in the postulated allocation. Following Step 1, choose any one of these players, say B, which received its top-ranked item 5 in the postulated allocation. Following Step 2, B becomes the first player in sequence *O*. According to Step 3, we remove item 5 from the rankings of all the players, which results in the following reduced rankings:

A: 1 2 3 4 6

B: 6 2 1 4 3

C: 3 6 4 1 2

Continuing in this manner by selecting arbitrarily players that received their top-ranked choices in the postulated allocation—but now with already chosen items removed—we generate a sequence of players of length 6. There may be different orders *O* that lead to the same allocation, including sequences that have players choosing two items in a row. For example, sequence AABBCC leads to the same allocation as ABCABC.[6]

[6] One might think that sequences in which there is a strict alternation of players, such as ABCABC, will produce fairer allocations than those in which a player gets two turns in a row, such as AABBCC, in which C, for example, gets only a 5th and 6th choice. But as Brams and Taylor (1999) show, a sequence called "balanced alternation," which in the present example is ABCCBA and gives late-choosing player C two turns in a row, may be superior. This question is complicated if the players have complete information about all players' rankings and, thereby, can make strategic calculations. In this situation, they may have good reason not to be sincere, as Brams and Straffin (1979) show for Example 2 in the context of teams that choose players in professional sports drafts. Indeed, the optimal strategies of players when the sequence is ABCABC produces a worse allocation for *everybody* than when the sequence is AABBCC, because the former sequence induces a 3-person Prisoners' Dilemma in Example 2. This example illustrates how strategic choices may not lead to efficient allocations; also, they may be non-monotonic, whereby choosing earlier in a sequence actually leads to a worse outcome for a player than choosing later (Brams and Kaplan, 2002)

We will not list all sequences for this example and indicate which produce what efficient allocations—this task is more manageable in the next example. Example 3 raises the question of whether the allocation that is generated by the most sequences is, in some sense, better than other efficient allocations.

Example 3

A: 1 2 3

B: 1 3 2

C: 2 1 3

We claim there are three efficient allocations—(1, 3, 2), (2, 1, 3), and (3, 1, 2)—in which each player receives one item. The first is the product of sincere choices in three sequences, ABC, ACB, and CAB; the second in one sequence, BAC; and the third in two sequences, BCA and CBA. It is apparent that different sequences may generate the same efficient allocation.

There are also sequences in which one player takes more than one turn, such as AAB or AAA. By Proposition 1, these also generate efficient allocations when the players make sincere choices, but they are obviously unfair to the player or players that receive no items.

It may seem that the more sequences that generate an allocation, the fairer it is. As a case in point, allocation (1, 3, 2), which is generated by three of the six sequences in which the players each receive one item, is the unique maximin and Borda maximin allocation.

But this is not the case in Example 1, in which the unique maximin and Borda maximin allocation to (A, B) is (13, 24), which is the product of two sequences (ABAB and BAAB). In addition, one sequence (AABB) leads to efficient maximin allocation (12, 34). However, there are three sequences (ABBA, BABA, BBAA) that generate the third efficient allocation, (14, 23), which is neither maximin nor Borda maximin.

Fortunately, for every profile there is always an efficient maximin and an efficient Borda maximin allocation:

Proposition 2. *There is at least one maximin and at least one Borda maximin allocation in the set of efficient allocations.*

Proof. We first prove Proposition 2 for Borda maximin allocations. Suppose $\boldsymbol{A}_0$ is an arbitrary Borda maximin allocation for a given preference profile. If $\boldsymbol{A}_0$ is efficient, we are done. If $\boldsymbol{A}_0$ is inefficient, then $\boldsymbol{A}_0$ is Pareto-dominated by another allocation, say $\boldsymbol{A}_1$. Hence, for some players the allocations they receive in $\boldsymbol{A}_1$ dominate their allocations in $\boldsymbol{A}_0$, though the minimum Borda score will remain the same (otherwise, $\boldsymbol{A}_0$ would not be a Borda maximin allocation).

Thus, the Borda scores for these players increase in going from $\boldsymbol{A}_0$ to $\boldsymbol{A}_1$. But for every other player, the allocation in $\boldsymbol{A}_1$ is the same as that in $\boldsymbol{A}_0$. Hence, the Borda scores of these players remain unchanged. Therefore, $\boldsymbol{A}_1$, like $\boldsymbol{A}_0$ itself, must be a Borda maximin allocation. Furthermore, the *sum* of the players' Borda scores for $\boldsymbol{A}_1$ is greater than that for $\boldsymbol{A}_0$.

If $\boldsymbol{A}_1$ is an efficient allocation, we are done. If not, we can repeat the preceding argument to obtain another Borda maximin allocation, $\boldsymbol{A}_2$, that has a Borda score sum greater than that of $\boldsymbol{A}_1$. If $\boldsymbol{A}_2$ is efficient, we are done. If $\boldsymbol{A}_2$ is inefficient, we can repeat the preceding argument again.

Continuing in this manner, we create a sequence, $\boldsymbol{A}_0$, $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, $\boldsymbol{A}_3$, …, of Borda maximin allocations with increasing Borda score sums. Because such sums are bounded from above, no such infinite sequence can occur. Thus, one of the $\boldsymbol{A}_i$ must be efficient. Hence, there exists an efficient Borda maximin allocation for every preference profile.

The argument for the existence of efficient maximin allocations proceeds along similar lines. We use the Borda score sum and note that an allocation $\boldsymbol{A}_1$ that Pareto-dominates another allocation $\boldsymbol{A}_0$ will raise the ranks of items of some players, and hence their Borda scores, though the minimum rank will remain the same (otherwise, $\boldsymbol{A}_0$ would not be a maximin allocation). Q.E.D.

We suggested earlier the *possibility* that a Borda maximin allocation may be inefficient in the proof of Proposition 2, but in none of our previous examples did we exhibit such an inefficient allocation. Accordingly, we now offer

Proposition 3. *There exist maximin and Borda maximin allocations that are inefficient*.

