

THE LOGIC OF LEIBNIZ'S *GENERALES INQUISITIONES DE
ANALYSI NOTIONUM ET VERITATUM*

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Abstract. The *Generales Inquisitiones de Analysi Notionum et Veritatum* is Leibniz's most substantive work in the area of logic. Leibniz's central aim in this treatise is to develop a symbolic calculus of terms that is capable of underwriting all valid modes of syllogistic and propositional reasoning. The present paper provides a systematic reconstruction of the calculus developed by Leibniz in the *Generales Inquisitiones*. We investigate the most significant logical features of this calculus and prove that it is both sound and complete with respect to a simple class of enriched Boolean algebras which we call auto-Boolean algebras. Moreover, we show that Leibniz's calculus can reproduce all the laws of classical propositional logic, thus allowing Leibniz to achieve his goal of reducing propositional reasoning to algebraic reasoning about terms.

The year 1686 marked an exceptionally fruitful period in the intellectual life of Gottfried Wilhelm Leibniz. In this one year, Leibniz produced four major treatises devoted, respectively, to the subjects of physics, metaphysics, theology, and logic. The last of these works, entitled *Generales Inquisitiones de Analysi Notionum et Veritatum*, is the centerpiece of Leibniz's logical writings and contains his most in-depth investigations into the foundations of the subject.

In the *Generales Inquisitiones*, Leibniz develops a symbolic calculus intended to underwrite all valid modes of syllogistic and propositional reasoning. This calculus takes the form of an algebra of terms, and prefigures in many important respects the algebraic systems of logic developed in the mid-19th century. While Leibniz was himself convinced of the success of the *Generales Inquisitiones*, noting on the first page of the manuscript that 'herein I have made excellent progress', students of Leibniz confront a number of challenges in trying to arrive at an exact understanding of what he accomplished in this work. One difficulty is due to the exploratory nature of the work. The *Generales Inquisitiones* does not take the form of a methodical presentation of an antecedently worked-out system of logic, but rather comprises a meandering series of investigations covering a wide range of topics. As a result, it can be difficult to discern the underlying currents of thought that shape the treatise amidst the varying terminology and conceptual frameworks adopted by Leibniz at different stages of its development.

In spite of such challenges, a closer inspection of the *Generales Inquisitiones* reveals it to be more coherent than it may at first appear. This is due, in large part, to the fact that the concluding sections of the *Generales Inquisitiones* culminate in an axiomatic system of

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principles from which one can derive all the central claims made throughout the treatise. By focusing on this final set of principles, the *Generales Inquisitiones* can be viewed as documenting the progressive development of Leibniz's thought towards a single, unified system of logic that brings together the various insights accumulated over the course of the preceding investigations.

The main aim of this paper is to provide a reconstruction of the calculus developed by Leibniz in the *Generales Inquisitiones*, and to conduct a detailed study of its most significant logical features. Our analysis builds upon previous attempts to reconstruct the logic of the *Generales Inquisitiones*, most notably in the work of Castañeda (1976, 1990) and Lenzen (1984a, 2004). While these earlier efforts have significantly advanced our overall understanding of Leibniz's logic, in our estimation they fail in crucial respects to give an adequate account of the calculus developed in the *Generales Inquisitiones*. The present paper aims to overcome these deficiencies and thereby to arrive at a tenable reconstruction of Leibniz's calculus that is faithful to the text. It is our hope that the analysis undertaken in this paper will help to give students of Leibniz a clearer sense of the overall unity of the *Generales Inquisitiones*, and thereby contribute to a more complete understanding of this landmark text in the history of logic.

The paper is divided into four parts. The first part describes the syntax of Leibniz's calculus. The language in which the calculus is formulated is extremely parsimonious in that the only well-formed propositions are those of the form $A = B$. These propositions assert that the terms A and B coincide, i.e., that they can be substituted for one another *salva veritate*. Additional propositional forms are then defined as coincidences between compound terms. Apart from Boolean compounds, Leibniz introduces a new kind of term generated from propositions. Specifically, for every proposition $A = B$, there is a corresponding term $\ulcorner A = B \urcorner$, referred to as a propositional term. By utilizing the device of propositional terms, Leibniz is able to define a number of other kinds of proposition in his calculus, including those expressing the noncoincidence of terms.

The second part of the paper presents the principles of Leibniz's calculus, which are enumerated in the concluding sections of the *Generales Inquisitiones*. We identify seven principles that form the axiomatic basis of the calculus and suffice to derive all the central theorems stated by Leibniz throughout the treatise.

The third part of the paper describes the semantics of Leibniz's calculus. We examine the algebraic structure imparted to the terms by the principles of the calculus. We show that Leibniz's calculus is both sound and complete with respect to a simple class of enriched Boolean algebras which we call auto-Boolean algebras. This completeness result obtains despite the fact that the calculus also has nonstandard models which are not auto-Boolean. Moreover, we show that every valid inference of classical propositional logic is derivable in Leibniz's calculus. The latter result affords us a proper understanding of the semantic significance of propositional terms, and thus allows us to overcome a number of difficulties that have beset previous attempts to reconstruct the logic of the *Generales Inquisitiones*.

The fourth part of the paper is an appendix supplying proofs of all the technical results stated throughout the first three parts. This appendix can be viewed as an optional supplement to be consulted at the reader's discretion.

In outline, the four parts of the paper are organized as follows:

§1. The Syntax of Leibniz's Calculus

- 1.1. Terms
- 1.2. Propositions
- 1.3. Containment

- 1.4. Falsehood
- 1.5. Propositional Terms
- 1.6. Summary

§2. The Principles of Leibniz's Calculus

- 2.1. The Principle of Substitution
- 2.2. Principles of Composition
- 2.3. Principles of Privation
- 2.4. The Principle of Propositional Containment
- 2.5. Leibniz's Principle
- 2.6. Summary

§3. The Semantics of Leibniz's Calculus

- 3.1. The Semilattice of Terms
- 3.2. The Boolean Algebra of Terms
- 3.3. The Auto-Boolean Algebra of Terms
- 3.4. Propositional Reasoning in Leibniz's Calculus
- 3.5. The Categorical Logic of Propositional Terms
- 3.6. Summary

§4. Appendix: Proofs of Theorems

- 4.1. Preliminaries
- 4.2. The Core Fragment
- 4.3. The Non-Propositional Fragment
- 4.4. Classical Propositional Logic
- 4.5. Auto-Boolean Semantics
- 4.6. The Auto-Boolean Completeness of Leibniz's Calculus

The *Generales Inquisitiones* remained unpublished for more than two centuries. The first edition of the text, prepared by Couturat, appeared in 1903. It was superseded by an edition published by Schupp in 1981 and revised in 1993. A new edition was published in 1999 in volume VI4A of the authoritative *Akademie-Ausgabe* of Leibniz's writings.¹ In what follows, all references to the *Generales Inquisitiones* are to this VI4A volume and all translations from the Latin are our own.

The *Generales Inquisitiones* consists of a series of introductory remarks followed by 200 numbered sections. To refer to these sections we use unaccompanied references such as §55 and §131. Where this is not possible, we provide the page number of the relevant passage in the VI4A volume.² In addition to the *Generales Inquisitiones*, this volume contains numerous other logical writings by Leibniz that we will have occasion to reference. When referring to these writings—which often take the form of brief notes and fragments—we do not provide the title of the piece but simply refer to the relevant page number in the VI4A volume.

More generally, we adopt the following abbreviations for published editions of Leibniz's writings:

¹ A free electronic copy of this volume is available from the Leibniz-Forschungsstelle Münster at <http://www.uni-muenster.de/Leibniz/>.

² The *Generales Inquisitiones* appears on pages 739–88 of this volume. Page references within this range refer to passages from the *Generales Inquisitiones* that do not properly appear in any of its 200 numbered sections, such as the introductory remarks preceding §1 and any marginal notes added by Leibniz throughout the text (which we refer to by their number in the VI4A edition).

- VI4A *Gottfried Wilhelm Leibniz: Sämtliche Schriften und Briefe*, vol. VI4A. Ed. by the Berlin Brandenburgische Akademie der Wissenschaften and the Akademie der Wissenschaften in Göttingen. Berlin: Akademie Verlag, 1999.
- C *Opuscules et fragments inédits de Leibniz*. L. Couturat, editor. Paris: Félix Alcan, 1903.
- G *Die philosophischen Schriften von Gottfried Wilhelm Leibniz*. Ed. by C. I. Gerhardt, 7 vols. Berlin: Weidmannsche Buchhandlung, 1857–90.

§1. The Syntax of Leibniz’s Calculus. The *Generales Inquisitiones* is a treatise about terms (*termini*) and propositions (*propositiones*). A term, for Leibniz, is a concept or idea while a proposition is a cognitive item composed of terms which is akin to a thought.³ Leibniz’s principal aim in the *Generales Inquisitiones* is to construct a symbolic calculus by means of which all valid inferences between propositions ‘might be directly ascertained from the words or symbols themselves’.⁴

Despite his emphasis on the symbolic nature of the calculus, Leibniz generally opts to characterize its syntax indirectly by describing the compositional structure of the terms and propositions signified by its expressions. In doing so, he assumes that the syntax of these expressions mirrors that of the items they signify:⁵

The art of symbolism is the art of composing and arranging the symbols in such a way that they replicate the thoughts, i.e., such that they stand in those relations between themselves that the thoughts stand in between themselves. (VI4A 916)

We thus begin our study of Leibniz’s calculus by examining the syntax of terms and propositions.

1.1. Terms. The calculus developed by Leibniz in the *Generales Inquisitiones* is a calculus for reasoning about terms. Specifically, it treats of those terms which ‘without any addition can be the subject or predicate of a proposition’.⁶ Elementary examples of such terms include *animal*, *rational*, and *learned*.

Some terms are simple in the sense of being unanalyzable (*irresolubiles*).⁷ Others are obtained as the result of applying certain operations to simpler terms. Two operations through which such compound terms can be generated are privation (*privatio*) and composition

³ ‘By a term I mean not a word but a concept or that which is signified by a word, you may also say a notion or idea’ (VI4A 288); see Kauppi 1960: 39–40, Ishiguro 1972: 20–2, Mates 1986: 58, Swoyer 1994: 4. For the cognitive nature of propositions, see G V 378; cf. Parkinson 1965: 8–11, Ishiguro 1972: 27–9, Castañeda 1974: 384–5, Mates 1986: 48–54.

⁴ VI4A 800; see also VI4A 643, 913, 920–1. While Leibniz does not use the phrase ‘symbolic calculus’ in the *Generales Inquisitiones*, he refers to this calculus as a *calculus Rationis* (§20; similarly, §26, §106), and takes it to be formulated within a symbolic system, or, *characteristica* (§18, §75, VI4A 741).

⁵ See Arndt 1971: 111, Heinekamp 1976: 529–33, Poser 1979: 313–14.

⁶ VI4A 740. Leibniz calls these terms ‘integral terms’. Excluding the introductory remarks at the beginning of the *Generales Inquisitiones*, Leibniz uses ‘term’ to refer exclusively to integral terms throughout the treatise.

⁷ See VI4A 742, 744–5. For Leibniz’s account of simple terms, see Ishiguro 1972: 42–7, Mates 1986: 59.

(*compositio*).⁸ Leibniz expresses privation by means of the particle 'non'. For example, *non-rational* is the privative of the term *rational*. Composition is not expressed by means of any particle but simply by concatenating the expressions signifying the terms to be composed. For example, *rational animal* is the composite of the terms *rational* and *animal*.

To avoid possible scope ambiguities, in what follows we write \bar{A} instead of *non-A*. Leibniz frequently appeals to composites of more than two terms (e.g., *ABC* and *ABCD*). He also makes free use of privatives of composites and of composites involving privatives (e.g., \overline{AB} and $A\bar{B}$). This suggests that he takes the set of terms to be closed under privation and composition. In other words, if *A* and *B* are terms, then \bar{A} and *AB* are terms.

1.2. Propositions. Terms are syntactic constituents of propositions. The propositions of Leibniz's calculus are first described in the *Generales Inquisitiones* as follows:

A proposition is: *A coincides with B, A is B (or B is in A . . .), A does not coincide with B.* (VI4A 750)

In this passage, Leibniz identifies three kinds of proposition:

- (i) *A coincides with B*
- (ii) *A does not coincide with B*
- (iii) *A is B*

To express propositions of the form (i), Leibniz writes $A = B$. He expresses propositions of the form (ii) by $A \text{ non} = B$, which we will write $A \neq B$. Leibniz signifies propositions of the form (iii) by various phrases such as *A is B, B is in A, and A contains B*. To express such propositions we will write $A \supset B$.

The above list of propositional forms is not exhaustive. A more comprehensive list is provided by Leibniz in §195:

A proposition is that which states which term is or is not contained in another. So a proposition can also affirm that some term is false, if it says that *Y non-Y* is contained in it; and it can affirm that a term is true if it denies this. A proposition is also that which says whether or not some term coincides with another. (§195)

Here, Leibniz reiterates the propositional forms (i)–(iii) and adds propositions of the form *A does not contain B*, which we will write $A \not\supset B$. In addition, he identifies two new kinds of proposition, one asserting that a term *A* is false and the other asserting that *A* is true. These latter propositions are described as follows:

A proposition is that which adds to a term that it is true or false; as, for example, if *A* is a term to which it is ascribed that *A is true* or *A is not true*. It is also often said simply that *A is* or *A is not*. (§198.5)

To express such propositions asserting the truth and falsehood of terms, we will write $\mathbf{T}(A)$ and $\mathbf{F}(A)$, respectively.⁹

⁸ See VI4A 740, 742; cf. Mates 1986: 58–9.

⁹ Leibniz uses a number of phrases interchangeably to express propositions of the form $\mathbf{T}(A)$. In addition to *A is true* and *A is*, he uses *A is a being* (§144, VI4A 807), *A is a thing* (§152, §155,

All told, then, Leibniz's calculus deals with the following six kinds of proposition:

- | | | |
|-------|-------------------|---|
| (i) | $A = B$ | <i>A coincides with B</i> |
| (ii) | $A \neq B$ | <i>A does not coincide with B</i> |
| (iii) | $A \supset B$ | <i>A is B, A contains B</i> |
| (iv) | $A \not\supset B$ | <i>A is not B, A does not contain B</i> |
| (v) | $\mathbf{F}(A)$ | <i>A is false, A is not</i> |
| (vi) | $\mathbf{T}(A)$ | <i>A is true, A is</i> |

In Leibniz's view, not all of these propositional forms are primitive. For example, in the above-quoted passage from §195 Leibniz indicates how propositions of the form $\mathbf{F}(A)$ can be reduced to those of the form $A \supset B$ by defining falsehood as the containment of a contradictory composite term. Accordingly, while Leibniz intends for the language of his calculus to be capable of expressing all six kinds of proposition, he does not posit a primitive means for expressing each of them. Indeed, as we shall see, Leibniz ends up positing only one kind of primitive proposition, $A = B$, and reduces all other kinds of proposition to those of this form.

This reduction in the number of propositional forms to just one takes place gradually over the course of the *Generales Inquisitiones* and constitutes one of the major accomplishments of the work. It proceeds in three distinct steps:

1. Propositions of the form $A \supset B$ are reduced to those of the form $A = B$.
2. Propositions of the form $\mathbf{F}(A)$ are reduced to those of the form $A \supset B$.
3. Propositions of the forms $A \neq B$, $A \not\supset B$, and $\mathbf{T}(A)$ are reduced to those of the form $\mathbf{F}(A)$.

Each step in this reduction exploits a different aspect of the syntactic structure of the terms of Leibniz's calculus. In what follows, we discuss each of these three steps in turn.

1.3. Containment. Propositions of the form $A \supset B$ are taken by Leibniz to assert that 'every A is B , or, the concept of B is contained in the concept of A ' (§28). He thus identifies these propositions with the universal affirmative statements appearing in Aristotelian syllogisms such as Barbara.¹⁰ Unlike Aristotle, however, Leibniz does not treat these statements as primitive, but instead suggests that 'it will be best to reduce (*reducere*) propositions from predication and from being-in-something to coincidence'.¹¹ In other words, Leibniz aims to reduce propositions of the form $A \supset B$ to those of the form $A = B$.

Leibniz's first attempt to reduce containment to coincidence in the *Generales Inquisitiones* appears in §16:

*A is B is the same as A is coincident with some B, or, $A = BY$.*¹² (§16)

⁹ §169), and *A is possible* (VI4A 744, 749 nn. 8 and 10, §35, §42, §§45–6, §55, §69). Leibniz employs a similarly diverse range of phrases to express propositions of the form $\mathbf{F}(A)$.

¹⁰ See, e.g., §47, §124, §129, §§190–1, VI4A 280.

¹¹ VI4A 622. This is in accordance with the fact that propositions of the form $A \supset B$ do not appear in Leibniz's initial presentation of the basic elements of his calculus in §§1–15 of the *Generales Inquisitiones* (although they are mentioned in marginal notes and later additions to the text; see the editorial notes at VI4A 749–50). Containment is first mentioned in §16, where it is defined by means of coincidence.

¹² See also §17, §189.4, and §198.9.

The letter 'Y' appearing in this definition is what Leibniz calls an indefinite letter. Leibniz utilizes indefinite letters throughout the *Generales Inquisitiones* as a means of expressing existential quantification.¹³ Their introduction into the calculus leads to complications analogous to those which arise in connection with the elimination of existential quantifiers in modern quantificational logic.¹⁴ Leibniz was aware of some of these complications and perhaps for this reason was never entirely satisfied with the use of indefinite letters, indicating that it would be preferable to omit them from his calculus.¹⁵ Leibniz ultimately succeeds in doing so, asserting about midway through the treatise that he has 'at last wholly eliminated the indefinite letter Y' (§128). This elimination is made possible, in part, by the following alternative definition of containment which avoids the use of indefinite letters:

A is B is the same as $A = AB$.¹⁶ (§83)

Leibniz is here appealing to the intuition that if *A* coincides with some kind of *B*, then one specific kind of *B* with which *A* coincides is the composite term *AB*. Note that this definition of containment relies on the fact that the terms of Leibniz's calculus are closed under the operation of composition. Thus, by positing this definition, Leibniz manages to decrease the number of primitive relations that obtain between terms by increasing the syntactic complexity of the terms themselves. In thereby shifting the focus from the relational structure of the terms to their compositional or algebraic structure, Leibniz's reduction of containment to coincidence marks a significant advance in the algebraic treatment of logic.