Proof. In Example 3, the allocation (2, 3, 1) to (A, B, C) is inefficient by Proposition 1: No player receives its top-ranked item, so (2, 3, 1) is not the product of sincere choices, whatever the sequence of player choices. Also note that a trade between players A and C that benefits each results in allocation (1, 3, 2), which is efficient. Both the inefficient allocation and the efficient allocation are maximin and Borda maximin, because there are no other allocations that give a worst-off player better than a 2^{nd}-best item, or a higher Borda score than 2. Q.E.D.

Henceforth, we will focus on *efficient* maximin and Borda maximin allocations, which by Proposition 2 always exist.

4. Maximin and Borda Maximin Allocations May Be Envy-Ensuring

In this section we will prove that maximin and Borda maximin allocations provide no insurance against envy. But first we show that our two maximin criteria may either converge or diverge in their choice of fair allocations:

Proposition 4. *Maximin and Borda maximin allocations may be disjoint or overlapping*.

Proof. To prove the first part of the proposition (disjointness), consider

Example 4

A: 1 2 3 4 5 6 7 8 9

B: 3 4 2 6 8 7 1 5 9

C: 5 8 2 7 9 1 3 4 6

There are three efficient maximin allocations, giving each player, at worst, a 5th-best item. These allocations to (A, B, C) and their Borda scores are shown below:

1. (124, 368, 579) [23, 20, 20]
2. (123, 468, 579) [24, 19, 20]
3. (125, 346, 879) [22, 23, 19]

Three sequences of sincere choices that generate allocations (1), (2), and (3), respectively, are AABCABBCC, AAABBBCC, and AABBABCCC. Showing that these allocations are the only maximin allocations is tedious, so we do not include a demonstration here.[7]

Observe that the minimum Borda scores for these allocations are 20 for allocation (1), 19 for allocation (2), and 19 for allocation (3), so it would seem that allocation (1) is the Borda maximin allocation. However, it turns out that there is a fourth allocation, which is not maximin (because A and B receive 6th-best items), in which the worst-off player receives a higher minimum Borda score (21 points):

4. (126, 347, 589) [21, 21, 22]

A computer calculation verifies that this is the unique Borda maximin allocation.[8]

That the maximin and Borda maximin allocations can overlap is shown by Example 1, in which allocation (1) is both maximin and Borda maximin, making it presumably the "fairest" allocation in this example.[9] Q.E.D.

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[7] We have written a computer program to check all unproved computational claims in this paper, which is available on request from the authors.

[8] This example well illustrates how the cardinalization of the ordinal ranks can matter. (By "cardinalization" we mean the attribution of cardinal scores to ranks that preserves player orderings by giving the most points to a player's highest-ranked item, the next-most points to its next highest-ranked item, etc.) Thus, instead of using Borda scores, assign each player's top 5 items its Borda score *plus* 2 additional points, but keep the old Borda scores for each player's bottom 4 items (under the presumption that these items, as a group, are worth 2 points less than the top 5). Then the Borda maximin allocation given in the text now has a minimum score of 25, whereas efficient maximin allocation (1) has a minimum score of 26, giving it the edge over all the others. We hypothesize that any cardinalization different from Borda, like the one just illustrated, will still lead to most of the difficulties identified in this paper—in particular, it will not preclude envy-ensuring allocations.

[9] In Example 1, there are two maximin allocations, one of which is Borda maximin. Below we give an example of the opposite—two Borda maximin allocations, one of which is maximin.

A: 1 2 3 4 5 6
B: 2 1 4 6 3 5
C: 4 5 6 3 1 2

The Borda maximin allocations, in which each player receives a minimum Borda score of 9, are as follows:

1. (13, 24, 56) [10, 10, 9]
2. (13, 26, 45) [10, 9, 11]

But what is the fairest allocation among the three maximin, and the disjoint Borda maximin, divisions in Example 4? Like allocation (1) in Example 1, allocation (1) in Example 4 maximizes the minimum Borda score *among the maximin allocations*; but unlike Example 1, non-maximin allocation (4) is the Borda maximin allocation.[10]

Whereas Borda maximin helps the worst-off player by maximizing its overall satisfaction, maximin's focus is narrower: It maximizes the rank of only the lowest-ranked item that any player receives, independent of whatever other items this player possesses. It is not surprising, therefore, that these criteria may clash over what allocation is fairest.

More surprising, we think, is that both a maximin and a Borda maximin allocation may ensure envy. In our ordinal framework, call an allocation *envy-ensuring* if, for some player A, A *assuredly envies* another player B, because B's allocation dominates A's allocation with respect to A's preferences.

To illustrate this concept, assume that A and B both rank four items in the order 1 2 3 4. If A receives items 24 and B receives items 13, this allocation is envy-ensuring: A envies B, because A prefers B's item 1 to its own item 2, and B's item 3 to its own item 4.

By contrast, if A receives items 14 and B receives items 23, this allocation is *envy-possible*: The presence of envy depends on the preferences of A and B for *sets* of items. In particular, one player will envy the other if and only if A prefers 23 to 14, or B prefers

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While only allocation (1) is maximin—giving each player an item ranked no lower than 3rd-best—it is allocation (2) that maximizes the sum of the players' Borda scores (i.e., is Borda maxsum), suggesting that allocation (2) is competitive with allocation (1), even though allocation (2) is not maximin.

[10] What explains this difference? Note that allocation (4) gives each player its two best items and a 6th-best item, whereas allocation (1) gives only player A its two best items (and a 5th-best item). While players B and C get their single best items in allocation (1), they also get only their 4th-best and 5th-best items, which

14 to 23. Thus, the presence of envy in an envy-possible allocation will depend on the cardinal utilities that the players attach to sets of items (see note 1 for an example).