1.4. Falsehood. Propositions of the form $\mathbf{F}(A)$ and $\mathbf{T}(A)$ assert of a term *A* that it is false and that it is true, respectively. This may seem odd since truth and falsehood are typically thought to apply to propositions rather than terms. However, by a false term Leibniz simply means one which 'contains a contradiction' (§57):

That term is false, or not true, which contains *A non-A*; that term is true which does not. (§198.4; similarly, §194)

In this definition of falsehood, '*A non-A*' is not meant to refer to a specific term. Instead, Leibniz's intended meaning is that a term is false if it contains *some* composite of a term and its privative (and true if it does not). Thus, if we were to avail ourselves of indefinite letters, the proposition $\mathbf{F}(A)$ could be defined as $A \supset Y\bar{Y}$.¹⁷ As we have seen, however, Leibniz wishes to avoid the use of indefinite letters in his calculus.¹⁸

¹³ See Lenzen 1984b: 7–13, 2004: 47–50, Hailperin 2004: 329.

¹⁴ In particular, care must be taken not to use one and the same indefinite letter in reducing distinct propositions $A \supset B$ and $C \supset D$ to coincidences. Otherwise, it might mistakenly be inferred that there is a single term *Y* such that $A = BY$ and $C = DY$. Leibniz is aware of this difficulty and resolves it by introducing a number of rules governing the use of indefinite letters in the calculus (see §§21–31).

¹⁵ See §162 and Schupp 1993: 153, 168, 182–3. Another possible source of dissatisfaction with the use of indefinite letters stems from the difficulties Leibniz encounters in attempting to establish the antisymmetry of containment in §§30–1.

¹⁶ See also §113, VI4A 751 n. 13, 808; cf. C 236.

¹⁷ For this definition of falsehood, see §61, §190, and §195.

¹⁸ Even using indefinite letters, $\mathbf{T}(A)$ cannot be defined as $A \not\supset Y\bar{Y}$, because this does not express the universally quantified claim that *A* does not contain *any* contradictory composite term. Leibniz entertains the possibility of introducing a new notation to express universal quantification in

The question therefore arises as to how falsehood can be defined without using indefinite letters. While Leibniz does not provide an explicit account of how this is to be done, the various claims he makes about falsehood in the *Generales Inquisitiones* provide a straightforward answer to this question. The proposed reduction relies on Leibniz's claim that a composite term AB is false just in case $A \supset \bar{B}$.¹⁹ In particular, AA is false just in case $A \supset \bar{A}$. But it is a basic principle of Leibniz's calculus that $A = AA$.²⁰ Consequently, A is false just in case AA is false, or, $A \supset \bar{A}$.²¹

The proposition $\mathbf{F}(A)$ can thus be defined as $A \supset \bar{A}$. Although this definition of falsehood does not appear in the *Generales Inquisitiones*, it is a direct consequence of Leibniz's claims about falsehood and has the advantage of being expressible without the use of indefinite letters.²² Given the above reduction of containment to coincidence, $\mathbf{F}(A)$ thus amounts to the proposition $A = A\bar{A}$. In this reduction of falsehood to coincidence, the number of primitive propositional forms is again decreased by exploiting the syntactic complexity of the terms—in this case, their closure under both composition and privation.

We have so far seen how both containment and falsehood can be reduced to coincidence. It remains to consider how propositions of the forms $\mathbf{T}(A)$, $A \not\supset B$, and $A \neq B$ can be reduced to coincidences. In light of the previous discussion, the first two of these propositional forms can be reduced to the third in the obvious way, namely, by defining $\mathbf{T}(A)$ as $A \not\supset \bar{A}$, and $A \not\supset B$ as $A \neq AB$. All that is left, then, is to show how propositions of the form $A \neq B$ can be reduced to those of the form $A = B$. It may not be obvious how, or even if, this can be done since it requires that the noncoincidence of two terms be expressed by the coincidence of two others. Nevertheless, as we shall see, Leibniz manages to achieve this by once more adding complexity to the syntactic structure of the terms.

1.5. Propositional Terms. In addition to privation and composition, Leibniz posits a third syntactic operation by which compound terms can be formed. This operation allows us to generate a new term from any given proposition, and is motivated by Leibniz's view that 'every proposition can be conceived of as a term':²³

If the proposition A is B is considered as a term, as we have explained that it can be, there arises an abstract term, namely A 's being B . And if from the proposition A is B the proposition C is D follows, then from this there comes about a new proposition of the following kind: A 's being B is (or contains) C 's being D ; or, in other words, the B -ness of A contains the D -ness of C , or the B -ness of A is the D -ness of C .²⁴ (§138)

§§80–1 and §112, but does not develop this idea in any detail (see Lenzen 1984b: 13–17, 2004: 50–5).

¹⁹ §§199–200; see Section 2.5 below.

²⁰ §18 and §198.2; cf. n. 46.

²¹ This last step of the argument relies on the fact that if $A = B$, then A is false just in case B is false. This follows from Leibniz's claims in §58 and §30.

²² Cf. Lenzen 1984a: 195 n. 23, 1987: 4–5. This definition turns out to be equivalent to Leibniz's existential definition of falsehood quoted above, since his claims in the *Generales Inquisitiones* imply that a term contains some contradictory composite just in case that term contains its own privative (see Theorems 4.30 and 4.31 in Section 4.3).

²³ See §75, §109, §197. When a proposition is conceived of as a term, Leibniz describes the proposition as giving rise to a 'new term' (*terminus novus*, §197 and §198.7; *contra* Parkinson 1966: 86 n. 2).

²⁴ 'Nempe si propositio A est B consideretur ut terminus, quemadmodum fieri posse explicuimus, oritur abstractum, nempe $\tau \delta A$ esse B , et si ex propositione A est B sequatur propositio D est C ,

Thus, a proposition such as $A \supset B$ gives rise to a new term, which Leibniz signifies by A 's *being B* or the *B-ness of A*.²⁵ In what follows, we shall refer to such terms as propositional terms.²⁶

As the above passage makes clear, the expressions used to signify propositional terms are distinct from those used to signify the propositions from which these terms are generated. In §§138–43, Leibniz considers the question of how the expression for a propositional term can be derived in various cases from the expression for the corresponding proposition. What is important for our present purposes is that, in Leibniz's view, the expressions for propositional terms can be derived from the expressions for the corresponding propositions in a systematic manner, so that the latter can be put in one-to-one correspondence with the former. We will use corner-quotes to indicate the application of this one-to-one mapping. Thus, for example, if $A = B$ is a proposition, the corresponding propositional term is $\ulcorner A = B \urcorner$.²⁷ In this notation, the complex proposition mentioned by Leibniz in the passage just quoted is written:

$$\ulcorner A \supset B \urcorner \supset \ulcorner C \supset D \urcorner.$$

Leibniz clearly means to include propositions such as this in the language of his calculus. It is less clear whether he also means to include 'mixed' propositions relating propositional

tunc inde fit nova propositio talis: $\tau\delta A$ esse B est vel continet $\tau\delta D$ esse C , seu *Beitas ipsius A continet Ceitatem ipsius D*, seu *Beitas ipsius A est Ceitas ipsius D*.' This text incorporates two reasonable emendations suggested by the editors of VI4A concerning the order of the letters 'C' and 'D'. The reverse order in which these letters are printed in the Latin text probably reflects Leibniz's desire to avoid the word '*Deitas*', which could be interpreted as deity rather than *D*-ness. Since there is no such ambiguity in English, we adopt the usual lexical order of 'C' and 'D' in our translation.

²⁵ See §§138–42 and VI4A 740. The expression '*A's being B*' translates ' $\tau\delta A$ esse B '. Sometimes Leibniz omits the Greek definite article $\tau\delta$, writing '*A esse B*' instead of ' $\tau\delta A$ esse B ' (e.g., §61, §109). In this case, the expression for the propositional term differs from the one for the original proposition only in using the infinitive form *esse* instead of the finite form *est*. In the case of propositions of the form $\mathbf{T}(A)$ such as *A verum est*, the expression for the propositional term is obtained either by replacing the finite verb by the infinitive form (*A verum esse*) or simply by deleting the finite verb (*A verum*); see §197 and §198.7. As a rule, expressions for propositional terms differ from those for propositions in that the latter but not the former contain the finite form of a verb.

²⁶ Leibniz does not use the phrase 'propositional term', but instead uses 'complex term' to refer to propositional terms and Boolean compounds thereof (§61, §65, §75; cf. VI4A 528–9). In a similar fashion, he uses 'incomplex term' to refer to simple terms and Boolean compounds thereof. Leibniz occasionally uses 'term' in a narrower sense applying only to incomplex terms (e.g., VI4A 754 n. 18). More often, however, he uses it in a broader sense applying to both incomplex and complex terms alike.

²⁷ It is sometimes maintained that, for Leibniz, propositions just are terms or concepts (Ishiguro 1972: 19–20, Castañeda 1974: 384 and 395, Mates 1986: 54, 125–6, and 176). Such a view, however, is not supported by the text of the *Generales Inquisitiones*. While Leibniz states that 'all propositions can be conceived of as terms' (*concipere omnes propositiones instar terminorum*, §75, §109, §197), this does not mean that they *are* terms. For example, Leibniz also claims that 'we can conceive of a term as a fraction' (*conciplamus terminum instar fractionis*, §187), but he obviously does not think that terms are fractions (rather, fractions 'represent' terms, §129). Moreover, Leibniz writes that propositional terms 'arise' (*oritur*, §138) from a proposition, and that a proposition 'becomes' (*fit*, §198.7) a term. Neither of these claims implies that a propositional term is identical with the corresponding proposition. For our purposes, it is not necessary to take a stand on these issues; all that is important is that the expressions used to signify propositional terms in the language of Leibniz's calculus are distinct from those used to signify the corresponding propositions.

to non-propositional terms, such as the following:

$$\ulcorner A \supset B \urcorner \supset C.$$

There is no evidence to suggest that Leibniz meant to exclude such mixed propositions from the language of his calculus.²⁸ Moreover, doing so would require the positing of additional syntactic constraints to prevent the mixing of distinct logical types of terms, and Leibniz does not posit such constraints in the *Generales Inquisitiones*. We therefore opt to include mixed propositions in the language, and likewise mixed composite terms such as $\ulcorner A \supset B \urcorner C$.²⁹ This, in effect, means that propositional terms function syntactically just like any other terms and can be substituted *salva congruitate* for non-propositional terms in any well-formed expression.

The device of propositional terms allows us to formulate within the language of Leibniz's calculus the various claims made in the *Generales Inquisitiones* concerning the truth and falsehood of propositions. For example:

A does not coincide with B is the same as *it is false that A coincides with B*.³⁰ (§5)

Here, Leibniz states that a proposition of the form $A \neq B$ is equivalent to the proposition *it is false that A coincides with B*.³¹ In the language of Leibniz's calculus, the latter proposition takes the form $\mathbf{F}(\ulcorner A = B \urcorner)$, asserting the falsehood of the propositional term $\ulcorner A = B \urcorner$. Thus, the equivalence stated in §5 asserts:³²

$$A \neq B \text{ if and only if } \mathbf{F}(\ulcorner A = B \urcorner).$$

Given the above reduction of falsehood to coincidence, $\mathbf{F}(\ulcorner A = B \urcorner)$ amounts to:

$$\ulcorner A = B \urcorner = \ulcorner A = B \urcorner \overline{\ulcorner A = B \urcorner}.$$

Hence, we have:

$$A \neq B \text{ if and only if } \ulcorner A = B \urcorner = \ulcorner A = B \urcorner \overline{\ulcorner A = B \urcorner}.$$

This last equivalence shows how propositions of the form $A \neq B$ can be reduced to those of the form $A = B$ by utilizing the device of propositional terms.³³ It follows that

²⁸ On the contrary, in §109 Leibniz countenances propositions such as *man's being animal is a reason* and *man's being animal is a cause*. He makes it clear that the subject term *man's being animal* is a propositional term, describing it as 'a proposition conceived of as a term' (§109). At the same time, the predicate terms *reason* and *cause* are presumably non-propositional terms. If so, then Leibniz allows for mixed propositions such as $\ulcorner \textit{man} \supset \textit{animal} \urcorner \supset \textit{reason}$ (and, given his reduction of containment to coincidence, also mixed composite terms such as $\ulcorner \textit{man} \supset \textit{animal} \urcorner \textit{reason}$).

²⁹ It would be equally open to us to develop Leibniz's calculus in a language that does not allow for the unrestricted mixing of propositional and non-propositional terms. With a few minor modifications, all of the main results of this paper would be unaffected by these syntactic restrictions (see n. 143).

³⁰ '*A non coincidit ipsi B idem est ac A coincidere ipsi B est falsum.*' We take the infinitive phrase '*A coincidere ipsi B*' to express the propositional term $\ulcorner A = B \urcorner$ (see n. 25).

³¹ Similarly, C 235, 421.

³² More generally, Leibniz holds that 'a negative statement is nothing other than that statement which says that the affirmative statement is false' (VI4A 811 n. 6). In particular, he takes $A \not\supset B$ to be equivalent to $\mathbf{F}(\ulcorner A \supset B \urcorner)$ (§32a, §84, VI4A 807 n. 1, 808, 811).

³³ This is in accordance with the fact that propositions of the form $A \neq B$ do not appear in Leibniz's final presentation of the calculus in §§198–200. The eventual omission of these propositions

all six kinds of proposition dealt with in Leibniz's calculus can be reduced to coincidences between terms.³⁴ In this way, Leibniz is able to achieve the aim, attributed to him by Castañeda, of formulating 'a strict equational calculus in which all propositions are about the coincidence of terms'.³⁵ As we shall see, notwithstanding its syntactic parsimony, Leibniz's calculus is capable of reproducing complex modes of reasoning pertaining to noncoincidence, containment, truth, and falsehood.

1.6. Summary. By utilizing the syntactic operations of privation, composition, and the formation of propositional terms, all the propositions dealt with in Leibniz's calculus are expressible in the form $A = B$. The terms and propositions of Leibniz's calculus can therefore be generated inductively from the simple terms as follows:

1. If A and B are terms, then \overline{A} and AB are terms.
2. If A and B are terms, then $A = B$ is a proposition.
3. If $A = B$ is a proposition, then $\ulcorner A = B \urcorner$ is a term.

In addition, we adopt the following definitions:³⁶

1. $A \supset B$ is the proposition $A = AB$.
2. $\mathbf{F}(A)$ is the proposition $A \supset \overline{A}$.
3. $A \neq B$ is the proposition $\mathbf{F}(\ulcorner A = B \urcorner)$.
4. $A \not\supset B$ is the proposition $\mathbf{F}(\ulcorner A \supset B \urcorner)$.
5. $\mathbf{T}(A)$ is the proposition $\mathbf{F}(\ulcorner \mathbf{F}(A) \urcorner)$.

§2. The Principles of Leibniz's Calculus. In the earlier portions of the *Generales Inquisitiones*, Leibniz explores the inferential relations that obtain between the propositions expressible in the language of his calculus without having specified a fixed axiom system. Toward the end of the treatise, however, Leibniz makes a deliberate effort to systematize the calculus by identifying a set of axiomatic principles from which the various theorems stated throughout the work can be derived. Leibniz intended this set of principles to be complete in the sense that 'whatever cannot be proved from these principles does not follow by virtue of logical form'.³⁷

In the concluding sections of the *Generales Inquisitiones*, Leibniz makes three increasingly refined attempts to formulate such a complete set of principles (*principia*). These

suggests that Leibniz took them to be reducible to other, more primitive propositions, so that there is no need to introduce any specific axioms or rules governing their usage.

³⁴ The fact that the only propositions in Leibniz's calculus are those expressing the coincidence between terms may seem to conflict with Leibniz's repeated statements to the effect that the letters 'A' and 'B' appearing in ' $A = B$ ' can signify either terms or propositions (VI4A 748 n. 6, §4, §13; similarly, §35, §55). These statements, however, all appear in the earlier sections of the *Generales Inquisitiones*, prior to Leibniz's introduction of propositional terms in §75. In light of this fact, we take Leibniz's earlier statements to the effect that 'A' can signify either a term or a proposition to mean that 'A' can signify any term, propositional or otherwise.

³⁵ Castañeda 1976: 483; similarly, Schupp 1993: 156. This view is further corroborated by Leibniz's characterization of propositions of the form $A = B$ as 'the most simple' (*simplicissimae*) of propositions (§157 and §163).

³⁶ For the last of these five definitions, see Leibniz's claim in §1 that 'these coincide: L is true and L 's being false is false'.

³⁷ §189.7; cf. §200 and VI4A 797 n. 1.

systems appear in §171, §189, and §§198–200.³⁸ Since the last of these systems is the most developed and supersedes the previous two, we will use it as a guide in our reconstruction of Leibniz’s calculus.

2.1. The Principle of Substitution. In §198 Leibniz provides a numbered list of nine principles. The first of these is:

1st. Coincidents can be substituted for one another. (§198.1)

This principle asserts that if two terms coincide then ‘one can be substituted for the other in every proposition *salva veritate*’.³⁹

Leibniz is well aware that there are contexts in ordinary language in which substitution of coincidents fails to preserve truth. He describes these contexts as those in which the claims we make ‘do not so much speak about a thing, as about our way of conceiving it’ (§19).⁴⁰ Since, however, in Leibniz’s calculus such opaque contexts are precluded, the proposition $A = B$ licenses free substitution of A for any occurrence of B (and vice versa) in any proposition expressible in the language of the calculus.⁴¹

In §§6–9, Leibniz identifies four corollaries (*corollaria*) of this principle of substitution. The first of these asserts the symmetry of coincidence:

If A coincides with B , B coincides with A . (§6)

The fact that Leibniz views this as a corollary of the substitution principle indicates that he intends this principle to allow for ‘reflexive’ substitution, i.e., substitution of one term for another in the very same coincidence that licenses the substitution.⁴²

In addition to symmetry, Leibniz observes that the principle of substitution entails the transitivity of coincidence.⁴³

If A coincides with B and B coincides with C , then also A coincides with C . (§8)

He also identifies the following corollary of the substitution principle:

If A coincides with B , non- A coincides with non- B . (§9)

³⁸ While the word ‘*principium*’ occurs frequently from §171 onwards, apart from a few unrelated occurrences it does not appear previously in the *Generales Inquisitiones*. The use of ‘*principium*’ in §171, §189, and §198 thus marks a clear shift by Leibniz toward an axiomatic approach to the calculus.

³⁹ VI4A 746; cf. §9, §19, VI4A 626, 816, 831, 846, 871. For Leibniz, coincidence does not necessarily amount to numerical identity. For example, there is evidence that he regards coincident terms such as *triangle* and *trilateral* as distinct from one another (Burkhardt 1974: 49–53, Castañeda 1974: 390–6, Mates 1986: 127; pace Ishiguro 1972: 41, 1990: 22–31).

⁴⁰ See also VI4A 672; cf. Burkhardt 1974: 49–53, Castañeda 1974: 394, Mates 1986: 130–1, Schupp 1993: 158–9.

⁴¹ There is a question as to whether or not the operation for forming propositional terms generates an opaque context. This amounts to the question of whether free substitution of coincident terms is legitimate when the substitution takes place within a propositional term. For example, given $A = B$ and $\mathbf{F}(\ulcorner A \supset C \urcorner)$, can one infer $\mathbf{F}(\ulcorner B \supset C \urcorner)$? Although Leibniz does not explicitly endorse such substitutions, it turns out that he is committed to their validity by the principles of his calculus (see n. 66). Thus, for Leibniz, propositional terms do not constitute opaque contexts.

⁴² Leibniz proves the symmetry of coincidence by applying reflexive substitution in VI4A 831. This proof also shows that Leibniz takes $A = B$ to license substitution in either direction, i.e., substitution of either A for B or B for A (cf. C 421).

⁴³ See also VI4A 815–16, 849; cf. Theorem 4.9, Section 4.2.

While Leibniz does not provide an explicit proof of this conditional, the most obvious way of deriving the consequent from the antecedent is by substituting A for the first occurrence of B in the proposition $\overline{B} = \overline{B}$. This, however, raises the question as to how this last proposition—and, more generally, the reflexivity of coincidence—is to be justified. Leibniz addresses this question in a parenthetical remark in the following passage:

Our principles, therefore, will be these: first, $AA = A$ (from which it is also evident that $non-B = non-B$, if we let $non-B = A$). (§189.1)

Leibniz here observes that the proposition $\overline{B} = \overline{B}$ can be derived by applying reflexive substitution to the proposition $\overline{B} \overline{B} = \overline{B}$.⁴⁴ More generally, the reflexivity of coincidence follows from the substitution principle provided that for any term A there is an axiom of the form $B = A$. Since, as we shall see, this latter condition is satisfied in Leibniz's calculus, the principle of substitution implies the reflexivity of coincidence.⁴⁵

Thus, in Leibniz's calculus, the principle of substitution entails that coincidence is symmetric, transitive, and reflexive. In other words, it entails that coincidence is an equivalence relation.

2.2. Principles of Composition. The second principle stated by Leibniz in §198 asserts the idempotence of composition:⁴⁶

2nd. $AA = A$. (§198.2)

Since this is equivalent to $A = AA$, given Leibniz's definition of containment, this principle amounts to the reflexivity of containment, i.e., $A \supset A$.⁴⁷

The principle of idempotence implies that, in any composite term, consecutive repetitions of the same term can be omitted. Leibniz also holds that nonconsecutive repetitions

⁴⁴ Some commentators replace the phrase ' $non-B = non-B$ ' in §189.1 with ' $non-B non-B = non-B$ ' (e.g., the editors of VI4A, Schupp 1993: 122–3, Rauzy 1998: 294–5, Mugnai 2008: 110). In doing so, they are assuming that this phrase is merely meant to provide a substitution instance of $AA = A$. There is, however, no textual support for this proposed emendation. Moreover, there is independent evidence for thinking that Leibniz's intent in §189.1 is to derive the reflexivity of coincidence from the principle $AA = A$. In his initial list of principles in §171, the first three principles stated are: $A = A$, $non-A = non-A$, and $AA = A$. In his revised list of principles in §189, Leibniz omits the first two of these principles. This strongly suggests that he regarded reflexivity as derivable from other principles, which is just what the unaltered text of §189.1 implies (see Castañeda 1990: 22, Doull 1991: 26 n. 8). This is further confirmed by the fact that, while the reflexivity of coincidence is asserted multiple times throughout the *Generales Inquisitiones* (§10, §156, §171.1), it does not appear in Leibniz's final list of principles in §§198–200.

⁴⁵ In some passages of the *Generales Inquisitiones*, Leibniz considers a variant of his calculus in which coincidence is reflexive for true terms but not for false terms (e.g., §§152–3, VI4A 784 n. 51). In the end, however, he prefers a calculus in which coincidence is reflexive for all terms, true and false alike (§§154–5, see Schupp 1993: 161–2, 204).

⁴⁶ Idempotence is also stated in §18, §24, §129, §156, §171.3, and §189.1; cf. also VI4A 289, 293, 808, 811, 834, C 235, 421. As Leibniz points out, idempotence marks one crucial respect in which the algebra of terms differs from the ordinary algebra of numbers, when composition is understood as multiplication (§129, VI4A 512 and 811). A similar observation was made by Boole (1854: 37; cf. Peckhaus 1997: 222–6).

⁴⁷ The reflexivity of containment is stated by Leibniz in §37 and §43; cf. also VI4A 120, 142–3, 147–8, 150, 274–5, 281, 292, 804. Comparison with the references listed in n. 46 reveals a gradual shift in Leibniz's thought from an emphasis in his earlier logical writings on the relation of containment to a later emphasis on the relation of coincidence.

can be omitted since ‘in forming composite terms it does not matter in which order the simple terms are collected’.⁴⁸ While the principles listed in §§198–200 do not justify the arbitrary permutation of composite terms, such a justification can be easily supplied by positing the commutativity of composition:

$$AB = BA.$$

This principle is stated by Leibniz in §147.⁴⁹ It suffices to justify the permutability of composite terms since composition, in the language of Leibniz’s calculus, is expressed by mere concatenation without the use of parentheses or other delimiters to indicate the order in which terms are composed.⁵⁰ Were such delimiters to be included in the language, then in order to derive permutability from commutativity a further principle of associativity for composition would be required. As it stands, however, there is no need for Leibniz to posit such a principle.⁵¹

One noteworthy consequence of commutativity is the antisymmetry of containment (with respect to coincidence):⁵²

$$\text{If } A \supset B \text{ and } B \supset A, \text{ then } A = B.$$

Since the two conjuncts of the antecedent amount to $A = AB$ and $B = BA$, it follows by commutativity that $A = B$.⁵³

Taken together, the principles of idempotence and commutativity entail that in a composite term neither the repetition nor the order of terms matters.⁵⁴ These principles therefore amount to the claim that, up to coincidence, a composite term can be identified with the set—as opposed to the sequence—of its noncomposite constituents.

2.3. Principles of Privation. The third principle stated by Leibniz in §198 asserts that a term coincides with the privative of its privative:⁵⁵

⁴⁸ VI4A 742 n. 4 (not printed in Parkinson 1966).

⁴⁹ Leibniz asserts that ‘ AB is the same as BA ’ (§147; see Schupp 1993: 150 and 161). Leibniz seems to rely on commutativity in §199, when he asserts that $\mathbf{F}(AB)$ just in case $\mathbf{F}(BA)$ (see Castañeda 1990: 21 and 24). Other possible appeals to commutativity occur in §19 and §30. For further statements of commutativity, see VI4A 280 n. 2, 289, 293, 834, C 235, 421.

⁵⁰ This implies that the terms of Leibniz’s calculus are not uniquely decomposable. Thus, for example, there are at least two distinct ways of constructing the term ABC : either by composing AB with C or by composing A with BC . Alternatively, ABC may be viewed as the result of a single application of ternary or variably polyadic composition applied to the terms A , B , and C (see Castañeda 1990: 21, Swoyer 1994: 20–1).

⁵¹ Associativity of composition is not mentioned anywhere in Leibniz’s writings; see Castañeda 1976: 488, Lenzen 1984a: 201, Peckhaus 1997: 48, Hailperin 2004: 327.

⁵² Leibniz asserts antisymmetry in §30 and he appeals to it in §88. Antisymmetry is also stated in VI4A 154, 275, 284, 285, 294, 552, 813.

⁵³ In some passages in the *Generales Inquisitiones*, Leibniz speaks as if coincidence is to be defined in terms of containment rather than the other way around (e.g., §195). One could adopt this approach by treating containment as primitive and defining $A = B$ as $\mathbf{T}(\ulcorner A \supset B \urcorner \ulcorner B \supset A \urcorner)$. While this approach is feasible, we will not pursue it in this paper but will instead follow the primary strategy adopted by Leibniz in the *Generales Inquisitiones*, according to which coincidence is treated as the only primitive relation between terms (see n. 35).

⁵⁴ See VI4A 293, 834, and 857–8; cf. Castañeda 1990: 21.

⁵⁵ This principle is also stated in VI4A 740, §2, §96, §171.4, §189.2, VI4A 146–7, 218, 624, 807, 811, 814, 877, 931, 935, 939, C 230, 235, 421.

3rd. *non-non-A = A*. (§198.3)

In addition to this principle of double privation, Leibniz endorses a second principle pertaining to privation, which he describes as follows:

If *B* is a proposition, *non-B* is the same as *B is false*, or, *B's being false*.⁵⁶
(§32a)

In this passage, Leibniz describes the effect of applying privation to propositional terms.⁵⁷ Specifically, if *B* is a propositional term, *non-B* coincides with the propositional term generated from the proposition *B is false*.⁵⁸ Since every propositional term is of the form $\ulcorner A = B \urcorner$, this can be stated in full generality as follows:

$$\overline{\ulcorner A = B \urcorner} = \ulcorner \mathbf{F}(\ulcorner A = B \urcorner) \urcorner.$$

We refer to this as the principle of propositional privation. It should be emphasized that this principle holds only for propositional terms and not for all terms in general, since it is not the case that \overline{A} coincides with $\ulcorner \mathbf{F}(A) \urcorner$ for any term *A*.⁵⁹

In §198, the principle of propositional privation is given a more colloquial expression in the form of Leibniz's remark that the addition of falsehood 'changes things into their opposite':

6th. The addition of truth or of being leaves things unchanged, but the addition of falsehood or nonbeing changes things into their opposite.
(§198.6)

In light of Leibniz's previous statement of propositional privation, the 'things' referred to in this passage are most likely propositional terms.⁶⁰ Moreover, by the 'opposite' of a term Leibniz means its privative.⁶¹ Hence, the second clause of the passage asserts that, if falsehood is applied to a propositional term, the result is the privative of that term—which is just what the principle of propositional privation states.

⁵⁶ 'Si *B* sit propositio, *non-B* idem est quod *B est falsum* seu $\tau\delta$ *B esse falsum*' (VI4A 753 n. 18; similarly, VI4A 809).

⁵⁷ The fact that Leibniz applies the operation of privation to *B* and that he takes *B* to be a constituent of a proposition (viz., $\mathbf{F}(B)$) indicates that *B* is a term. The further fact that he characterizes *B* as a proposition makes it clear that *B* is, more specifically, a propositional term (see n. 34).

⁵⁸ Leibniz signifies this latter term by the phrase ' $\tau\delta$ *B esse falsum*' (VI4A 753 n. 18). This is in accordance with his customary convention of signifying propositional terms by means of infinitive phrases prefixed with the Greek definite article (see n. 25).

⁵⁹ With regard to the principle of propositional privation Leibniz writes that 'this is not so in the case of incomplex terms' (VI4A 753 n. 18), thus indicating that the principle does not hold for non-propositional terms (see n. 26). For example, it is not the case that *non-animal* coincides with the propositional term $\ulcorner animal\ is\ false \urcorner$. Intuitively, this is because both *animal* and *non-animal* are true terms, i.e., neither of them contains a contradiction. Since *animal* is true, the propositional term $\ulcorner animal\ is\ false \urcorner$ is itself false (by the definition of truth given in Section 1.6). But then this propositional term cannot coincide with *non-animal*, since otherwise there would be a term that is both true and false (see n. 101).

⁶⁰ It is clear from §198.4–5 that the 'things' referred to in this passage are terms; but for the reasons discussed in n. 59, it is unlikely that Leibniz intends this claim to apply to non-propositional terms (*contra* Castañeda 1990: 23; see n. 119).

⁶¹ In §194, Leibniz refers to *A* and *non-A* as 'opposites' (*oppositos*); cf. §190 and §196.

In a similar fashion, Leibniz's claim that the addition of truth 'leaves things unchanged' can be formulated as follows:⁶²

$$\ulcorner A = B \urcorner = \ulcorner \mathbf{T}(\ulcorner A = B \urcorner) \urcorner.$$

This latter claim is a consequence of the principles of double privation and propositional privation. For, given the definition of truth introduced above, $\ulcorner \mathbf{T}(\ulcorner A = B \urcorner) \urcorner$ is the term:

$$\ulcorner \mathbf{F}(\ulcorner \mathbf{F}(\ulcorner A = B \urcorner) \urcorner) \urcorner.$$

By two applications of propositional privation, this coincides with $\overline{\overline{\ulcorner A = B \urcorner}}$, which by double privation coincides with $\ulcorner A = B \urcorner$.

Having stated propositional privation in §198.6, Leibniz proceeds to identify four consequences of this principle describing the interaction of truth and falsehood:

So if it is said to be true that something is true or false, it remains true or false; but if it is said to be false that it is true or false, it becomes false from being true and true from being false. (§198.6)

The four claims in this passage can be formulated as follows:⁶³

$$\ulcorner \mathbf{T}(\ulcorner \mathbf{T}(A) \urcorner) \urcorner = \ulcorner \mathbf{T}(A) \urcorner$$

$$\ulcorner \mathbf{T}(\ulcorner \mathbf{F}(A) \urcorner) \urcorner = \ulcorner \mathbf{F}(A) \urcorner$$

$$\ulcorner \mathbf{F}(\ulcorner \mathbf{T}(A) \urcorner) \urcorner = \ulcorner \mathbf{F}(A) \urcorner$$

$$\ulcorner \mathbf{F}(\ulcorner \mathbf{F}(A) \urcorner) \urcorner = \ulcorner \mathbf{T}(A) \urcorner.$$

Given the definition of truth in terms of falsehood, the last of these claims follows from the reflexivity of coincidence. Moreover, given this definition, the first three claims are straightforward instances of the law just established, that $\ulcorner \mathbf{T}(\ulcorner A = B \urcorner) \urcorner$ coincides with $\ulcorner A = B \urcorner$. Thus, all the laws of truth and falsehood stated by Leibniz in §198.6 follow from his two principles of privation: double privation and propositional privation.

2.4. The Principle of Propositional Containment. In addition to propositional privation, Leibniz endorses a second principle pertaining to propositional terms. This principle, which characterizes the relationship between propositional terms and the propositions from which they are generated, is stated in §198 as follows:

8th. For a proposition to follow from a proposition is nothing other than for the consequent to be contained in the antecedent as a term in a term. By this method we reduce consequences to propositions, and propositions to terms. (§198.8)

According to this passage, one proposition follows from another just in case the propositional term corresponding to the former is contained in the propositional term corresponding to the latter.⁶⁴ When Leibniz says that one proposition follows (*sequitur*)

⁶² In §1, Leibniz formulates this claim as follows: 'These coincide: the (direct) statement L and the (reflexive) statement: L is true.' In order to represent L is true as a proposition of Leibniz's calculus, the letter ' L ' must be taken to stand for a propositional term (see n. 34; cf. Schupp 1993: 172–3).

⁶³ These four claims are also stated in §1 for the special case where A is a propositional term.

⁶⁴ Similarly, Leibniz writes that 'whatever is said of a term which contains a term can also be said of a proposition from which another proposition follows' (§189.6). See also §138, VI4A 809, 811 n. 6, 863; cf. Swoyer 1995: 110.

from another, he means that the latter is derivable from the former in his calculus.⁶⁵ So, if the symbol '⊢' is used to denote derivability in Leibniz's calculus, the above passage asserts that:

$$A = B \vdash C = D \quad \text{if and only if} \quad \vdash \ulcorner A = B \urcorner \supset \ulcorner C = D \urcorner.$$

We refer to this as the principle of propositional containment. In Leibniz's calculus, this principle plays a role analogous to that of a deduction theorem in modern proof systems: it allows facts concerning the inferential relations between propositions in the calculus to be expressed by propositions in the language of the calculus itself.⁶⁶ Thus, by means of this principle we can, as Leibniz puts it, 'reduce consequences to propositions'.

In the left-to-right direction, the principle of propositional containment underwrites a method of hypothetical reasoning by means of which a containment claim between propositional terms can be established by supplying a suitable subproof. A convenient notation for representing such subproofs is that employed in modern Fitch-style natural deduction systems. In this notation, a simple example of this sort of hypothetical reasoning is:

$$\begin{array}{l|l} 1 & \begin{array}{l} A = B \\ \hline B = A \end{array} \\ 2 & \\ 3 & \ulcorner A = B \urcorner \supset \ulcorner B = A \urcorner \end{array}$$

Line 1 of this proof initiates a subproof by assuming $A = B$ as a hypothesis. In line 2, $B = A$ is inferred from this hypothesis by the symmetry of coincidence. The initial hypothesis is then discharged and, in line 3, the proposition $\ulcorner A = B \urcorner \supset \ulcorner B = A \urcorner$ is inferred from the existence of the subproof in lines 1–2 by the left-to-right direction of propositional containment.