If each player receives only one item, as might occur in a gift exchange, an allocation may be envy-ensuring even though the players all have different rankings of the items, as illustrated by Example 3. As we showed in section 3, there are three allocations to (A, B, C) that are efficient—(1, 3, 2), (2, 1, 3), and (3, 1, 2)—wherein at least one player receives its best item. None is envy-free, however, because the player that receives item 1 in each allocation (A or B) is envied by at least one of the other two players. Allocation (1, 3, 2), in which no player receives its worst item, is the unique efficient maximin and Borda maximin allocation, but it is envy-ensuring because B envies A.

Before showing that maximin and Borda maximin allocations cannot, in general, prevent assured envy (at an ordinal level), we analyze the one exceptional case—when there are exactly two players, and each receives two or more items. In Example 1, recall that allocation (12, 34) is Borda maximin (each player gets a Borda score of 6); the fact that this allocation does not ensure envy—in fact, it is envy-free—is no accident.

Proposition 5. *Assume there are $n = 2$ players and $k \geq 2$ items that each receives. Then an efficient Borda maximin allocation is never envy-ensuring.*

Proof. Suppose, for the sake of contradiction, that there exists an efficient Borda maximin allocation ***A*** that is envy-ensuring. Suppose players A and B receive sets *X* and *Y*, containing *k* items each. For concreteness, suppose that B envies A.

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lowers their Borda scores to 20 points rather than the minimum of 21 points that allocation (4) gives the players in Example 4.

Let $Y = \{y_1, y_2, \ldots, y_k\}$, where B prefers y_i to y_j if $i < j$. Because B envies A, set X dominates set Y for B. Thus, there exists an ordering the elements in X, say $X = \{x_1, x_2, \ldots, x_k\}$, such that B prefers x_i to y_i for all i. For B, therefore, the Borda score of X exceeds that of Y by at least the number of items in Y:

$$\text{Borda}_\text{B}X \geq \text{Borda}_\text{B}Y + k. \qquad (1)$$

Because the original allocation is efficient, it must also be the case that A prefers x_i to y_i for all i; otherwise, $\boldsymbol{A}$ would be Pareto-dominated by another allocation. Hence we have

$$\text{Borda}_\text{A}X \geq \text{Borda}_\text{A}Y + k. \qquad (2)$$

Because there are only two players, we know

$$\text{Borda}_\text{B}X + \text{Borda}_\text{B}Y = \text{Borda}_\text{A}X + \text{Borda}_\text{A}Y. \qquad (3)$$

From (1) and (2) it follows that

$$\text{Borda}_\text{B}Y - \text{Borda}_\text{B}X + k \leq \text{Borda}_\text{A}X - \text{Borda}_\text{A}Y - k. \qquad (4)$$

Adding (3) and (4),

$$2\text{Borda}_\text{B}Y + k \leq 2\text{Borda}_\text{A}X - k$$

$$\text{Borda}_\text{A}X \geq \text{Borda}_\text{B}Y + k. \qquad (5)$$

Let $d_i = \text{Borda}_\text{A}\{x_i\} - \text{Borda}_\text{A}\{y_i\}$ for $i = 1, 2, \ldots, n$. Let d_i^0 be the minimal value of d_i for $i = 1, 2, \ldots, n$. Because there are $2k$ items altogether, $d_i^0 \leq k$, with equality holding only if and only if A prefers all items in X to all items in Y. Now consider two cases:

Case 1. Suppose $d_i^0 < k$. Let $\boldsymbol{A}'$ be the allocation that results when A and B swap items x_i^0 and y_i^0; and let X' and Y' be the allocations to A and B, respectively, in allocation $\boldsymbol{A}'$. Clearly, $\text{Borda}_B Y' > \text{Borda}_B Y$, because B is receiving a higher-ranked item in the trade. Furthermore, because A ranks x_i^0 fewer than k units above y_i^0 (because $d_i^0 < k$), we have $\text{Borda}_A X' > \text{Borda}_A X - k \geq \text{Borda}_B Y$ from (5). Thus, both players receive a higher Borda score from $\boldsymbol{A}'$ than that received by B from $\boldsymbol{A}$. This contradicts the assumption that $\boldsymbol{A}$ is a Borda maximin allocation.

Case 2. Suppose $d_i^0 = k$. In this case, A prefers all items in X to all items in Y. Hence, if $k \geq 2$, inequality (1) is strict. This implies that inequality (5) is strict as well:

$$\text{Borda}_A X > \text{Borda}_B Y + k. \tag{6}$$

As in case (1), let $\boldsymbol{A}'$ be the allocation that results when A and B swap items x_i^0 and y_i^0, and let X' and Y' be the allocations to A and B, respectively, in allocation $\boldsymbol{A}'$. Clearly, $\text{Borda}_B Y' > \text{Borda}_B Y$, because B is receiving a higher-ranked item in the trade. Furthermore, because $d_i^0 = k$, we have $\text{Borda}_A X' = \text{Borda}_A X - k > \text{Borda}_B Y$ from (6). Thus, both players receive a higher Borda score from $\boldsymbol{A}'$ than that received by B from $\boldsymbol{A}$. This contradicts the assumption that $\boldsymbol{A}$ is a Borda maximin allocation. Q.E.D.

Next we turn to our main negative findings. To avoid trivialities, we assume that there are enough items so that each player can obtain at least one item. If this were not the case, then any player receiving no item would envy those players that receive some items. In addition, we assume that the number of items is such that they can be equally divided among all the players.

We begin by showing that there always exists a preference profile in which an efficient *maximin* allocation is envy-ensuring (Proposition 6). We then show that the same conflict arises if the allocation is an efficient *Borda maximin* allocation (Proposition 7), except when $n = 2$, as we just demonstrated (Proposition 5). Finally, we show that Borda maximin allocations that are envy-ensuring may or may not be unique (Proposition 8).