More complex applications of this kind of hypothetical reasoning arise when the subproof appeals to propositions occurring outside of it. For example:

$$\begin{array}{l|l} 1 & \begin{array}{l} A = B \\ \hline \begin{array}{l} B = C \\ \hline A = C \end{array} \end{array} \\ 2 & \\ 3 & \\ 4 & \ulcorner B = C \urcorner \supset \ulcorner A = C \urcorner \end{array}$$

⁶⁵ To justify the claim that one proposition follows (*sequitur*) from another, Leibniz often provides a derivation of the former from the latter in his calculus (e.g., §49, §§52–4, §100). Moreover, there is no indication that Leibniz entertained the possibility that a proposition might follow from another despite not being derivable from it in his calculus. On the contrary, he asserts the completeness of his calculus in §189.7.

⁶⁶ Since $A = B$ entails both $A \supset B$ and $B \supset A$ (see Theorems 4.15 and 4.16, Section 4.2), it follows that coincidence between propositional terms expresses mutual derivability of the corresponding propositions in the calculus. This is in accordance with Leibniz's claim that 'statements coincide if one can be substituted for the other *salva veritate*, or if they reciprocally entail one another' (VI4A 748; similarly, VI4A 810 n. 3). One consequence of this is that coincident terms can be freely substituted for one another even within propositional terms (this can be verified by a simple induction). Thus, given the principle of propositional containment, our current unrestricted version of the principle of substitution could be replaced without loss by a more restricted version which does not license free substitution into propositional terms (cf. n. 41).

This proof purports to establish that $\lceil B = C \rceil \supset \lceil A = C \rceil$ is derivable in Leibniz's calculus from $A = B$. The latter proposition is posited as a premise in line 1. Line 2 initiates a subproof, the conclusion of which is stated in line 3. Crucially, this conclusion is derived not only from the hypothesis in line 2, but also from the premise in line 1 (by the transitivity of coincidence). In line 4, the desired containment proposition is inferred from the existence of the subproof in lines 2–3.

The proof thus relies on the following assumption:

$$\text{If } A = B, B = C \vdash A = C, \text{ then } A = B \vdash \lceil B = C \rceil \supset \lceil A = C \rceil.$$

Note that this conditional is not justified by the principle of propositional containment. Instead, it relies upon a generalization of this principle which allows for hypothetical reasoning to appeal to superordinate premises appearing outside of the subproof. If Γ is a set of propositions, this generalized version of the principle states that:

$$\Gamma \cup \{A = B\} \vdash C = D \quad \text{if and only if} \quad \Gamma \vdash \lceil A = B \rceil \supset \lceil C = D \rceil.$$

As it turns out, there is no need to posit this stronger version of propositional containment as a principle of Leibniz's calculus. This is because, given the other principles of the calculus, the stronger version of propositional containment can be shown to follow from its weaker counterpart stated above, in which Γ is taken to be empty.⁶⁷ Consequently, the more complex mode of hypothetical reasoning illustrated above is admissible in Leibniz's calculus.

2.5. Leibniz's Principle. At this point, we have considered all of the principles listed in §198.⁶⁸ There is, however, one remaining principle, presented by Leibniz in §§199–200, that plays a crucial role in his calculus. This final principle arises in the context of Leibniz's analysis of syllogistic reasoning and relates, specifically, to the formalization of the universal negative proposition *No A is B*.

In the *Generales Inquisitiones*, Leibniz proposes two distinct ways of rendering universal negative propositions in the language of his calculus:

No A is B is the same as *A is non-B*. (§87)

No A is B, i.e., *AB is not*, or *AB is not a thing*. (§149)

According to the first of these two passages, the universal negative can be expressed as $A \supset \overline{B}$.⁶⁹ According to the second, it can be expressed as $\mathbf{F}(AB)$.⁷⁰

Both of these formulations have their advantages and disadvantages when it comes to modeling syllogistic reasoning. Any proposed formulation of the universal negative must satisfy two main desiderata. First, it must validate the conversion rule licensing the inference from *No A is B* to *No B is A*; second, it must validate the syllogism

⁶⁷ See Theorem 4.53 (Section 4.4).

⁶⁸ We have so far discussed the principles of substitution (§198.1), idempotence (§198.2), double privation (§198.3), propositional privation (§198.6), and propositional containment (§198.8). The remaining items listed in §198 describe the syntax of Leibniz's calculus, which we discussed in Sections 1.1–1.5.

⁶⁹ See also §§127–9, §190, VI4A 300.

⁷⁰ See also VI4A 774 n. 47, §151, §152, §169, §199.

Celarent:⁷¹

$$\begin{array}{c} \text{No } B \text{ is } A \\ \text{Every } C \text{ is } B \\ \hline \text{No } C \text{ is } A \end{array}$$

As Leibniz points out, the first of these desiderata is clearly satisfied by his second formulation of the universal negative, since $\mathbf{F}(BA)$ follows from $\mathbf{F}(AB)$ by commutativity.⁷² By contrast, it is unclear whether this desideratum is satisfied by Leibniz's first formulation of the universal negative. For, under this formulation, the conversion rule for the universal negative amounts to the claim that $B \supset \bar{A}$ is derivable from $A \supset \bar{B}$, and this law of contraposition does not follow from the principles that we have stated so far.

With respect to the second desideratum the situation is reversed. Under the first formulation of the universal negative, Celarent amounts to the following inference:⁷³

$$\begin{array}{c} B \supset \bar{A} \\ C \supset B \\ \hline C \supset \bar{A} \end{array}$$

This is valid given the transitivity of containment.⁷⁴ Under the second formulation of the universal negative, however, Celarent is:

$$\begin{array}{c} \mathbf{F}(BA) \\ C \supset B \\ \hline \mathbf{F}(CA) \end{array}$$

Again, the conclusion of this argument does not follow from its premises by the principles of Leibniz's calculus that we have stated so far.

Thus, each formulation of the universal negative proposed by Leibniz satisfies one of the two desiderata, but neither satisfies both. To overcome this difficulty, Leibniz posits an additional principle asserting the equivalence of these two formulations:

If I say *AB is not*, it is the same as if I were to say *A contains non-B*.
(§200)

This equivalence can be expressed by the following bi-directional rule of inference:⁷⁵

$$\mathbf{F}(AB) \dashv\vdash A \supset \bar{B}.$$

Owing to the vital role that this principle plays in Leibniz's calculus, we will refer to it as Leibniz's Principle.

⁷¹ Leibniz states these two desiderata in C 302–3, 306.

⁷² See §199 and VI4A 774 n. 47.

⁷³ The universal affirmative minor premise of Celarent, *Every C is B*, is expressed by the proposition $C \supset B$ (see n. 10).

⁷⁴ The transitivity of containment is a direct consequence of the principle of substitution (see Theorem 4.18, Section 4.2).

⁷⁵ For similar statements of this equivalence, see VI4A 862–3 and C 237. We write ' $\dashv\vdash$ ' to indicate mutual derivability in Leibniz's calculus.

Given Leibniz's Principle, the law of contraposition for containment can now be established as follows:

1	$A \supset \bar{B}$	
2	$\mathbf{F}(AB)$	from 1 by Leibniz's Principle
3	$AB = BA$	Commutativity
4	$\mathbf{F}(BA)$	from 2, 3 by Substitution
5	$B \supset \bar{A}$	from 4 by Leibniz's Principle

This proof of contraposition utilizes the right-to-left direction of Leibniz's Principle to transform $A \supset \bar{B}$ into the convertible formulation of the universal negative, $\mathbf{F}(AB)$. A and B are then commuted and the result is transformed back, by the left-to-right direction of Leibniz's Principle, into $B \supset \bar{A}$.⁷⁶ By an analogous maneuver, Leibniz's Principle can be used to validate Celarent in its less obvious formulation.

Leibniz's Principle thus makes possible a single formulation of universal negative propositions that satisfies the two desiderata imposed by the theory of the syllogism. Even apart from this, as we shall see, Leibniz's Principle has far-reaching consequences that extend well beyond its role in formalizing syllogistic reasoning.⁷⁷

2.6. Summary. The principles of Leibniz's calculus comprise the following axiom schemata and rules of inference:

AXIOM SCHEMATA

Idempotence:	$AA = A$
Commutativity:	$AB = BA$
Double Privation:	$\bar{\bar{A}} = A$
Propositional Privation:	$\overline{\lceil A = B \rceil} = \lceil \mathbf{F}(\lceil A = B \rceil) \rceil$

RULES OF INFERENCE

Substitution of Coincidents:	$A = B, C = D \vdash C^* = D^*$, where $C^* = D^*$ is the result of substituting B for an occurrence of A , or vice versa, in $C = D$.
Propositional Containment:	$A = B \vdash C = D$ iff $\vdash \lceil A = B \rceil \supset \lceil C = D \rceil$
Leibniz's Principle:	$A \supset \bar{B} \dashv\vdash \mathbf{F}(AB)$

⁷⁶ In the earlier portions of the *Generales Inquisitiones*, Leibniz struggles to provide a proof of contraposition (see §77, §§93–5, §99; cf. also C 236–7). His informal argument for contraposition in §99 appeals to the conversion of universal negative propositions; however, prior to the introduction of Leibniz's Principle, there is no way to prove in the calculus that $A \supset \bar{B}$ is equivalent to a proposition convertible in A and B .

⁷⁷ One important consequence of Leibniz's Principle is that $\mathbf{F}(A)$ is provably equivalent to $A \supset \bar{A}$ independently of how falsehood is defined in the calculus (see Section 1.4). As a result, our definition of falsehood by means of this equivalence is conservative in that it does not allow us to establish any new theorems (i.e., theorems that could not be established if falsehood were treated as an undefined primitive in the calculus).

Each of these seven principles is explicitly stated by Leibniz in the *Generales Inquisitiones*. With the sole exception of commutativity, they are all included in his final list of axioms presented in §§198–200.

In referring to the calculus constituted by these principles as ‘Leibniz’s calculus’, we do not mean to imply that Leibniz stated—or was even aware of—all of its significant logical features. Indeed, as we shall see, several theorems of the calculus that are helpful in explicating its semantics do not appear anywhere in Leibniz’s writings. Nevertheless, it is clear that Leibniz regarded this calculus as the culminating achievement of the *Generales Inquisitiones*, remarking in the concluding sentence of the treatise that ‘in these few principles are contained the fundamentals of logical form’ (§200). Accordingly, if we wish to arrive at a more complete understanding of the *Generales Inquisitiones*, it is important to examine Leibniz’s calculus in detail and identify its most significant logical features.⁷⁸

§3. The Semantics of Leibniz’s Calculus. In Leibniz’s view, the principles of his calculus describe a ‘universal algebra’ (*algebra universalis*) of terms.⁷⁹ In what follows, we develop a semantics for Leibniz’s calculus by examining the type of algebraic structure characterized by its principles. We will do so in three distinct stages, focusing at each stage on an increasingly large fragment of the calculus. As we shall see, the calculus as a whole has a very natural algebraic interpretation. In particular, Leibniz’s calculus is both sound and complete with respect to a simple class of enriched Boolean algebras which we call auto-Boolean algebras (and which, in a different signature, are known as simple monadic algebras).

3.1. The Semilattice of Terms. As a first step in the construction of an algebraic semantics for Leibniz’s calculus, we consider the core fragment of the calculus consisting of the principles of idempotence, commutativity, and substitution. The semantic significance of these principles is best understood by considering the constraints they impose on the containment relation.

As we have seen, given Leibniz’s definition of containment in terms of coincidence, these three principles imply that containment is both reflexive and antisymmetric.⁸⁰ Moreover, the principle of substitution implies that containment is transitive:

$$A \supset B, B \supset C \vdash A \supset C.$$

The proof is as follows:⁸¹

1	$A = AB$	
2	$B = BC$	
	$A = ABC$	from 1, 2 by Substitution
4	$A = AC$	from 1, 3 by Substitution

⁷⁸ In deriving theorems of Leibniz’s calculus, we will assume that the derivability relation \vdash satisfies the three usual structural laws of reflexivity, monotonicity, and cut (see Definition 4.3, Section 4.1). There is strong evidence that Leibniz endorses both reflexivity and monotonicity (VI4A 149). The rule of cut, which allows for the construction of complex derivations, is not explicitly stated by Leibniz but is clearly presupposed throughout the *Generales Inquisitiones*.

⁷⁹ For Leibniz’s conception of a ‘universal algebra’, see his letter to Lange (5 June 1716), published in Dutens 1768: 405; cf. Couturat 1901: 319–20.

⁸⁰ See Section 2.2.

⁸¹ This proof of transitivity is given by Leibniz in C 229–30; cf. Couturat 1901: 347.

Thus, idempotence, commutativity, and substitution imply that containment is a partial order on the set of terms.

A further consequence of these three principles is that the composite term AB is a lower bound of both A and B :⁸²

$$\vdash AB \supset A$$

$$\vdash AB \supset B.$$

In addition, it follows from the principle of substitution that AB is an upper bound of any lower bound of both A and B :⁸³

$$C \supset A, C \supset B \vdash C \supset AB.$$

Hence, AB is the greatest lower bound of A and B .⁸⁴

In sum, the terms of Leibniz's calculus, partially ordered by containment, constitute a semilattice (i.e., a partially ordered set in which any pair of elements has a greatest lower bound). It can, moreover, be shown that every inference valid in the class of all semilattices can be proven by means of the principles of idempotence, commutativity, and substitution, and conversely that each of these principles is valid in any semilattice. In other words, the core fragment of Leibniz's calculus is both sound and complete with respect to the class of semilattices.⁸⁵

3.2. The Boolean Algebra of Terms. As a second step in the construction of an algebraic semantics for Leibniz's calculus, we extend the core fragment of the calculus by adding to it the principle of double privation and Leibniz's Principle. Since this extended fragment excludes only those principles that pertain specifically to propositional terms (i.e., propositional privation and propositional containment), we refer to it as the non-propositional fragment of Leibniz's calculus.

As we have seen, one of the consequences of Leibniz's Principle is the following law of contraposition:⁸⁶

$$A \supset \bar{B} \vdash B \supset \bar{A}.$$

This law, in conjunction with double privation, entails that the set of terms is not only a semilattice but a lattice (i.e., a semilattice in which any two elements have a least upper bound).⁸⁷

⁸² See Theorems 4.20 and 4.21 (Section 4.2). In calling AB a lower (rather than an upper) bound, we adopt the convention that $A \supset B$ means that A is beneath (rather than above) B in the ordering defined by containment.

⁸³ Theorem 4.22 (Section 4.2).

⁸⁴ Our use of the definite article in descriptions such as '*the* greatest lower bound' indicates that the terms satisfying the description are unique up to coincidence (which, in this case, follows by antisymmetry).

⁸⁵ See Theorem 4.24 (Section 4.2). This completeness result bears a close correspondence to the well-known characterization of semilattices as commutative idempotent semigroups. Swoyer (1994: 14–20 and 29–30) establishes a similar result, albeit for a language which includes a primitive means for expressing the noncoincidence of terms.

⁸⁶ See Section 2.5; cf. Theorem 4.26 (Section 4.3).

⁸⁷ Given that $\bar{A}\bar{B}$ is the greatest lower bound of \bar{A} and \bar{B} (Section 3.1), it follows by contraposition and double privation that:

A further consequence of Leibniz's Principle is the law of explosion, according to which a false term contains any term:⁸⁸

$$\mathbf{F}(A) \vdash A \supset B.$$

By contraposition, it follows that the privative of a false term is contained in any term. Thus, any false term is a lower bound of the entire lattice, and the privative of any false term is an upper bound. Moreover, under the assumption that there is at least one term (which we shall henceforth take for granted), Leibniz's Principle implies that there is at least one false term, namely, the composite of this term and its privative. This is because, for any term A , Leibniz's Principle entails that $\mathbf{F}(A\bar{A})$.⁸⁹ It follows that the lattice of terms has both a least and a greatest element (i.e., an element which contains every term and an element which is contained in every term). Since, by antisymmetry, these elements are unique up to coincidence, we will refer to them as 0 and 1, respectively.⁹⁰

Furthermore, it can easily be shown that the privative of a term is its complement in the lattice (i.e., that the greatest lower bound of a term and its privative is 0, and their least upper bound is 1).⁹¹ Thus, since the set of terms is closed under privation, it follows that this set is a complemented lattice (i.e., a lattice in which every element has a complement). Now, a complemented lattice is a Boolean algebra just in case it satisfies the law of distributivity. As it turns out, this law is in fact derivable in the non-propositional fragment of Leibniz's calculus, although the proof is somewhat lengthy and involved.⁹²

All told, then, the terms of Leibniz's calculus constitute a Boolean algebra. It can, moreover, be shown that every inference valid in the class of all Boolean algebras can be proven by means of the principles of idempotence, commutativity, double privation, substitution, and Leibniz's Principle. Since, conversely, each of these principles is valid in any Boolean algebra, it follows that the non-propositional fragment of Leibniz's calculus is both sound and complete with respect to the class of Boolean algebras.⁹³

$$\begin{aligned} & \vdash A \supset \overline{\overline{A} \overline{B}} \\ & \vdash B \supset \overline{\overline{A} \overline{B}} \\ & A \supset C, B \supset C \vdash \overline{\overline{A} \overline{B}} \supset C. \end{aligned}$$

Hence, $\overline{\overline{A} \overline{B}}$ is the least upper bound of A and B . It should be emphasized that none of these theorems appear in the *Generales Inquisitiones*. Indeed, Leibniz does not make any substantial use of this definition of the least upper bound, or join, in his logical writings (see Mugnai 2005: 170). We cite the above three theorems not in order to attribute them to Leibniz, but merely to facilitate a grasp of the semantic significance of Leibniz's Principle.

⁸⁸ Theorem 4.31 (Section 4.3). This law is not stated by Leibniz, but is a straightforward consequence of claims that he explicitly endorses in the *Generales Inquisitiones* (see Lenzen 1983: 135 n. 17a, 1984a: 195–6, 2004: 16).

⁸⁹ Theorem 4.29 (Section 4.3).

⁹⁰ We leave open the possibility that 0 coincides with 1.

⁹¹ In other words, the non-propositional fragment of Leibniz's calculus entails that $A\bar{A} = 0$ and $\overline{\overline{A} \overline{A}} = 1$ (see n. 87).

⁹² For a proof of distributivity, see Theorem 4.36 (Section 4.3).

⁹³ See Theorem 4.35 (Section 4.3). Lenzen (1984a: 200, 2004: 10–16) establishes a similar result showing that Leibniz's calculus is complete with respect to the class of Boolean algebras. However, Lenzen's reconstruction of Leibniz's calculus differs in important respects from our own. For one, the axioms chosen by Lenzen are not based on any of the systems of principles presented by Leibniz toward the end of the *Generales Inquisitiones*. In addition, Lenzen regards Leibniz's

Despite the Boolean completeness of his calculus, Leibniz does not explicitly state many of the laws that figure most prominently in modern axiomatizations of Boolean algebra (e.g., distributivity, complementation, and absorption). This is to be expected since Leibniz does not adopt the syntactic signature typically employed by these latter axiomatizations. In the language of Leibniz's calculus, the only primitive Boolean operations are composition (meet) and privation (complementation). By contrast, most contemporary axiomatizations posit additional redundant primitives such as the join operation and the constants 0 and 1. Many laws that have an apparent significance when formulated in this enriched signature appear cryptic and unmotivated when expressed in Leibniz's more parsimonious syntax, and are thus of little interest as either axioms or theorems of his calculus. Accordingly, the non-propositional fragment of Leibniz's calculus bears almost no resemblance to most contemporary axiomatizations of Boolean algebra. Instead, it closely resembles axiomatizations that adopt a more parsimonious signature such as those put forward by Huntington 1904, Lewis 1918, and Byrne 1946.⁹⁴ When compared with these systems, the non-propositional fragment of Leibniz's calculus constitutes an economical and remarkably elegant axiomatization of the theory of Boolean algebras.

3.3. The Auto-Boolean Algebra of Terms. While the non-propositional fragment of Leibniz's calculus entails that the set of terms is a Boolean algebra, it does not specify the place of propositional terms in the algebra. For this, we need to take into account all of the principles of Leibniz's calculus, including propositional privation and propositional containment.

We already know from explosion that if a propositional term is false, it is the least element of the algebra:

$$\mathbf{F}(\ulcorner A = B \urcorner) \vdash \ulcorner A = B \urcorner \supset C.$$

Correspondingly, by propositional privation, we can infer that if a propositional term is true, it is the greatest element of the algebra:⁹⁵

$$\mathbf{T}(\ulcorner A = B \urcorner) \vdash C \supset \ulcorner A = B \urcorner.$$

Given the definition of truth in terms of falsehood, the proof of this last theorem proceeds as follows:

calculus as a theory formulated within propositional logic, and is thus free to avail himself of propositional modes of reasoning in deriving the axioms of Boolean algebra (Lenzen 1983: 131–6, 1984a: 193 and 201–2, 2004: 4 and 10). For further discussion of this last point, see n. 114.

⁹⁴ Huntington 1904: 297–305, Lewis 1918: 118–32, Byrne 1946: 269–70. Each of these systems includes among its axioms a close variant of Leibniz's Principle (Huntington's postulate 9 on p. 297, Lewis's postulate 1.61 on p. 119, and Byrne's axiom I on p. 270).

⁹⁵ This implies that every term contains every true propositional term. For example, the term *animal* contains the propositional term $\ulcorner 2+2 = 4 \urcorner$. Leibniz does not explicitly discuss such seemingly odd containment claims relating non-propositional and propositional terms, and so it is an open question whether or not he would find such claims problematic. If we wish to prevent such claims from appearing among the theorems of Leibniz's calculus, we can do so by restricting the language of the calculus so as to exclude any mixed propositions relating propositional and non-propositional terms (see nn. 29 and 143).

1	$\mathbf{F}(\ulcorner \mathbf{F}(\ulcorner A = B \urcorner) \urcorner)$	
2	$\ulcorner A = B \urcorner = \ulcorner \mathbf{F}(\ulcorner A = B \urcorner) \urcorner$	Propositional Privation
3	$\mathbf{F}(\ulcorner A = B \urcorner)$	from 1, 2 by Substitution
4	$\ulcorner A = B \urcorner \supset \overline{C}$	from 3 by Explosion
5	$C \supset \ulcorner A = B \urcorner$	from 4 by Contraposition

In sum, then, every false propositional term coincides with 0 and every true propositional term coincides with 1.⁹⁶

The principles of Leibniz's calculus thus suffice to determine the place of a given propositional term in the Boolean algebra provided that this term is either true or false. It has not been established, however, that every propositional term is either true or false. This turns out to be a substantive issue. On the one hand, it is a theorem of Leibniz's calculus that a propositional term $\ulcorner A = B \urcorner$ is true just in case A coincides with B .⁹⁷

$$A = B \dashv\vdash \mathbf{T}(\ulcorner A = B \urcorner).$$

There is, on the other hand, no corresponding theorem stating that $\ulcorner A = B \urcorner$ is false just in case A and B do not coincide. This is because the language of Leibniz's calculus does not have an independent means of expressing the noncoincidence of two terms apart from the proposition stating the falsehood of their coincidence, e.g., $\mathbf{F}(\ulcorner A = B \urcorner)$.⁹⁸

It is thus an open question whether the proposition $\mathbf{F}(\ulcorner A = B \urcorner)$ holds whenever A and B do not coincide. As it turns out, the answer to this question is not decided by the principles of Leibniz's calculus. These principles are compatible with there being a propositional term $\ulcorner A = B \urcorner$ that is not false even though A does not coincide with B . Since in this case $\ulcorner A = B \urcorner$ is also not true, this implies that the principles of Leibniz's calculus are compatible with there being propositional terms that are neither true nor false.⁹⁹ In this sense, the law of bivalence for propositional terms is not a consequence of the principles of Leibniz's calculus.

Nevertheless, it is clear that Leibniz subscribes to the law of bivalence in the *Generales Inquisitiones*:¹⁰⁰

Every proposition is either true or false. In other words, if L is not true, it is false; if it is true, it is not false; if it is not false, it is true; if it is false, it is not true. (§4)

⁹⁶ It follows that all true propositional terms coincide. This is in accordance with Leibniz's claim that 'a true proposition is that which coincides with this proposition: AB is B ' (§40). By contrast, non-propositional true terms need not coincide, e.g., *animal* and *non-animal* (see n. 59).

⁹⁷ See Theorem 4.37, Section 4.4.

⁹⁸ Recall that in the language of Leibniz's calculus the proposition $A \neq B$ is merely shorthand for $\mathbf{F}(\ulcorner A = B \urcorner)$ (see Section 1.6). The main reason for not including in the language a primitive means of expressing noncoincidence is to realize Leibniz's aim of reducing all propositions to the form $A = B$ (see nn. 33 and 35). There is a similar rationale for not including in the language a primitive operation of propositional negation (*pace* Lenzen 1983: 131–6, 1984a: 193, 2004: 10, Swoyer 1994: 14).

⁹⁹ For a model of Leibniz's calculus in which there are propositional terms that are neither true nor false, see Definition 4.63 and Theorem 4.69 (Section 4.5).

¹⁰⁰ For similar formulations of the law of bivalence, see VI4A 670, 672, 804, G V 343, G VII 420. Cf. Mates 1986: 153–4, Swoyer 1994: 6, Cover & O'Leary-Hawthorne 1999: 191.

If, following Leibniz, we assume that the law of bivalence holds for propositional terms, then the place of all propositional terms in the Boolean algebra is uniquely determined by the principles of the calculus. Specifically, $\ulcorner A = B \urcorner$ coincides with 1 if A coincides with B , and it coincides with 0 if A does not coincide with B .

In addition to bivalence, the passage just quoted also asserts the law of noncontradiction, according to which no proposition is both true and false.¹⁰¹ This law is likewise not a consequence of the principles of Leibniz's calculus. These principles are compatible with there being propositional terms that are both true and false.¹⁰² Since every true propositional term coincides with 1 and every false propositional term with 0, the existence of such terms amounts to the claim that 0 coincides with 1. Thus, in endorsing the law of noncontradiction for propositional terms, Leibniz is, in effect, asserting that 0 does not coincide with 1 (i.e., that there are at least two noncoincident terms).

Assuming, then, that the laws of bivalence and noncontradiction hold for propositional terms, the principles of Leibniz's calculus entail that the algebra of terms satisfies the following three conditions:

1. If A coincides with B , then $\ulcorner A = B \urcorner$ coincides with 1.
2. If A does not coincide with B , then $\ulcorner A = B \urcorner$ coincides with 0.
3. It is not the case that 0 coincides with 1.

We refer to any Boolean algebra which satisfies these three conditions as an 'auto-Boolean' algebra.¹⁰³

In an auto-Boolean algebra, as in any Boolean algebra, a term is false just in case it coincides with 0. Moreover, it is a special feature of auto-Boolean algebras that a term is true just in case it does not coincide with 0.¹⁰⁴ Thus, in an auto-Boolean algebra, every

¹⁰¹ For similar formulations of the law of noncontradiction, see G V 343, G VII 355; cf. Mates 1986: 153–4, Rodriguez-Pereyra 2013. If the law of noncontradiction holds for propositional terms, it holds for all terms in general. For, suppose that $\mathbf{T}(A)$ is satisfied in a given model. This means that $\ulcorner \mathbf{F}(A) \urcorner$ is false, and so, by the law of noncontradiction for propositional terms, $\ulcorner \mathbf{F}(A) \urcorner$ is not true. But since, in any model of Leibniz's calculus, a proposition is satisfied iff the propositional term generated from this proposition is true, the proposition $\mathbf{F}(A)$ is not satisfied in the model.

¹⁰² See the discussion following Theorem 4.62 (Section 4.5).

¹⁰³ More precisely, an auto-Boolean algebra is a structure $\langle \mathbb{S}, \wedge, \prime, \circ \rangle$, where $\langle \mathbb{S}, \wedge, \prime \rangle$ is a Boolean algebra in which $0 \neq 1$, and \circ is the binary operation on \mathbb{S} defined by:

$$x \circ y = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

Up to a change of signature, auto-Boolean algebras just are what are known as simple monadic algebras (cf. Halmos 1962: 40–8, Goldblatt 2006: 14). These are Boolean algebras equipped with an additional unary operation f defined by:

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise.} \end{cases}$$

An auto-Boolean algebra is a simple monadic algebra in which the operation f is defined by $f(x) = (x \circ 0) \circ 0$. Conversely, a simple monadic algebra is an auto-Boolean algebra in which the operation \circ is defined by $x \circ y = (f(x'y \vee y'x))'$. Auto-Boolean algebras can also be viewed as a special kind of cylindric algebra of dimension 1, in which the cylindrification is given by the function f . For an explanation of why we refer to these algebraic structures as 'auto-Boolean', see Section 3.4.

¹⁰⁴ This follows from the fact that $\mathbf{T}(A)$ is the proposition:

$$\ulcorner A = A\bar{A} \urcorner = \ulcorner A = A\bar{A} \urcorner \ulcorner A = A\bar{A} \urcorner.$$

term is either true or false but not both. In other words, all auto-Boolean algebras satisfy the laws of bivalence and noncontradiction for terms in general.

Since neither of these laws is entailed by the principles of Leibniz's calculus, it does not follow from these principles that the terms constitute an auto-Boolean algebra. In other words, there are 'nonstandard' models of the calculus in which the three auto-Boolean conditions stated above are not satisfied.¹⁰⁵ Despite this fact, it can be shown that Leibniz's calculus is both sound and complete with respect to the class of auto-Boolean algebras. In other words, a proposition is derivable in Leibniz's calculus from a set of premises just in case these premises imply the proposition in every auto-Boolean algebra.¹⁰⁶

This completeness result is possible, despite the existence of non-auto-Boolean models, because the language of Leibniz's calculus is not expressive enough to formulate the auto-Boolean conditions stated above.¹⁰⁷ In particular, the second and third auto-Boolean conditions cannot be formulated because the language possesses no general means of expressing the noncoincidence of terms. That is to say, there is no proposition of the language that holds in any given model of Leibniz's calculus just in case A does not coincide with B . If, however, we restrict ourselves to auto-Boolean models, noncoincidence becomes expressible by the proposition $\mathbf{F}(\ulcorner A = B \urcorner)$. The above soundness and completeness results show that, despite the existence of nonstandard models, we are free to interpret $\mathbf{F}(\ulcorner A = B \urcorner)$ as expressing the noncoincidence of A and B without thereby exposing ourselves to any risk of error or omission. Since, moreover, the only algebraic models of Leibniz's calculus that satisfy the laws of bivalence and noncontradiction are auto-Boolean, this is the most natural semantics for the calculus developed by Leibniz in the *Generales Inquisitiones*.

Leibniz's calculus can thus be viewed as a calculus for reasoning about terms insofar as they constitute an auto-Boolean algebra.¹⁰⁸ The distinctive feature of such an algebra is a binary operation which determines the place in the algebra of the propositional term

Since the right-hand side of this coincidence is 0, it follows by the auto-Boolean conditions that this proposition holds just in case A does not coincide with \overline{AA} , i.e., just in case A does not coincide with 0.

¹⁰⁵ See nn. 99 and 102.

¹⁰⁶ For the proof of soundness, see Theorem 4.62 (Section 4.5). For the proof of completeness, see Theorem 4.94 (Section 4.6).

¹⁰⁷ In this respect, the auto-Boolean completeness of Leibniz's calculus differs from the Boolean completeness of the non-propositional fragment of the calculus (see Section 3.2). In the latter case, there are no nonstandard models, i.e., models in which the terms do not constitute a Boolean algebra. This is because the language of Leibniz's calculus is rich enough to express the axioms of Boolean algebra but not those of auto-Boolean algebra.

¹⁰⁸ This algebraic semantics of Leibniz's calculus is abstract in the sense that it does not determine what the semantic value of a term is apart from specifying its place in an abstract algebra. As such, this semantics allows for a number of distinct concrete instantiations. For example, it is compatible with either an extensional approach, on which the semantic value of a term is taken to be the set of individuals that fall under the term, or an intensional approach, on which this value is taken to be the set of concepts that are, in some sense, conceptual parts of the term. While Leibniz tends to prefer the intensional over the extensional approach, he maintains that both approaches yield valid interpretations of his calculus (see §§122–3, VI4A 199–200, 247–8, 838–9). Accordingly, he intends his calculus to be an abstract calculus admitting of both extensional and intensional readings (see Swoyer 1995: 104–6, Bassler 1998: 132–6; cf. also Lenzen 1983). Since our main concern in this paper is to reconstruct Leibniz's calculus in its full generality, we will not enter into a discussion of the metaphysical commitments Leibniz would incur by endorsing the principles of his calculus on either of these readings.

$\lceil A = B \rceil$ as a function of the terms A and B . This operation, which maps any two coincident terms to 1 and any two noncoincident terms to 0, is not Boolean. That is to say, except in the degenerate case in which every term coincides with either 0 or 1, this operation cannot be defined by means of composition and privation alone.¹⁰⁹ Accordingly, the methods of reasoning licensed by the propositional principles of Leibniz's calculus constitute a genuine extension of Boolean reasoning. As we shall see in the next section, these additional methods of auto-Boolean reasoning allow us to derive all the laws of classical propositional logic in Leibniz's calculus.

3.4. Propositional Reasoning in Leibniz's Calculus. The only propositions in Leibniz's calculus are simple propositions of the form $A = B$. While the language of the calculus includes operations for forming compound terms, it does not include any special operations for forming compound propositions. Nevertheless, as we shall see, the calculus is capable of reproducing all valid inferences of classical propositional logic by utilizing Leibniz's device of propositional terms.

In an auto-Boolean algebra, every propositional term coincides with either 0 or 1. The same is also true for any Boolean compound formed from propositional terms by means of privation or composition. The place in the algebra of such Boolean compounds is given by the following tables, where φ and ψ are any propositions of the language of Leibniz's calculus:

$\lceil \varphi \rceil$	$\overline{\lceil \varphi \rceil}$	$\lceil \varphi \rceil$	$\lceil \psi \rceil$	$\lceil \varphi \rceil \lceil \psi \rceil$
1	0	1	1	1
0	1	1	0	0
		0	1	0
		0	0	0

Thus, in an auto-Boolean algebra, propositional terms and their Boolean compounds are strictly analogous to formulae of classical propositional logic: they can take one of two semantic values, and the semantic value of any Boolean compound is determined by the classical truth-tables for propositional negation and conjunction.¹¹⁰

To make this analogy more precise, it is helpful to consider a language of propositional logic whose atomic formulae are the propositions of Leibniz's calculus and whose only

¹⁰⁹ This can be most easily seen by representing the elements of the algebra as subsets of some universal set X . Under this representation, every binary Boolean operation $*$ is point-wise in the following sense: for any $x \in X$ and A, B in the algebra, whether or not $x \in A * B$ is determined by whether or not (i) $x \in A$ and (ii) $x \in B$. But if there are more than two elements in the algebra, the auto-Boolean operation \circ that maps A and B to 1 if $A = B$ and to 0 otherwise is not point-wise in this sense. For suppose $x \in A$ and $x \in B$. This is compatible with both $A = B$ and $A \neq B$, unless \emptyset and X are the only two elements of the algebra. Hence, the conjunction of $x \in A$ and $x \in B$ is compatible with both $x \in A \circ B$ and $x \notin A \circ B$.