Proposition 6. *Assume the number of items to be allocated is an integer multiple k of the number of players n. There is always an efficient maximin allocation that is envy-ensuring for some preference profile.*

Proof. We start with the n-person, n-item case (i.e., $k = 1$). Assume players P_1, P_2, P_3, …, P_n rank items as follows:

Example 5

P_1: 1 2 3 4 5 . . . n-1 n

P_2: 1 2 3 4 5 . . . n-1 n

P_3: 3 1 2 4 5 . . . n-1 n

P_4: 4 1 2 3 5 . . . n-1 n

.
.
.

P_n: n 1 2 3 4 . . . n-2 n-1

It is apparent that an efficient maximin allocation is one in which all players get their first choices—except *both* P1 and P2, which have the same ranking—one of which (say, P_2) must get its second choice. But then P_2 will envy P_1 in the allocation (1, 2, 3, …, n).

This example can readily be extended to the *n*-player, *kn*-item case ($k \geq 2$), which we illustrate with a 4-player, 8-item example, in which we have grouped items into two-sets (e.g., 12, which indicates a ranking of item 1 ahead of item 2):

Example 6

A: 12 34 56 78

B: 12 34 56 78

C: 56 12 34 78

D: 78 12 34 56

In the efficient maximin allocation (12, 34, 56, 78), which is the product of sincere choices in sequence AABBCCDD (among others), B envies A.[11] Q.E.D.

Proposition 7. *Assume the number of items to be allocated is an integer multiple k of the number of players n. If n ≠ 2, there is always an efficient Borda maximin allocation that is envy-ensuring for some preference profile. If n = 2, such a division can exist only if k = 1.*

Proof. Consider the *n*-person, *n*-item case (i.e., $k = 1$) analyzed in the proof of Proposition 6. The efficient maximin allocations in Example 5, whereby each player receives its best choice—except for one of either P_1 or P_2—is also a Borda maximin allocation that ensures envy.[12]

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[11]There is another efficient maximin allocation (34, 12, 56, 78), which is the product of sincere choices in sequence BBAACCDD (among others), in which A envies B. But because A and B have the same preferences, this is isomorphic to the one given in the text. Henceforth, we will ignore isomorphic allocations (i.e., when two or more players have the same preferences).

[12] But envy-ensuringness does not extend to allocation (12, 34, 56, 78) in Example 6, wherein each player receives two items. The Borda maximin allocation in this example is (14, 23, 56, 78), which gives the players—in particular, A and B—minimum Borda scores of 13 points. These scores are greater than the minimum Borda scores of 11 points each that envy-ensuring maximin allocation, (12, 34, 56, 78), gives A

We now offer a construction when $k \geq 2$ that yields Borda maximin allocations that are envy-ensuring for $n \geq 3$ (recall from Proposition 5 that there exist no such allocations when $n = 2$). We begin the construction with three players, A, B, and C, and later show how additional players can be added:

1. A and B have the same preference ranking of items; let numbers 1, 2, 3,…, n identify these items, ranked from best to worst by both players.

2. A receives the first k–1 odd-numbered items, 1 3 5…2k–3; its k^{th} item is item 3k–1, its next-to-last ranked item.

3. B receives the first k even-numbered items, 2 4 6…2k.

4. C's 1^{st}, 3^{rd}, 5^{th}, …, $2k{-}1^{st}$-ranked items are those that A receives (in numerically increasing order). C's 2^{nd}, 4^{th}, 6^{th}, …, $2k^{th}$-ranked items are those that neither A nor B receives (in numerically increasing order); C receives these items. C ranks the items that B receives below its $2k^{th}$-ranked item (in numerically increasing order).

The simplest example that illustrates this construction, which we call CON, is for n = 3, k = 2:

Example 7

A: 1 2 3 4 5 6

B: 1 2 3 4 5 6

C: 1 3 5 6 2 4

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and B. Example 6, therefore, does *not* show that there exists a Borda maximin allocation that is envy-ensuring when the players get more than one item each.

CON gives allocation (15, 24, 36), with Borda scores of [8, 8, 8]. Note that because the Borda scores of B and C are given by their 2nd and 4th-ranked items, each player receives 5 + 3 = 8 points. To ensure that A also receives 8 points, we ask how many points it must receive, in addition to the 6 points it receives from obtaining item 1, to give it 8 points. The answer of 2 points means that it must receive its 5th-ranked item (item 5), which is what step (2) of CON specifies.

To show that A always can be given a k^{th} item that equates its Borda score with the (equal) Borda scores of B and C, suppose that A's k^{th} item is *not* that which equalizes its Borda score but instead the k^{th} odd-numbered item (item 3 in Example 7). Then its Borda score would be k points greater than those of B and C, which receive items at even-numbered ranks. But because there are k+1 items below A's k^{th} odd-numbered item (items 4, 5, and 6 in Example 7), we can drop k items below this item (i.e., go from item 3 to item 5) to award A—instead of its k^{th} odd-numbered item—one that equalizes the Borda scores of all three players. The award of this item to A does not conflict with the items that C receives, because C's even-ranked items are determined after A and B receive their items, according to step (4) of CON.

To show that CON yields an efficient allocation, note that the allocation is the product of sincere sequence ABCBAC in Example 7. In general, sequence ABCABCABC…BAC, which gives A the first k-1 odd-numbered items, B the first k even-numbered items, and C the remaining items, except for item 3k–1, which A takes before C does on the last round.

Before showing that CON yields a Borda-maximin allocation, observe that the allocation it gives in Example 7 ensures that C envies A. Because, in general, the k items that C receives at ranks 2, 4, 6, … are preceded by the k items that A receives, C will envy A. Later we will show that this result can be extended to the case of more than three players.

We begin with the $n = 3$ case, in which each player receives k items, where $k \geq 2$. The allocation to players A, B, and C, according to CON, is

$$\boldsymbol{A} = (1\ 3\ 5 \ldots 2k{-}3\ \ 3k{-}1, \quad 2\ 4\ 6 \ldots 2k{-}2\ \ 2k, \quad 2k{-}1\ \ 2k{+}1\ \ 2k{+}2 \ldots 3k{-}2\ \ 3k).$$

The Borda scores of the players, in which A receives its 1st, 3rd, 5th…-best items, except for its k^{th} item that it ranks next-to-last, and B and C receive their 2nd, 4th, 6th…, k^{th}-best items, are

$$s(\text{A}) = 3k + 3k\text{-}2 + 3k{-}4 + \ldots + 3k{-}(2k{-}4) + 2$$

$$s(\text{B}) = s(\text{C}) = 3k{-}1 + 3k{-}3 + 3k{-}5 + \ldots + 3k{-}(2k{-}3) + 3k{-}(2k{-}1).$$

Combining terms and simplifying, it is not difficult to show that each of these sums is $2k^2$, giving the three players identical Borda scores.

We know that a Borda maxsum allocation is achieved if and only if each item is allocated to a player that ranks it highest. In fact, this is true of all the items received by the players in CON, except for the k^{th} item received by A, item $3k - 1$, which it ranks next-to-last, contributing 2 points to A's Borda score. If C, which ranks this item the highest of the three players, had received it, it would have contributed $3k - (2k - 2) = k + 2$ points to C's Borda score (1 more point than the last term of s(C) above). The addition of the k points to the sum of the Borda scores that the players receive from the CON allocation gives a Borda maxsum value of $6k^2 + k$ points.

Now for an allocation to achieve this Borda maxsum value, it is necessary for C to receive all the items shown in ***A*** above *plus* item $3k - 1$, which it ranks higher than A or B. If it does not receive one of these items, the reduction in the maxsum value will be at least k points. Postulating ***A′*** to be any other allocation, we consider two cases:

Case 1. Suppose that C does not receive in ***A′*** all the items it receives in ***A***. Then the maximal possible Borda sum will be $6k^2 + k - k = 6k^2$. In this case, an allocation cannot possibly give each player a Borda score strictly greater than $2k^2$ points.

Case 2. Suppose that C does receive in ***A′*** all the items it receives in ***A*** (as provided by CON). These items contribute $2k^2$ points to the Borda score of C. The remaining items, if allocated only to A or B, contribute $4k^2$ points to the Borda sum of the allocation. Such an allocation cannot give each player a Borda score strictly greater than $2k^2$ points.

Because no allocation in ***A′*** can provide a Borda score strictly greater than $2k^2$ to each player, we conclude that ***A***, which gives each player exactly this score, is Borda maximin.[13]

It remains only to show that ***A*** can be extended to situations in which there are more than three players. Consider any number n–3 of additional players D, E, F,…, all of which rank (n–3)k additional items in such a manner that D ranks k of these items best, E a different set of k items best, F the next k items best, and so on. A, B, and C rank these additional items below items 1, 2, …, $3k$, which they rank according to CON. More concretely, consider the following example in which $n = 6$ and $k = 2$:

Example 8

A: 1 2 3 4 5 6 7 8 9 10 11 12

B: 1 2 3 4 5 6 7 8 9 10 11 12

C: 1 3 5 6 2 4 7 8 9 10 11 12

D: 7 8 9 10 11 12 1 2 3 4 5 6

E: 9 10 11 12 7 8 1 2 3 4 5 6

F: 11 12 7 8 9 10 1 2 3 4 5 6

We claim that the following allocation, in which A, B, and C receive their CON allocation for the first 6 items, and D, E, and F receive their 2 best items from among the remaining 6, is Borda maximin:

(1 5, 2 4, 3 6, 7 8, 9 10, 11 12) [20, 20, 23, 23, 23]

In order to try to raise the minimum Borda score of A, B, and C above 20, and hurt D, E, and F as little as possible, we could give each of A, B, and C an item possessed by D, E, and F. But giving them the next-best item of each of the latter players would lower D, E, and F's Borda scores to 12, far below the 23 points they now receive and the present minimum Borda score of 20. This example can be generalized to show that a CON allocation to A, B, and C of the first $3k$ items, and an allocation to each of the remaining $n - 3$ players of its k best items, is not only Borda maximin but also envy-ensuring (C envies A). Q.E.D.

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[13] There may be other Borda maximin allocations, including unequal ones, as we will show in Example 9. The one generated by CON is distinguished by its always ensuring envy, which other efficient allocations that equalize the Borda scores of players may not do.

The next proposition shows that sometimes an escape from efficient Borda maximin allocations that are envy-ensuring is possible.

Proposition 8. *Efficient Borda maximin allocations that are envy-ensuring may or may not be the only efficient Borda maximin allocations.*

Proof. Proposition 7 shows by construction that there is always a preference profile that renders an efficient Borda maximin allocation envy-ensuring if $n \neq 2$. But there may be other efficient Borda maximin allocations that are not envy-ensuring, as illustrated by the following example:

Example 9

A: 1 2 3 4 5 6 7 8 9 10 11 12

B: 1 2 3 4 5 6 7 8 9 10 11 12

C: 1 7 3 9 5 10 11 12 2 4 6 8

Because this preference profile is the product of CON, the allocation

1. (1 3 5 11, 2 4 6 8, 7 9 10 12) [32, 32, 32, 32]

is Borda maximin and envy-ensuring. However, there are three other allocations, which give the same (equal) Borda scores, that are not envy-ensuring:

2. (1 2 6 11, 3 4 5 8, 7 9 10 12) [32, 32, 32, 32]
3. (1 5 6 8, 2 3 4 11, 7 9 10 12) [32, 32, 32, 32]
4. (1 2 4, 3 5 6 8 11, 7 9 10 12) [32, 32, 32, 32]

In all four allocations, notice that C receives the same four items, but A and B receive different allocations. Only in allocation (4) do the players receive different numbers of items (A—3, B—5, C—4).