¹¹⁰ The fact that propositional terms are analogous in this way to formulae of classical propositional logic is compatible with either an extensional or an intensional interpretation of Leibniz's calculus, as described in n. 108. On an extensional interpretation, the semantic value of a true propositional term is the set of all individuals in the domain of discourse, and that of a false propositional term is the empty set (see Barnes 1983: 314–15). On an intensional interpretation, the semantic value of a true propositional term is the set of concepts each of which is a conceptual part of every term, and that of a false propositional term is the set of all concepts which are conceptual parts of some term (cf. Lenzen 1983: 145 and van Rooij 2014: 188–92).

connectives are \neg and $\&$. We associate each formula p of this language with a term $[p]$ of Leibniz's calculus, as follows:

1. If p is an atomic formula, $[p]$ is the propositional term $\ulcorner p \urcorner$.
2. $[\neg p]$ is the term $\overline{[p]}$.
3. $[p \& q]$ is the term $[p][q]$.

If the material conditional $p \rightarrow q$ is defined in the usual way as $\neg(p \& \neg q)$, the following is a theorem of Leibniz's calculus:¹¹¹

$$\vdash [p \rightarrow q] = \ulcorner [p] \supset [q] \urcorner.$$

Hence, containment, when applied to propositional terms and their Boolean compounds, behaves exactly like material implication.

It is clear from the above tables that if a formula p of propositional logic is a classical tautology, the term $[p]$ coincides with 1 in any auto-Boolean algebra. Moreover, in an auto-Boolean algebra, a term of the form $[p]$ coincides with 1 just in case that term is true. Thus, if p is a tautology, $\mathbf{T}([p])$ holds in any auto-Boolean algebra and is therefore provable in Leibniz's calculus. So, for example, since $\neg(p \& \neg p)$ is a tautology, we have:

$$\vdash \mathbf{T}(\overline{[p][\overline{p}]}) .$$

Leibniz's calculus is thus capable of deriving all tautologies of classical propositional logic. More generally, it can be shown that every propositionally valid inference can be established in Leibniz's calculus. In other words, if the formula q follows from the formulae p_1, p_2, \dots in classical propositional logic, then:

$$\mathbf{T}([p_1]), \mathbf{T}([p_2]), \dots \vdash \mathbf{T}([q]).$$

In this sense, Leibniz's calculus is complete with respect to classical propositional logic.¹¹²

It is a consequence of this completeness result that standard methods of proof that are valid in classical propositional logic become admissible in Leibniz's calculus. One example is the method of proof by multiple hypotheses:

$$\Gamma \cup \{\varphi, \psi\} \vdash \chi \quad \text{if and only if} \quad \Gamma \vdash \ulcorner \varphi \urcorner \ulcorner \psi \urcorner \supset \ulcorner \chi \urcorner.$$

This biconditional follows from the strong principle of propositional containment and the following theorem of Leibniz's calculus:¹¹³

$$\ulcorner \varphi \urcorner \supset \ulcorner \ulcorner \psi \urcorner \supset \ulcorner \chi \urcorner \urcorner \dashv\vdash \ulcorner \varphi \urcorner \ulcorner \psi \urcorner \supset \ulcorner \chi \urcorner.$$

¹¹¹ See Theorem 4.58, Section 4.4. Given the auto-Boolean completeness of Leibniz's calculus, this theorem can be established on semantic grounds by considering the following table:

$[p]$	$[q]$	$\overline{[p][\overline{q}]}$	$\ulcorner [p] = [p][q] \urcorner$
1	1	1	1
1	0	0	0
0	1	1	1
0	0	1	1

¹¹² For a direct proof of this completeness result that does not rely on the auto-Boolean completeness of Leibniz's calculus, see Theorem 4.59 (Section 4.4).

¹¹³ For a proof of this theorem, see Theorem 4.51 (Section 4.4). For the strong principle of propositional containment, see Theorem 4.53; cf. Section 2.4.

Given the propositional completeness of Leibniz's calculus, this latter theorem follows from the validity of importation and exportation in classical propositional logic. Thus, despite the fact that the language of Leibniz's calculus does not include any conjunctive propositions, the calculus is capable of reproducing classically valid patterns of conjunctive reasoning.¹¹⁴

Another important method of proof that is valid in Leibniz's calculus is proof by *reductio*.¹¹⁵

$$\text{If } \Gamma \cup \{\varphi\} \vdash \psi \text{ and } \Gamma \cup \{\varphi\} \vdash \mathbf{F}(\ulcorner \psi \urcorner), \text{ then } \Gamma \vdash \mathbf{F}(\ulcorner \varphi \urcorner).$$

The method of *reductio* facilitates the proof of a number of theorems stated in the *Generales Inquisitiones*, including the following theorem from §55:

$$A \supset B, \mathbf{T}(A) \vdash \mathbf{T}(B).$$

This theorem asserts the upward monotonicity of truth with respect to containment; it is dual to the downward monotonicity of falsehood. Whereas the downward monotonicity of falsehood is readily provable in the non-propositional fragment of Leibniz's calculus, the upward monotonicity of truth requires an appeal to the propositional principles of the calculus, since truth is defined as the falsehood of a certain propositional term.¹¹⁶ By *reductio*, the upward monotonicity of truth can be derived from the downward monotonicity of falsehood as follows:

1	$A \supset B$					
2	$\mathbf{T}(A)$					
3	<table style="border-collapse: collapse; margin-left: 10px;"> <tr> <td style="border-left: 1px solid black; padding-left: 10px; border-bottom: 1px solid black;">$\mathbf{F}(B)$</td> <td style="padding-left: 10px;">assumption for <i>reductio</i></td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 10px;">$\mathbf{F}(A)$</td> <td style="padding-left: 10px;">from 1, 3 by downward monotonicity of falsehood</td> </tr> </table>	$\mathbf{F}(B)$	assumption for <i>reductio</i>	$\mathbf{F}(A)$	from 1, 3 by downward monotonicity of falsehood	
$\mathbf{F}(B)$	assumption for <i>reductio</i>					
$\mathbf{F}(A)$	from 1, 3 by downward monotonicity of falsehood					
5	$\mathbf{T}(B)$	from 2 and 3–4 by <i>reductio</i>				

Leibniz clearly accepts the validity of *reductio*.¹¹⁷ In the *Generales Inquisitiones*, however, he does not posit it as a primitive rule, but rather assumes that it can be derived from the principles of his calculus.¹¹⁸ While Leibniz does not himself provide such a derivation,

¹¹⁴ Leibniz's calculus also allows for classically valid patterns of disjunctive reasoning, such as proof by cases; see Theorem 4.55 (Section 4.4). In this sense, the calculus is capable of reproducing propositional logic from within. By contrast, Lenzen introduces the machinery of propositional logic from the outside by treating Leibniz's calculus as a theory formulated within a system of propositional logic (Lenzen 1983: 131–6, 1984a: 193 and 201–2, 2004: 4 and 10). He thus presupposes methods of propositional reasoning without first establishing their validity in the calculus. On this approach, Leibniz's project of reducing propositional logic to a logic of terms is open to the charge of circularity, in that Leibniz's calculus 'is in the first instance based upon the propositional calculus, but that it afterwards serves as a basis for propositional logic' (Lenzen 2004: 4). Our reconstruction of Leibniz's calculus avoids such circularity since it does not presuppose any laws of propositional logic.

¹¹⁵ See Theorem 4.54 (Section 4.4).

¹¹⁶ For a proof of the downward monotonicity of falsehood, see Theorem 4.28 (Section 4.3).

¹¹⁷ See, e.g., VI4A 499–502, 815, C 208, 304, 307; cf. Swoyer 1994: 6.

¹¹⁸ In §91, Leibniz writes: 'I believe that this mode of reasoning, i.e., *reductio ad absurdum*, has already been established in what precedes' (VI4A 766 n. 36). However, no explicit justification

this assumption turns out to be correct. Thus, his occasional appeals to *reductio* in the *Generales Inquisitiones*, e.g., in §§93–99, are justified.

Although Leibniz's calculus includes only simple propositions of the form $A = B$, as we have seen, it is capable of encoding complex patterns of propositional reasoning. Key to this encoding is Leibniz's device of propositional terms. The propositional principles of Leibniz's calculus guarantee that complex claims about the algebra of terms can be mapped in a homomorphic manner to the terms themselves. In this way, calculations performed within the algebra can encode propositional reasoning about the algebra itself. It is because of this additional capacity for self-reflection, so to speak, that we call the Boolean algebra of terms an 'auto-Boolean' algebra.

3.5. The Categorical Logic of Propositional Terms. The principles of propositional privation and containment impart to the terms of Leibniz's calculus their distinctively auto-Boolean structure. They do so by bestowing upon the propositional terms certain logical properties that set them apart from terms in general. Previous attempts to reconstruct the logic of the *Generales Inquisitiones* have failed to take proper account of the special role played by propositional terms in Leibniz's calculus. The two most prominent examples are the reconstructions proposed by Castañeda (1976, 1990) and Lenzen (1984a, 2004). To the extent that they acknowledge propositional terms at all, neither Castañeda nor Lenzen take these terms to be subject to any special laws. Consequently, they both have difficulty accommodating various passages in the *Generales Inquisitiones* in which Leibniz makes claims that apply exclusively to propositional terms but not to terms in general.

Consider, for example, Leibniz's statement of propositional privation:

If B is a proposition, $\text{non-}B$ is the same as B is false, or, B 's being false.
(§32a)

On our reading, this passage asserts that the following coincidence holds for any propositional term $\ulcorner \varphi \urcorner$:

$$\overline{\ulcorner \varphi \urcorner} = \ulcorner \mathbf{F}(\ulcorner \varphi \urcorner) \urcorner.$$

Thus, on our view, propositional privation is a principle that applies exclusively to propositional terms. If, however, there are no special laws governing propositional terms, the principle cannot be interpreted this way but must instead either hold for all terms or fail to hold even for propositional terms. Castañeda adopts the first of these positions while Lenzen adopts the second. However, neither of the two resulting views is satisfactory.

Castañeda endorses the following generalized variant of propositional privation, where A is any term:¹¹⁹

$$\overline{A} = \ulcorner \mathbf{F}(A) \urcorner.$$

If this were to be added to the principles of Leibniz's calculus, then every term would coincide with a propositional term and therefore be subject to the laws of classical propositional logic. In fact, Castañeda accepts this latter consequence, claiming that the calculus developed by Leibniz in the *Generales Inquisitiones* amounts to a complete axiomatization of classical propositional logic. More precisely, he maintains that *all* terms, propositional

of *reductio* appears in the *Generales Inquisitiones*. Elsewhere, Leibniz feels compelled to posit *reductio* as a primitive, albeit reluctantly (C 309).

¹¹⁹ Castañeda 1990: 23 (Ax 6.2). Castañeda thinks that this is supported by §198.6, whereas we read this passage as applying exclusively to propositional terms (see Section 2.3).

and non-propositional alike, are subject to the laws of classical propositional logic when composition is interpreted as conjunction, containment as material implication, and both privation and falsehood as negation.¹²⁰

Castañeda's completeness result is, of course, much stronger than the propositional completeness result that we established above, and is clearly not what Leibniz intends in the *Generales Inquisitiones*. For example, since $\neg(p \rightarrow \neg q)$ implies $p \rightarrow q$ in classical propositional logic, Castañeda's completeness result entails that the following inference is valid for any terms A and B :

$$\mathbf{F}(\ulcorner A \supset \overline{B} \urcorner) \vdash A \supset B.$$

Leibniz, however, rejects this inference. As he points out, the inference does not hold if A is the term *animal* and B is the term *man*.¹²¹

The following inference is invalid: if *A is not non-B*, then *A is B*. Indeed, *it is false that every animal is a non-man*, but it does not follow from this that *every animal is a man*. (§92)

This passage is consistent with our completeness result established above. For, this result only entails that the laws of classical propositional logic hold for those terms which provably coincide with a propositional term, and this is not the case for ordinary terms such as *man* and *animal*.¹²²

In endorsing his extreme variant of propositional completeness, Castañeda in effect collapses Leibniz's distinction between propositional and non-propositional terms.¹²³ He thus denies the existence of non-propositional terms that occupy an intermediate place in the algebra (i.e., that do not coincide with either 0 or 1). The principles of Leibniz's calculus, however, do not rule out there being such genuinely non-propositional terms. Indeed, such terms are the focus of Leibniz's discussion throughout most of the *Generales Inquisitiones*. So, contrary to Castañeda's view, Leibniz's calculus is not merely a system of classical propositional logic. It is rather a comprehensive logic of terms of which classical propositional logic is only a proper part—namely, the part that deals specifically with propositional terms and their Boolean compounds.

Whereas Castañeda's reconstruction of Leibniz's calculus is too strong, the reconstruction proposed by Lenzen is too weak. This is because Lenzen does not include in his system

¹²⁰ Castañeda 1990: 24–5; a similar position is endorsed by Doull 1991: 21–2.

¹²¹ Similarly, C 231 (cf. Lenzen 1983: 139–40). For various complications concerning Leibniz's discussion of the above inference, see Lenzen 1986: 26–36.

¹²² See n. 59. Moreover, the following inference is valid in Castañeda's system:

$$\mathbf{T}(A) \vdash \mathbf{F}(\overline{A}).$$

Given Leibniz's commitment to the law of noncontradiction for all terms (see n. 101), Castañeda's endorsement of this inference conflicts with Leibniz's claim that both a term and its privative can be true (VI4A 810 n. 5).

¹²³ This conflation is also manifest in the fact that Castañeda (1976: 491) defines the propositional term $\ulcorner A = B \urcorner$ as the Boolean compound:

$$\overline{\overline{A} \overline{B}} \overline{\overline{B} \overline{A}}.$$

Similarly, Doull (1991: 21–2) defines $\ulcorner A \supset B \urcorner$ as $\overline{\overline{A} \overline{B}}$. As we have seen, such purely Boolean definitions of propositional terms are inadequate since they imply that the algebra of terms is degenerate in the sense that every term coincides with either 0 or 1 (see n. 109).

any version of the principle of propositional privation, not even for propositional terms. As a result, Lenzen cannot establish the restricted completeness result stated above, according to which all propositional terms obey the laws of classical propositional logic. Lenzen is well aware of this and maintains that the propositional logic to which Leibniz's calculus gives rise is weaker than full-blown classical logic. In particular, he holds that containment when applied to propositional terms does not obey the laws of material implication but only those of some weaker system of strict implication.¹²⁴

Lenzen takes this reading of containment as strict implication to be motivated by Leibniz's Principle in conjunction with a modal interpretation of truth and falsehood for propositional terms.¹²⁵ Specifically, he interprets $\mathbf{T}(\ulcorner \varphi \urcorner)$ as the modal claim that the proposition φ is possible (i.e., $\diamond\varphi$), and $\mathbf{F}(\ulcorner \varphi \urcorner)$ as the modal claim that φ is impossible (i.e., $\neg\diamond\varphi$).¹²⁶ Now, Leibniz's Principle states that the proposition $\ulcorner \varphi \urcorner \supset \ulcorner \psi \urcorner$ is equivalent to:

$$\mathbf{F}(\ulcorner \varphi \urcorner \overline{\ulcorner \psi \urcorner}).$$

Accordingly, Lenzen interprets $\ulcorner \varphi \urcorner \supset \ulcorner \psi \urcorner$ as the claim:

$$\neg\diamond(\varphi \ \& \ \neg\psi).$$

But this is the standard definition of strict implication in the framework of propositional modal logic.

The precise sort of strict implication expressed by containment in Lenzen's system is determined by the nature of the modal operator \diamond . In particular, if this operator is trivial in the sense that $\diamond\varphi$ is equivalent to φ , then strict implication as defined above collapses into material implication. Thus, if Lenzen is correct that containment does not express material implication, \diamond cannot be a trivial modal operator. Since $\diamond\varphi$ is Lenzen's way of expressing $\mathbf{T}(\ulcorner \varphi \urcorner)$, the requirement that \diamond be nontrivial means that $\mathbf{T}(\ulcorner \varphi \urcorner)$ must not be equivalent to φ . This, however, conflicts with Leibniz's claim that 'the addition of truth or of being leaves things unchanged' (§198.6). Or, as Leibniz puts it in §1:

These coincide: the (direct) statement L and the (reflexive) statement: L is true. (§1)

In this passage, Leibniz states that propositional truth is disquotational:

$$\varphi \dashv\vdash \mathbf{T}(\ulcorner \varphi \urcorner).$$

This law of disquotation excludes any nontrivial modal interpretation of truth. Likewise, any nontrivial modal interpretation of falsehood is excluded by Leibniz's endorsement of the principle of propositional privation. For, on Lenzen's modal interpretation of falsehood, this principle amounts to the claim that $\neg\varphi$ is equivalent to $\neg\diamond\varphi$, but this means that \diamond is trivial.

Thus, Lenzen's contention that the logic of propositional terms is a nonclassical logic of strict implication fails to account for fundamental laws of truth and falsehood that

¹²⁴ Lenzen 1987, 2004: 35–9; similarly, Juniewicz 1987: 51.

¹²⁵ See Lenzen 1987: 13, 2004: 36, 2005: 352; similarly, Rescher 1954: 10, Levey 2011: 122–3, Bonevac and Dever 2012: 202–4.

¹²⁶ Lenzen 1987: 4, 2004: 13–14, 2005: 347–8. Lenzen focuses primarily on those passages from Leibniz's writings in which $\mathbf{T}(A)$ is expressed by the phrases *A is*, *A is a being*, and *A is possible* (cf. n. 9). It is clear, however, that Lenzen takes these phrases to be synonymous with *A is true* (see his disussion of §55 in 2004: 15–16 n. 17).

are stated by Leibniz in the *Generales Inquisitiones* (e.g., propositional privation and disquotation).¹²⁷ These laws indicate that, for Leibniz, truth and falsehood when applied to propositional terms do not express possibility and impossibility but instead purely assertoric notions that impart no modal force to the propositions of the calculus.