It is easy to show that allocations (2), (3), and (4) are not envy-ensuring. The fact that there are Borda maximin allocations different from that generated by CON mitigates somewhat the conflict between Borda maximin and envy-ensuringness. However, this is not the case in both Example 7 and the next-larger example generated by CON ($n = 3$, $k = 3$):

Example 10

A: 1 2 3 4 5 6 7 8 9

B: 1 2 3 4 5 6 7 8 9

C: 1 5 3 7 8 9 2 4 6

The CON allocation, (138, 246, 579), which has Borda scores of [18, 18, 18], is the *only* (non-isomorphic) Borda maximin allocation. In both this example and Example 7, therefore, an escape from the conflict between Borda maximin and envy-ensuringness is impossible. Indeed, there are other examples, not generated by CON, in which every Borda maximin allocation is envy-ensuring:

Example 11

A: 1 2 3 4 5 6

B: 1 2 3 4 5 6

C: 1 5 4 6 2 3

There are two allocations—(14, 23, 56) and (13, 24, 56), each giving a minimum Borda score of 8—which are Borda maximin as well as being maximin (no player gets worse than its 4th-best item). In addition, there is a third maximin allocation, (12, 34, 56), which is not Borda maximin. *All* these allocations ensure envy.[14] Example 11 can be generalized to all *n* x 2 cases, in a manner analogous to the extension pattern used in Example 8, to show that all maximin and Borda maximin allocations are envy-ensuring. Q.E.D.

We have shown in this section that there are always preference profiles that render maximin and Borda maximin allocations envy-ensuring (except when there are two players in the case of Borda maximin allocations). Indeed, there is a profile in which every maximin allocation is envy-ensuring, whatever the number of players *n* and the number of items *k* that each player receives. While there is always a Borda maximin allocation that ensures envy, except when $n = 2$, there may be other Borda maximin allocations that do not ensure envy.

That neither of the maximin criteria rules out assured envy highlights the dilemma of helping worst-off players—doing so may actually guarantee that they envy other players. As we will show next, escaping envy may require abandoning maximin or Borda maximin allocations.

5. Finding Envy-Unensuring Allocations

With the knowledge that both maximin and Borda maximin allocations can ensure envy, we turn next to the problem of finding efficient allocations that do not lead to this

[14] This 3-player, 6-item example is actually stronger than Example 7, which has maximin allocations, like (14, 23, 56), that are not envy-ensuring. In Example 11 there are none—all maximin and Borda maximin

unhappy outcome. Fortunately, in all situations in which maximin and Borda maximin allocations are envy-ensuring, there is always at least one efficient allocation that is *envy-unensuring*—that is, either envy-possible or envy-free—and so does not ensure envy.

Proposition 9. *For any preference profile, there is always an efficient allocation that is envy-unensuring if there are at least twice as many items as players.*

Proof. Consider an arbitrary ordering of the players, $P_1P_2P_3\ldots P_n$, such that each player appears exactly once in the ordering. Assume that each player makes a sincere choice of an item when its turn comes up. These are the Round 1 choices. On Round 2, reverse the order of choice, and assume again that each player makes a sincere choice on this round when its turn comes up. If there are more items to be allocated, assume the players make sincere choices in any order.

This selection order appears below as the Round 1 sub-sequence (terminated by the first slash), followed by the Round 2 sub-sequence (terminated by the second slash) and, if necessary, an arbitrary ordering of players to complete the selection order (indicated by asterisks):

$$P_1P_2\ldots P_n \,/\, P_nP_{n-1}\ldots P_1 \,/\, ***.$$

The resulting allocation is efficient, which follows immediately from Proposition 1 because sincere choices are made throughout the sequence. In addition, we claim this allocation is envy-unensuring. For $i < j$, P_i will choose, as its first item, one that it ranks higher than any item chosen subsequently by P_j. Thus, the allocation to P_j will not dominate that to P_i with respect to P_i's preferences, so P_i will not assuredy envy P_j. By

allocations are envy-ensuring.

the same token, P_j will not assuredly envy P_i, because P_j makes a second choice before P_i, precluding P_i's allocation from dominating P_j's with respect to P_j's preferences. Since the order of players' making sincere choices in Round 1 was chosen arbitrarily, the existence of envy-unensuring allocations is general. Q.E.D.

The construction in the proof of Proposition 9 gives one method for finding an efficient envy-possible or envy-free allocation. It also makes perspicuous why each player must receive at least two items; if not, the reversing of the first sub-sequence to obtain the second sub-sequence, which gives all players undominated allocations, would not be possible.

While the construction provides a sufficient condition for an allocation to be efficient and envy-unensuring, it is not necessary—other constructions are possible. For example, assume there are 3 players and 6 items, and all the players rank the items exactly the same, which is a worst-case preference profile (i.e., one most likely to induce envy). Then the sequence starting ABCCB…, *without* an A following the last B, will result in an envy-unensuring allocation, whether the sequence ends with a third B (ABCCBB) or a third C (ABCCBC). Indeed, there are three other structurally different sequences, in which A makes only a first appearance in a sequence (ABBCCC, ABCCCB, ABCBCC), that yield allocations that are envy-unensuring, even in the worst-case scenario in which the three players have identical preferences.

In all five sequences, A's sincere first choice is sufficient to preclude its envying, with certainty, any other player (because its top-ranked item might be more valuable to it than all the other items combined). But how can we be sure that the other two players will not be envious? In fact, it is not difficult to show the following:

Condition for Envy-Unensuringness. *Given sincere choices, a sequence yields an envy-unensuring allocation, whatever the preference profile of the players, if, for each pair of players S and T, there is not a matching so that the first S to appear in the sequence is ahead of the first T to appear, the second S to appear is ahead of the second T to appear, and so on.*

No such one-to-one matching is possible if the number of appearances of the players is different, so the condition presumes the same number of appearances. In the envy-unensuring sequence given by the construction in the proof of Proposition 9 (e.g., ABCCBA for 3 players and 6 items), the players do appear the same number of times. Moreover, because in this sequence there is *not* the stated matching for each pair, the Condition for Envy-Unensuringness is satisfied. Thus in sequence ABCCBA, the first A precedes the first B, but the second A follows the second B.

The Condition for Envy-Unensuringness, while necessary and sufficient if the players' preferences are identical, is not necessary if their preferences are different. That is, there may be other sequences that result in envy-unensuring allocations when players make sincere choices.

In Example 1, for instance, sequence ABBA gives (14, 23), and sequence BAAB gives (13, 24), to (A, B);[15] the first allocation is envy-possible and the second is envy-free. However, there is a third efficient division, (12, 34), induced by sequence AABB, that is envy-possible. Clearly, if the preferences of the two players were identical, this latter sequence would generate an envy-ensuring allocation.

[15] While ABBA and BAAB are structurally the same sequences, they produce different allocations because the preferences of the players are different, unlike the worst-case scenario we assumed earlier in the text.

To summarize, while the construction given in the proof of Proposition 9 suffices to give an efficient, envy-unensuring allocation, it may not be the only one to do so. This is true even if each player appears the same number of times in a sequence (e.g., as in AABB), which makes a matching possible but does not satisfy the Condition for Envy-Unensuringness.

A unique envy-free allocation that is maximin and Borda maximin, like (13, 24) in Example 1, would seem to be the fairest of efficient allocations. Unfortunately, two problems can beset such envy-free allocations:

Proposition 10. *For a given preference profile, an envy-free allocation may not exist. If one exists, it may be inefficient and the only envy-free allocation.*

Proof. Consider

Example 12

A: 1 2 3 4

B: 1 3 4 2

wherein there are two efficient maximin and Borda maximin allocations, both of which are envy-possible:[16]

1. (12, 34) [7, 5]
2. (23, 14) [5, 6].

There is also one efficient, envy-ensuring allocation:

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[16] This and the next two examples are drawn from Brams, Edelman, and Fishburn (2001), wherein they are discussed as "paradoxes of fair division." Here we present them as further obstacles in precluding envy via efficient or equal divisions.

3. (24, 13)   [4, 7]

In allocation (3), A envies B because A prefers B's item 1 to its item 2 and B's item 3 to its item 4.

Using Proposition 1, it is not hard to show that these three allocations exhaust the efficient allocations that give each player two items. Because none of the inefficient or unequal allocations in this example is envy-free either, this example yields no envy-free allocation.

Consider next

Example 13

A: 1 2 3 4 5 6

B: 4 3 2 1 5 6

C: 5 1 2 6 3 4

The unique envy-free allocation to (A, B, C) is (13, 42, 56), whereby A and B get their 1st-best and 3rd-best items, and C gets its 1st-best and 4th-best items. Clearly, A prefers its two items to those of B (which are A's 2nd-best and 4th-best items) and those of C (A's two worst items). Likewise, B and C prefer their items to those of the other two players. Consequently, all three players prefer their items to those of the other two players, so the allocation is envy-free.

Compare this allocation with (12, 43, 56), whereby A and B receive their two best items, and C receives, as before, its 1st-best and 4th-best items. This allocation Pareto-dominates (13, 42, 56), because two of the three players (A and B) prefer their items in (12, 43, 56), whereas both allocations give player C the same two items (56). It is easy to see that allocation (12, 43, 56) is efficient: It is the product of sincere choices in sequence AABBCC as well as several other sequences.

So far we have shown that efficient allocation (12, 43, 56), which is envy-possible (because C may envy A), Pareto-dominates inefficient allocation (13, 42, 56), which is envy-free. To show that allocation (13, 42, 56) is uniquely envy-free, note first that an envy-free allocation must give each player its best item (because a player not receiving its best item may envy the player that does receive it). Thus, an allocation that does not give a player its best item is either envy-possible or envy-ensuring. Second, even if each player receives its best item, this allocation cannot be the only item it receives, because then that player might envy any player that receives two or more items, *whatever* these items are.

By this reasoning, then, the only possible envy-free allocations in Example 13 are those in which each player receives two items, including its top choice. In the six possible allocations in which each player receives its top choice and one other item, only the allocation (13, 42, 56) is envy-free. Therefore, the unique envy-free allocation is inefficient. Q.E.D.

We provided in this section a sufficient condition for an allocation to be envy-unensuring. In fact, the proof of Proposition 9 showed how to generate an efficient, envy-unensuring allocation. However, because an envy-unensuring allocation may not be envy-free but only envy-possible, we then investigated the existence of envy-free allocations. Our main finding was negative: An envy-free allocation may not exist; even if one does, it may be inefficient (Proposition 10).

6. Unequal Allocations and Statistics

In this section, we will show that the best way to achieve maximin and Borda maximin allocations may require that the players receive unequal numbers of items. We will also present statistics on the relative frequency with which maximin and Borda maximin allocations, when they are equal, ensure envy. These statistics will give us insight into how serious a conflict there is between helping the worst off and avoiding envy.

In section 2 we argued that the maximin property is not meaningful in comparing allocations that give players different numbers of items (though we suggested an extension of this property that makes unequal allocations more comparable; see note 4). By contrast, because Borda maximin takes into account *all* items in a player's set, valuing them by the same standard, it renders comparable sets that contain different numbers of items.

One might suppose that Borda maximin as well as Borda maxsum will always favor equal allocations, but this is not the case.