It should be acknowledged, of course, that there are certain *prima facie* considerations that seem to speak in favor of a modal interpretation of truth and falsehood in the *Generales Inquisitiones*. For example, as we have seen, Leibniz defines a false term as one which contains a contradiction.¹²⁸ If a propositional term contains a contradiction, it is not unreasonable to suppose that the corresponding proposition is not only false but necessarily false, i.e., impossible. For Leibniz, however, this is not the case. In his view, every false propositional term contains a contradiction; an impossible propositional term has the further property that one of the contradictions it contains can be disclosed through a finite as opposed to an infinite process of analysis.¹²⁹

That term or proposition is false which contains opposites, however they are proved. That term or proposition is impossible which contains opposites that are proved by reduction to finitely many terms. So, $A = AB$, if a proof has been produced through a finite analysis, must be distinguished from $A = AB$, if a proof has been produced through an analysis *ad infinitum*. From this there already results what has been said about the necessary, the possible, the impossible, and the contingent. (§130b)

As this passage implies, there are false propositional terms which are not impossible since the contradictions they contain cannot be revealed through any finite process of analysis. Likewise, there are true propositional terms which fail to be necessary since the fact that they do not contain a contradiction cannot be discovered through any finite analysis.¹³⁰

Thus, for Leibniz, possibility and impossibility are not definable by the criterion of whether or not a term contains a contradiction. Rather, such modal notions are defined by appeal to the independent distinction between finite and infinite analysis, which is not expressible in the language of Leibniz's calculus. Hence, while Leibniz makes use of modal notions at various points throughout the *Generales Inquisitiones*, his calculus is not a calculus for reasoning about such modal notions but only about assertoric truth and falsehood.¹³¹ In short, Leibniz's calculus is not a modal logic.¹³²

¹²⁷ Another difficulty for Lenzen's modal interpretation of truth and falsehood derives from Leibniz's statement in §1 that $\mathbf{T}(\ulcorner \varphi \urcorner)$ is equivalent to $\mathbf{F}(\ulcorner \mathbf{F}(\ulcorner \varphi \urcorner) \urcorner)$ (see n. 36). On Lenzen's interpretation, this amounts to the equivalence of $\diamond\varphi$ and $\Box\diamond\varphi$, which is characteristic of the modal system S5. Lenzen, however, denies that Leibniz's modal logic satisfies this S5-equivalence (Lenzen 2004: 42; similarly, Adams 1994: 47).

¹²⁸ See Section 1.4.

¹²⁹ See also §§60–1 and Schupp 1993: 224–9.

¹³⁰ See §74 and §§133–4; cf. Schupp 1993: 224–31, Adams 1994: 25–30.

¹³¹ Leibniz sometimes expresses $\mathbf{T}(A)$ by the phrase *A is possible* (see n. 9). This use of the phrase, however, is limited to the earlier portions of the *Generales Inquisitiones* prior to Leibniz's discussion of necessity, possibility, and impossibility in §§130–8. One plausible explanation of why Leibniz refrains from this use of the phrase in the later portions of the text is that he wishes to avoid any potential conflation of the notions of truth and possibility (cf. Schupp 1993: 226–9).

¹³² *Pace* Lenzen, who maintains that Leibniz's calculus 'yields a modal logic of strict implication' (Lenzen 2004: 4; similarly, Lenzen 1987: 21–6 and the authors mentioned in n. 125 above). We likewise do not agree with the view that Leibniz's calculus 'makes little sense as a theory of

One of Leibniz's central ambitions in the *Generales Inquisitiones* is to develop a single unified calculus for reasoning about both propositions and terms:

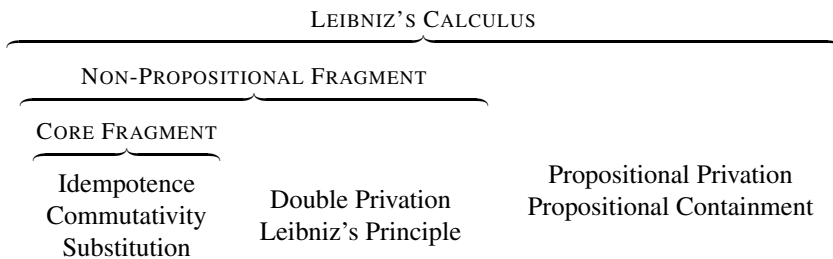
If, as I hope, I can conceive of all propositions as terms, and of all hypothetical propositions as categorical, and if I can give a universal treatment of them all, this promises a wonderful ease in my symbolism and analysis of concepts, and will be a discovery of the greatest moment. (§75)

By a categorical proposition Leibniz means a proposition expressing a containment between terms. By a hypothetical proposition he means a conditional statement whose antecedent and consequent are categorical propositions.¹³³ As the above passage makes clear, Leibniz aims to reduce hypothetical to categorical propositions.¹³⁴ The key innovation that allows him to achieve this aim is the introduction into his calculus of propositional terms. With this device in hand, a proposition φ can be transformed into the term $\ulcorner \varphi \urcorner$, and a hypothetical proposition *if φ then ψ* into the categorical proposition $\ulcorner \varphi \urcorner \supset \ulcorner \psi \urcorner$. The fundamental difficulty confronting both Castañeda and Lenzen stems from the fact that, in Leibniz's calculus, propositional terms are subject to special principles that do not hold of terms in general. These principles guarantee that hypothetical propositions, conceived of categorically as containments between propositional terms, obey all the laws of classical propositional logic.

Thus, by positing only a few simple principles, Leibniz succeeds in constructing a calculus in which propositional logic is reduced to a pure logic of terms. In particular, the logic of hypothetical propositions is reduced to that of categorical ones. Crucially, Leibniz manages to effect this reduction while preserving the important differences that exist between the logic of propositions and the logic of terms. This remarkable success, in addition to the many other accomplishments of the *Generales Inquisitiones*, justifies Leibniz's claim to have

discovered many secrets of great moment for the analysis of all of our thoughts and for the discovery and proof of truths, including . . . how absolute and hypothetical truths have one and the same laws and are contained in the same general theorems, so that all syllogisms become categorical. (§137)

3.6. Summary. We have constructed an algebraic semantics for Leibniz's calculus in three stages, based on the following partitioning of its seven principles:



categorical propositions, for it gives all universal propositions modal force' (Bonevac and Dever 2012: 202).

¹³³ See VI4A 127–31, 864, 992.

¹³⁴ Cf. VI4A 811 n. 6, 863, 992.

The main results presented in this part of the paper are as follows:

1. The core fragment is both sound and complete with respect to the class of semilattices.
2. The non-propositional fragment is both sound and complete with respect to the class of Boolean algebras.
3. Leibniz's calculus is both sound and complete with respect to the class of auto-Boolean algebras.
4. There exist nonstandard (i.e., non-auto-Boolean) models of Leibniz's calculus in which the laws of bivalence and noncontradiction do not hold.
5. All the laws of classical propositional logic are derivable in Leibniz's calculus.

§4. Appendix: Proofs of Theorems. In this appendix, we supply proofs of the various technical results that have been stated throughout the paper. The material has been organized in a self-contained manner and does not presuppose familiarity with the preceding discussion.

We begin by recapitulating the syntax and principles of Leibniz's calculus (Section 4.1). We then prove that the core fragment of Leibniz's calculus is complete with respect to the class of semilattices (Section 4.2), and that the non-propositional fragment of the calculus is complete with respect to the class of Boolean algebras (Section 4.3). We proceed to demonstrate that the laws of classical propositional logic are derivable in Leibniz's calculus (Section 4.4). Next, we introduce the notion of an auto-Boolean interpretation and show that Leibniz's calculus has nonstandard, i.e., non-auto-Boolean, models (Section 4.5). Finally, we show that Leibniz's calculus is complete with respect to the class of auto-Boolean interpretations (Section 4.6).

4.1. Preliminaries. We introduce a formal language for Leibniz's calculus. The well-formed expressions of this language are called terms and propositions.¹³⁵ We take as given a nonempty countable set of primitive expressions referred to as simple terms.¹³⁶

DEFINITION 4.1. *Terms and propositions are the expressions defined inductively as follows:*

1. *Every simple term is a term.*
2. *If A and B are terms, then \overline{A} and AB are terms.*
3. *If A and B are terms, then $A = B$ is a proposition.*
4. *If $A = B$ is a proposition, then $\ulcorner A = B \urcorner$ is a term.*

We refer to AB as the *composite* of A and B , and to \overline{A} as the *privative* of A . We define A^n ($n \geq 1$) inductively as follows: A^1 is the term A ; A^n is the composite of A and A^{n-1} (i.e., AA^{n-1}).

¹³⁵ By contrast, Leibniz uses 'term' and 'proposition' to refer not to expressions of a formal language but to cognitive items signified by such expressions (see n. 3 above). While we have thus far followed Leibniz in this nonlinguistic usage, we now shift to a linguistic usage in order to avoid such locutions as 'term-expression' and 'proposition-expression'.

¹³⁶ We assume that the simple terms are chosen in such a way that no simple term is identical to any compound term or proposition as defined in Definition 4.1.

DEFINITION 4.2. We adopt the following shorthand for expressing propositions:

1. $A \supset B$ is an abbreviation for $A = AB$.
2. $\mathbf{F}(A)$ is an abbreviation for $A \supset \bar{A}$.
3. $\mathbf{T}(A)$ is an abbreviation for $\mathbf{F}(\ulcorner \mathbf{F}(A) \urcorner)$.

DEFINITION 4.3. A calculus is a relation \vdash between sets of propositions and propositions, which satisfies:

1. $\{\varphi\} \vdash \varphi$.
2. If $\Gamma \vdash \varphi$, then $\Gamma \cup \Delta \vdash \varphi$.
3. If $\Gamma \cup \{\varphi\} \vdash \psi$ and $\Delta \vdash \varphi$, then $\Gamma \cup \Delta \vdash \psi$.

Here, $\Gamma \vdash \varphi$ indicates that the set of propositions Γ stands in the relation \vdash to the proposition φ . In what follows, $\vdash \varphi$ is shorthand for $\emptyset \vdash \varphi$, and $\psi_1, \dots, \psi_n \vdash \varphi$ is shorthand for $\{\psi_1, \dots, \psi_n\} \vdash \varphi$.

DEFINITION 4.4. Leibniz's calculus is the smallest calculus \vdash satisfying the following principles:

1. Idempotence (ID): $\vdash AA = A$
2. Commutativity (CM): $\vdash AB = BA$
3. Double Privation (DP): $\vdash \bar{\bar{A}} = A$
4. Propositional Privation (PP): $\vdash \ulcorner \bar{A} = \bar{B} \urcorner = \ulcorner \mathbf{F}(\ulcorner A = B \urcorner) \urcorner$
5. Substitution of Coincidents (SC):

$$A = B, C = D \vdash C^* = D^*,$$

where $C^* = D^*$ is the result of substituting B for an occurrence of A , or vice versa, in $C = D$.

6. Propositional Containment (PC):

$$A = B \vdash C = D \quad \text{iff} \quad \vdash \ulcorner A = B \urcorner \supset \ulcorner C = D \urcorner$$

7. Leibniz's Principle (LP): $A \supset \bar{B} \dashv\vdash \mathbf{F}(AB)$

DEFINITION 4.5. The core fragment of Leibniz's calculus, \vdash_c , is the smallest calculus satisfying ID, CM, and SC. The non-propositional fragment of Leibniz's calculus, \vdash_{np} , is the smallest calculus satisfying ID, CM, DP, SC, and LP.

We henceforth use \vdash without a subscript to denote Leibniz's calculus as specified in Definition 4.4. The following definition provides a general algebraic semantics for the language of Leibniz's calculus:

DEFINITION 4.6. An interpretation (\mathfrak{A}, μ) consists of (i) an algebraic structure $\mathfrak{A} = \langle \mathbb{A}, \wedge, ' \rangle$, where \mathbb{A} is a nonempty set, \wedge is a binary operation on \mathbb{A} , and $'$ is a unary operation on \mathbb{A} ; and (ii) a function μ mapping each term to an element of \mathbb{A} such that:¹³⁷

$$\mu(\bar{A}) = \mu(A)'$$

¹³⁷ In the following two equations, the symbol '=' indicates the identity (or algebraic equivalence) of elements of \mathbb{A} . Although the same symbol is used to indicate the coincidence of terms in Leibniz's calculus, it will always be clear from the context which of these two uses of '=' is intended.

$$\mu(AB) = \mu(A) \wedge \mu(B).$$

A proposition $A = B$ is satisfied in an interpretation (\mathfrak{A}, μ) iff $\mu(A) = \mu(B)$. We write $(\mathfrak{A}, \mu) \models A = B$ to indicate that (\mathfrak{A}, μ) satisfies $A = B$.

4.2. The Core Fragment. In this section, we establish various theorems of the core fragment of Leibniz's calculus, \vdash_c (i.e., the fragment consisting of the principles ID, CM, and SC). When textual references are included in the statement of a theorem, these refer to the passages in which Leibniz asserts said theorem (or an obvious variant of it).¹³⁸

THEOREM 4.7. $\vdash_c A = A$ (§10, §115, §156, §171.1, VI4A 774 n. 47, VI4A 811, C 235)

1	AA = A	ID
2	A = A	SC: 1

THEOREM 4.8. $A = B \vdash_c B = A$ (§6, VI4A 746, 831)

1	A = B	
2	A = A	SC: 1
3	B = A	SC: 1, 2

THEOREM 4.9. $A = B, B = C \vdash_c A = C$ (§8, VI4A 815–16, 849)

1	A = B	
2	B = C	
3	A = C	SC: 1, 2

THEOREM 4.10. $\vdash_c A^n = A$, for all $n \geq 1$ (§18)

The proof proceeds by induction on n . If $n = 1$, then the result follows from Theorem 4.7. Now, suppose $\vdash_c A^{n-1} = A$, for $n > 1$. Then:

1	$A^{n-1} = A$	by induction hypothesis
2	$A^n = AA^{n-1}$	T4.7
3	$A^n = AA$	SC: 1, 2
4	AA = A	ID
5	$A^n = A$	SC: 3, 4

¹³⁸ Such references are not meant to indicate that our proof of the relevant theorem corresponds exactly to the proof which Leibniz gives for this theorem in any particular passage. In fact, Leibniz rarely proves theorems directly from the calculus specified in Definition 4.4. This is because this calculus is not fully in place until the concluding sections of the *Generales Inquisitiones* (§§198–200). Consequently, most of the proofs that Leibniz gives in the *Generales Inquisitiones* are not based on his final system of principles, but proceed from whatever set of postulates he provisionally adopts at that particular stage of the treatise.

THEOREM 4.11. $A = B \vdash_c \bar{A} = \bar{B}$ (§2, §9, §78, §157, §171.6, VI4A 749 n. 9, C 235, 421–2)

1	$A = B$ <hr style="width: 100%;"/>	
2	$\bar{A} = \bar{A}$	T4.7
3	$\bar{A} = \bar{B}$	SC: 1, 2

THEOREM 4.12. $A = B \vdash_c AC = BC$ (§171.5, VI4A 812, C 236, 422)

1	$A = B$ <hr style="width: 100%;"/>	
2	$AC = AC$	T4.7
3	$AC = BC$	SC: 1, 2

To shorten proofs, we will henceforth allow ourselves both to invoke a coincidence claim and to substitute one coincident term for the other one or more times in a previously occurring proposition, in a single step. So, for example, instead of:

1	$A = BC$ <hr style="width: 100%;"/>	
2	$BC = CB$	CM
3	$A = CB$	SC: 1, 2

we will simply write:

1	$A = BC$ <hr style="width: 100%;"/>	
2	$A = CB$	CM=: 1

THEOREM 4.13. $A \supset B \vdash_c AC \supset BC$ (VI4A 291)

1	$A = AB$ <hr style="width: 100%;"/>	
2	$AC = ABC$	T4.12: 1
3	$AC = ABCC$	ID=: 2
4	$AC = ACBC$	CM=: 3

THEOREM 4.14. $A \supset B, C \supset D \vdash_c AC \supset BD$ (VI4A 150, 291)

1	$A = AB$	
2	$C = CD$	
3	$AC = ABC$	T4.12: 1
4	$AC = ACB$	CM=: 3
5	$AC = ACDB$	SC: 2, 4
6	$AC = ACBD$	CM=: 5

THEOREM 4.15. $A = B \vdash_c A \supset B$ (§36, VI4A 813)

1	$A = B$	
2	$A = AA$	ID
3	$A = AB$	SC: 1, 2

THEOREM 4.16. $A = B \vdash_c B \supset A$ (VI4A 813)

1	$A = B$	
2	$B = BB$	ID
3	$B = BA$	SC: 1, 2

THEOREM 4.17. $\vdash_c A \supset A$ (§37, §43, VI4A 120, 142–3, 147–8, 150, 274–5, 281, 292, 804)

1	$AA = A$	ID
2	$A = AA$	T4.8: 1

THEOREM 4.18. $A \supset B, B \supset C \vdash_c A \supset C$ (§19, VI4A 275, 281, 293, 813, C 229)

1	$A = AB$	
2	$B = BC$	
3	$A = ABC$	SC: 1, 2
4	$A = AC$	SC: 1, 3

THEOREM 4.19 (Antisymmetry). $A \supset B, B \supset A \vdash_c A = B$ (§30, VI4A 154, 275, 284, 285, 294, 552, 813)

1	$A = AB$	
2	$B = BA$	
3	$A = BA$	CM=: 1
4	$A = B$	SC: 2, 3

THEOREM 4.20. $\vdash_c AB \supset B$ (§38, §39, §46, VI4A 274–5, 281)

1	$AB = AB$	T4.7
2	$AB = ABB$	ID=: 1

THEOREM 4.21. $\vdash_c AB \supset A$ (§77, VI4A 274, 280, 292, 813)

1	$BA = BAA$	T4.20
2	$AB = ABA$	CM=: 1

THEOREM 4.22. $A \supset B, A \supset C \vdash_c A \supset BC$ (§102, VI4A 150, 290, 754 n. 19, 808, 813)

1	$A = AB$	
2	$A = AC$	
3	$A = ABC$	SC: 1, 2

In the remainder of this section, we establish that the core fragment of Leibniz's calculus is both sound and complete with respect to the class of semilattices. We first introduce the notion of a semilattice interpretation (for the general notion of an interpretation, see Definition 4.6).