Proposition 11. *Allocations that give unequal numbers of items to players may be the only ones that are Borda maximin; they may also be Borda maxsum.*

Proof. Consider

Example 14

A: 1 2 3 4 5 6 7 8 9

B: 3 1 2 4 5 6 7 8 9

C: 4 1 2 3 6 5 7 8 9

There are exactly two *unequal* allocations, (12, 357, 4689) and (12, 3589, 467), that maximize the minimum Borda scores of players, which are [17, 17, 17] for both allocations.[17]

On the other hand, among *equal* allocations there are two allocations, (129, 357, 468) and (129, 358, 467), that maximize the minimum Borda scores of players, which are [18, 17, 16] for the first allocation and [18, 16, 17] for the second allocation. Because the worst-off player in the equal allocations receives fewer points (16) than the worst-off (and best-off) player in the two unequal allocations (17 points), the unequal allocations are Borda maximin among *all* allocations. Also, Borda scores for all the aforementioned allocations (equal and unequal) sum to 51, which, it can be shown, is the maximum sum among all possible allocations in Example 14 (i.e., Borda maxsum). Q.E.D.

Whereas an unequal allocation may be both egalitarian and benefit-inducing (e.g., the Borda maximin and the Borda maxsum allocation in Example 14), Borda maxsum allocations may also be very inegalitarian (e.g., Example 1). These examples illustrate that a wide range of allocations is possible satisfying different normative criteria.

To obtain a better idea of the fundamental trade-off discussed in this paper, we calculated in a computer simulation the relative frequencies that efficient maximin and Borda maximin allocations are envy-ensuring, based on a sample of several hundred randomly generated preference profiles. The percentages of *equal* efficient (n, k)-

[17] Unlike Example 9, in which the unequal Borda maximin allocation was accompanied by three equal ones, the unequal Borda maximin allocations in Example 14 are the only ones.

allocations that are (i) envy-ensuring, (ii) envy-ensuring and maximin, and (iii) envy-ensuring and Borda maximin are shown below:[18]

(*n*, *k*)-Allocation:	(2, 2)	(2,3)	(2, 4)	(2,5)	(3, 2)	(3, 3)	(4, 2)
Envy-ensuring	29	24	8	5	53	25	68
Maximin	22	8	4	3	37	18	52
Borda maximin	0	0	0	0	6	0	12

Reading down the columns, maximin, but especially Borda maximin, act like sieves to reduce the proportion of envy-ensuring allocations. In fact, we already know from Proposition 5 that there are no 2-item Borda maximin allocations that are envy-ensuring (first four columns).

Reading across the rows in the 2-player and 3-player cases, we see that an increase in the number of items also reduces the proportion of allocations that are envy-ensuring. Although the (3, 3) percentage for Borda maximin is 0, Example 10 shows that this is not a null set—as is no other Borda maximin (*n*, *k*)-allocation in which there are 3 or more players equally dividing 6 or more items.

It is evident that envy-ensuringness is more worrisome the more players, and the fewer the number of items they must divide equally, are. Indeed, in the case in which there are *n* players that must divide exactly *n* items (not shown in the table), envy becomes a virtual certainty as the number of players increases. While there is welcome relief from envy when the allocation is Borda maximin for two players, beyond two

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[18] We focus on equal allocations, because maximin allocations are not meaningful for unequal allocations (see note 4). While Borda maximin allocations render unequal allocations comparable with equal ones, unequal Borda maximin allocations are rare, Examples 8 and 14 notwithstanding. More to the point of making comparisons, however, there seem to be no examples of unequal Borda maximin allocations that

players there is always a preference profile in which both maximin and Borda maximin allocations are envy-ensuring.

7. Conclusions

While maximin and Borda maximin allocations cannot prevent envy (except for Borda maximin when there are two players), the good news is that these allocations reduce the probability that it occurs. The other good news is that there is always an efficient allocation that is not envy-ensuring, given that each player receives at least two items, but it will not necessarily be a maximin, a Borda maximin, or a maxsum allocation.

That an efficient, envy-unensuring allocation may not satisfy either of the two maximin criteria may require that a wrenching choice be made between helping the worst off and avoiding envy. This choice is even more difficult when each player is entitled to only one item, because in this case envy is most difficult to escape (each player must desire a different item). To the best of our knowledge, the fact that Rawlsian (Rawls, 1971) criteria like maximin and Borda maximin may ensure envy rather than prevent it has not heretofore been investigated in our minimal-information framework, in which only simple preference rankings are assumed.

While this clash is reminiscent of Arrow's (1951) impossibility theorem, wherein the conflict among several reasonable conditions for aggregating individual choices into a social choice is ineradicable, fair division is not generally a problem of collective choice. Rather, it is usually a private issue, as when parents decide how a family estate will be divided among their children.

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are envy-ensuring. For this to happen, a player would have to envy another receiving at least as many

But larger entities, including countries, may also face such issues, in which case millions of people may be affected by how fair-division questions are resolved. A case in point is the breakup of Czechoslovakia into the Czech Republic and Slovakia in 1993, in which the two sides had to split a number of indivisible items, including certain military assets. Likewise, the division of both Germany and Berlin after World War II, while ostensibly involving the Allies' drawing borders over the divisible good of land, also included, by extension, indivisible goods like universities and opera houses that were situated on this land.

In such cases, allocations may be inefficient because of practical constraints that players place on the division. More surprising is that independent of any practical constraints, requiring that an allocation be envy-free may force it to be inefficient.

Although the symbolic value of giving players equal numbers of items, such as landing slots at an airport, may be important, equal allocations may violate the two maximin criteria and, as well, ensure envy. To be sure, unequal divisions can also force a violation of these criteria, so neither equality nor inequality can be held up as an egalitarian ideal.

Our computer simulation showed that Borda maximin allocations ensure envy less often than maximin allocations. The chances of assured envy also become less as the number of items increases, but they never fall to zero as long as there are not two players. While envy may be ineradicable if one desires to help the worst off, it is not clear that abandoning the maximin criteria to avoid it is a better alternative.

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items; we have found no example of this and suspect one might not exist.

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