DEFINITION 4.23. *An interpretation $(\langle \mathbb{A}, \wedge, ' \rangle, \mu)$ is a semilattice interpretation if the algebraic structure $\langle \mathbb{A}, \wedge \rangle$ is a semilattice. If Γ is a set of propositions and φ is a proposition, we write $\Gamma \models_{sl} \varphi$ to indicate that, for any semilattice interpretation (\mathfrak{A}, μ) : if $(\mathfrak{A}, \mu) \models \psi$ for all $\psi \in \Gamma$, then $(\mathfrak{A}, \mu) \models \varphi$.*

The following completeness proof employs the standard technique introduced by Birkhoff (1935) in his proof of the algebraic completeness of equational calculi.

THEOREM 4.24. $\Gamma \vdash_c \varphi$ iff $\Gamma \models_{sl} \varphi$.

Proof. The left-to-right direction follows straightforwardly from the fact that the principles ID, CM, and SC (as well as the general properties of calculi stated in Definition 4.3) are valid in every semilattice interpretation.

For the right-to-left direction, let Γ be a set of propositions and let \equiv be the binary relation between terms defined by:

$$A \equiv B \quad \text{iff} \quad \Gamma \vdash_c A = B.$$

By T4.7, T4.8, and T4.9, it follows that \equiv is an equivalence relation on the set of all terms. Now, let μ be the function mapping each term to its equivalence class under \equiv , and let \mathbb{T} be the set $\{\mu(A) : A \text{ is a term}\}$. By T4.11 and the fact that:

$$A = B, C = D \vdash_c AC = BD$$

it follows that \equiv is a congruence relation on the algebra of terms formed by the operations of privation and binary composition. Hence, there is a binary operation \wedge on \mathbb{T} and a unary operation $'$ on \mathbb{T} , such that for any terms A and B :

$$\mu(\overline{A}) = \mu(A)'$$

$$\mu(AB) = \mu(A) \wedge \mu(B).$$

The latter equation implies both:

$$\mu(ABC) = \mu(AB) \wedge \mu(C) = (\mu(A) \wedge \mu(B)) \wedge \mu(C)$$

and

$$\mu(ABC) = \mu(A) \wedge \mu(BC) = \mu(A) \wedge (\mu(B) \wedge \mu(C)).$$

It thus follows that:

$$(\mu(A) \wedge \mu(B)) \wedge \mu(C) = \mu(A) \wedge (\mu(B) \wedge \mu(C)).$$

In other words, \wedge is an associative relation on \mathbb{T} . Moreover, it follows by ID and CM that \wedge is both idempotent and commutative. Hence, $\langle \mathbb{T}, \wedge \rangle$ is a commutative idempotent semigroup, i.e., a semilattice. Accordingly, if \mathfrak{T} is the algebraic structure $\langle \mathbb{T}, \wedge, ' \rangle$, then (\mathfrak{T}, μ) is a semilattice interpretation. Moreover, by the definition of (\mathfrak{T}, μ) , we have:

$$(\mathfrak{T}, \mu) \models \varphi \text{ iff } \Gamma \vdash_c \varphi.$$

Now, suppose $\Gamma \not\vdash_c \varphi$. Then, $(\mathfrak{T}, \mu) \not\models \varphi$. But since $\Gamma \vdash_c \psi$ for all $\psi \in \Gamma$, it follows that $(\mathfrak{T}, \mu) \models \psi$ for all $\psi \in \Gamma$. Hence, since (\mathfrak{T}, μ) is a semilattice interpretation, $\Gamma \not\models_{sl} \varphi$. \square

4.3. The Non-Propositional Fragment. In this section, we establish various theorems of the non-propositional fragment of Leibniz's calculus, \vdash_{np} (i.e., the fragment consisting of the principles ID, CM, DP, SC, and LP). We write $\varphi \dashv\vdash_{np} \psi$ to indicate that both $\varphi \vdash_{np} \psi$ and $\psi \vdash_{np} \varphi$.

To shorten our proofs, we will make use of the obvious fact that SC licenses the direct substitution of one coincident term for another within the abbreviations $A \supset B$, $\mathbf{F}(A)$, and $\mathbf{T}(A)$ introduced in Definition 4.2. Thus, for example, $C \supset B$ can be derived from $C = A$ and $A \supset B$ as follows:

$$\begin{array}{l|l} 1 & C = A \\ 2 & A = AB \\ \hline 3 & C = AB \quad \text{SC: 1, 2} \\ 4 & C = CB \quad \text{SC: 1, 3} \end{array}$$

More generally, we have:

$$\begin{aligned} A = B, C \supset D &\vdash_{np} C^* \supset D \\ A = B, D \supset C &\vdash_{np} D \supset C^* \\ A = B, \mathbf{F}(C) &\vdash_{np} \mathbf{F}(C^*) \\ A = B, \mathbf{T}(C) &\vdash_{np} \mathbf{T}(C^*), \end{aligned}$$

where C^* is the result of substituting B for an occurrence of A , or vice versa, in C .¹³⁹ Since each of these substitutions can be justified by multiple applications of SC, we will henceforth use the label 'SC' to indicate direct substitutions into both abbreviated and unabbreviated forms alike. Labels such as 'DP=' will be used in a similar fashion, as in the proof of the following theorem:

THEOREM 4.25. $A \supset B \dashv\vdash_{np} \mathbf{F}(A\overline{B})$ (§169, §199, §200)

For the \vdash direction:

$$\begin{array}{l|l} 1 & A \supset B \\ 2 & A \supset \overline{\overline{B}} \quad \text{DP=: 1} \\ 3 & \mathbf{F}(A\overline{B}) \quad \text{LP: 2} \end{array}$$

¹³⁹ This includes cases in which the term C contains abbreviated propositional terms such as $\ulcorner A \supset E \urcorner$, $\ulcorner \mathbf{F}(A) \urcorner$, and $\ulcorner \mathbf{T}(A) \urcorner$. So, for example, $\ulcorner \mathbf{F}(B) \urcorner \supset D$ can be derived from $A = B$ and $\ulcorner \mathbf{F}(A) \urcorner \supset D$ (see n. 41).

The proof of the \neg direction is the reverse of the one just given.

THEOREM 4.26 (Contraposition). $A \supset B \vdash_{\text{np}} \overline{B} \supset \overline{A}$ (§77, §93, §95, §189.5, C 422)

1	$A \supset B$	
2	$\mathbf{F}(\overline{A}\overline{B})$	T4.25: 1
3	$\mathbf{F}(\overline{B}A)$	CM=: 2
4	$\overline{B} \supset \overline{A}$	LP: 3

THEOREM 4.27. $\vdash_{\text{np}} \overline{A} \supset \overline{AB}$ (§76b, §104, §189.5, C 237, 422)

1	$AB \supset A$	T4.21
2	$\overline{A} \supset \overline{AB}$	T4.26: 1

THEOREM 4.28. $A \supset B, \mathbf{F}(B) \vdash_{\text{np}} \mathbf{F}(A)$ (§58)

1	$A \supset B$	
2	$B \supset \overline{B}$	
3	$\overline{B} \supset \overline{A}$	T4.26: 1
4	$B \supset \overline{A}$	T4.18: 2, 3
5	$A \supset \overline{A}$	T4.18: 1, 4

THEOREM 4.29. $\vdash_{\text{np}} \mathbf{F}(A\overline{A})$ (§32b, §171.8)

1	$A \supset A$	T4.17
2	$\mathbf{F}(A\overline{A})$	T4.25: 1

THEOREM 4.30. $A \supset B\overline{B} \vdash_{\text{np}} \mathbf{F}(A)$ (§34, §194, §198.4, VI4A 808 n. 2, 810 n. 4)

1	$A \supset B\overline{B}$	
2	$\mathbf{F}(B\overline{B})$	T4.29
3	$\mathbf{F}(A)$	T4.28: 1, 2

THEOREM 4.31 (Explosion). $\mathbf{F}(A) \vdash_{\text{np}} A \supset B$

1	$\mathbf{F}(A)$	
2	$\overline{A}\overline{B} \supset A$	T4.21
3	$\mathbf{F}(A\overline{B})$	T4.28: 1, 2
4	$A \supset B$	T4.25: 3

THEOREM 4.32. $\mathbf{F}(A), \mathbf{F}(B) \vdash_{\text{np}} A = B$ (VI4A 817)

1	$\mathbf{F}(A)$	
2	$\mathbf{F}(B)$	
3	$A \supset B$	T4.31: 1
4	$B \supset A$	T4.31: 2
5	$A = B$	T4.19: 3, 4

In the remainder of this section, we establish that the non-propositional fragment of Leibniz's calculus is both sound and complete with respect to the class of Boolean algebras.

DEFINITION 4.33. An interpretation $(\langle \mathbb{A}, \wedge, ' \rangle, \mu)$ is a Boolean interpretation if the algebraic structure $\langle \mathbb{A}, \wedge, ' \rangle$ is a Boolean algebra. If Γ is a set of propositions and φ is a proposition, we write $\Gamma \models_{\text{ba}} \varphi$ to indicate that, for any Boolean interpretation (\mathfrak{A}, μ) : if $(\mathfrak{A}, \mu) \models \psi$ for all $\psi \in \Gamma$, then $(\mathfrak{A}, \mu) \models \varphi$.

If x and y are elements of a Boolean algebra, we write $x \leq y$ to indicate that $x = x \wedge y$, and we write 0 and 1 for the least and greatest elements of the algebra, respectively. We allow for the possibility that $0 = 1$, i.e., that a Boolean algebra has only one element.

Our proof of completeness will rely upon a concise axiomatization of Boolean algebra discovered by Byrne (1946). The main principle in this axiomatization is:

THEOREM 4.34 (Byrne's Principle). $A\bar{B} = C\bar{C} \dashv\vdash_{\text{np}} A \supset B$

For the \vdash direction:

1	$A\bar{B} = C\bar{C}$	
2	$\mathbf{F}(C\bar{C})$	T4.29
3	$\mathbf{F}(A\bar{B})$	SC: 1, 2
4	$A \supset B$	T4.25: 3

For the \dashv direction:

1	$A \supset B$	
2	$\mathbf{F}(A\bar{B})$	T4.25: 1
3	$\mathbf{F}(C\bar{C})$	T4.29
4	$A\bar{B} = C\bar{C}$	T4.32: 2, 3

THEOREM 4.35. $\Gamma \vdash_{\text{np}} \varphi$ iff $\Gamma \models_{\text{ba}} \varphi$

Proof. The left-to-right direction follows straightforwardly from the fact that the principles of the non-propositional fragment of Leibniz's calculus are valid in any Boolean interpretation. This is obvious for ID, DP, CM, and SC. The validity of LP follows from the Boolean fact that $x \wedge y = 0$ iff $x \leq y'$.

For the right-to-left direction, let Γ be a set of propositions. Following the proof of T4.24, we construct an interpretation $(\mathfrak{I}, \mu) = ((\mathbb{T}, \wedge, '), \mu)$ such that $\mathbb{T} = \{\mu(A) : A \text{ is a term}\}$, \wedge is a commutative and associative operation on \mathbb{T} , and:

$$(\mathfrak{I}, \mu) \models \varphi \text{ iff } \Gamma \vdash_{\text{np}} \varphi.$$

By T4.34, it follows that, for any terms A, B , and C :

$$\mu(A) \wedge \mu(B)' = \mu(C) \wedge \mu(C)' \text{ iff } \mu(A) = \mu(A) \wedge \mu(B).$$

Byrne (1946: 269–271) has shown that this law, in conjunction with the commutativity and associativity of \wedge , implies that \mathfrak{I} is a Boolean algebra.¹⁴⁰ Hence, (\mathfrak{I}, μ) is a Boolean interpretation.

Now, suppose $\Gamma \not\vdash_{\text{np}} \varphi$. Then, $(\mathfrak{I}, \mu) \not\models \varphi$. But since $\Gamma \vdash_{\text{np}} \psi$ for all $\psi \in \Gamma$, it follows that $(\mathfrak{I}, \mu) \models \psi$ for all $\psi \in \Gamma$. Hence, since (\mathfrak{I}, μ) is a Boolean interpretation, $\Gamma \not\models_{\text{ba}} \varphi$. \square

Theorem 4.35 implies that all the laws of Boolean algebra are provable in Leibniz's calculus. One notable example of such a law that does not appear in Leibniz's writings is the law of distributivity: $A(B + C) = AB + AC$.¹⁴¹ In spite of the fact that Leibniz never states this law, we conclude the current section with an explicit proof of it. We do so not only to illustrate the power of the non-propositional fragment of Leibniz's calculus, but also because proofs of distributivity tend to be elusive in the literature in comparable axiomatizations of Boolean algebra that are based on some variant of Leibniz's principle.¹⁴²

THEOREM 4.36 (Distributivity). $\vdash_{\text{np}} \overline{\overline{AB} \overline{C}} = \overline{\overline{AB} \overline{AC}}$

1	$\overline{B} \supset \overline{BA}$	T4.27
2	$\overline{B} \supset \overline{AB}$	CM=: 1
3	$\overline{C} \supset \overline{CA}$	T4.27
4	$\overline{C} \supset \overline{AC}$	CM=: 3
5	$\overline{B} \overline{C} \supset \overline{AB} \overline{AC}$	T4.14: 2, 4
6	$\overline{\overline{AB} \overline{AC}} \supset \overline{\overline{B} \overline{C}}$	T4.26: 5
7	$\overline{A} \supset \overline{AB}$	T4.27
8	$\overline{A} \supset \overline{AC}$	T4.27
9	$\overline{A} \supset \overline{AB} \overline{AC}$	T4.22: 7, 8

¹⁴⁰ Byrne proves this result under the assumption that $0 \neq 1$, but this assumption plays no role in his proof and can therefore be omitted.

¹⁴¹ Here, $A + B$ is the least upper bound of A and B , i.e., $\overline{\overline{A} \overline{B}}$.

¹⁴² For example, Peirce forgoes any attempt to prove distributivity on the grounds that 'the proof is too tedious to give' (Peirce 1880: 33). Similarly, Lewis and Langford (1932: 36) write that 'the proof is long and complex, and is omitted for that reason'. One of the earliest proofs of distributivity that is based on a version of Leibniz's Principle is given by Huntington, who acknowledges that this proof was communicated to him by Peirce (see Huntington 1904: 300–2; cf. Houser 1991, Badesa 2004: 21–5).

10	$\overline{\overline{AB} \overline{AC}} \supset \overline{\overline{A}}$	T4.26: 9
11	$\overline{\overline{AB} \overline{AC}} \supset A$	DP=: 10
12	$\overline{\overline{AB} \overline{AC}} \supset \overline{\overline{AB} \overline{C}}$	T4.22: 6, 11
13	$\mathbf{F}(A\overline{B}\overline{A}\overline{B})$	T4.29
14	$\mathbf{F}(B\overline{A}\overline{A}\overline{B})$	CM=: 13
15	$B \supset \overline{\overline{A}\overline{A}\overline{B}}$	LP: 14
16	$\overline{\overline{\overline{A}\overline{A}\overline{B}}} \supset \overline{B}$	T4.26: 15
17	$\overline{\overline{A}\overline{A}\overline{B}} \supset \overline{B}$	DP=: 16
18	$\mathbf{F}(A\overline{C}\overline{A}\overline{C})$	T4.29
19	$\mathbf{F}(C\overline{A}\overline{A}\overline{C})$	CM=: 18
20	$C \supset \overline{\overline{A}\overline{A}\overline{C}}$	LP: 19
21	$\overline{\overline{\overline{A}\overline{A}\overline{C}}} \supset \overline{C}$	T4.26: 20
22	$\overline{\overline{A}\overline{A}\overline{C}} \supset \overline{C}$	DP=: 21
23	$\overline{\overline{A}\overline{A}\overline{B}\overline{A}\overline{A}\overline{C}} \supset \overline{\overline{B} \overline{C}}$	T4.14: 17, 22
24	$\overline{\overline{A}\overline{A}\overline{B}\overline{A}\overline{A}\overline{C}} \overline{\overline{B} \overline{C}} \supset \overline{\overline{B} \overline{C}} \overline{\overline{B} \overline{C}}$	T4.13: 23
25	$\mathbf{F}(\overline{\overline{B} \overline{C}} \overline{\overline{B} \overline{C}})$	T4.29
26	$\mathbf{F}(\overline{\overline{A}\overline{A}\overline{B}\overline{A}\overline{A}\overline{C}} \overline{\overline{B} \overline{C}})$	T4.28: 24, 25
27	$\mathbf{F}(\overline{\overline{A}\overline{A}\overline{B} \overline{C}} \overline{\overline{A}\overline{B} \overline{A}\overline{C}})$	CM=: 26
28	$\mathbf{F}(\overline{\overline{A}\overline{B} \overline{C}} \overline{\overline{A}\overline{B} \overline{A}\overline{C}})$	ID=: 27
29	$\overline{\overline{A}\overline{B} \overline{C}} \supset \overline{\overline{A}\overline{B} \overline{A}\overline{C}}$	LP: 28
30	$\overline{\overline{A}\overline{B} \overline{C}} = \overline{\overline{A}\overline{B} \overline{A}\overline{C}}$	T4.19: 12, 29

4.4. Classical Propositional Logic. In this section, we examine the role played by propositional terms in Leibniz's calculus by exploring the deductive consequences of the principles PP and PC. In what follows, we use ' φ ', ' ψ ', and ' χ ' to stand for arbitrary propositions of Leibniz's calculus.

THEOREM 4.37 (Disquotation). $\mathbf{T}(\ulcorner \varphi \urcorner) \dashv\vdash \varphi$ (§1, §4, §198.6, VI4A 737, C 235, 421)

1	$\ulcorner \mathbf{F}(\ulcorner \varphi \urcorner) \urcorner = \ulcorner \mathbf{F}(\ulcorner \mathbf{F}(\ulcorner \varphi \urcorner) \urcorner) \urcorner$	PP
2	$\overline{\overline{\ulcorner \varphi \urcorner}} = \ulcorner \mathbf{F}(\ulcorner \mathbf{F}(\ulcorner \varphi \urcorner) \urcorner) \urcorner$	PP=: 1
3	$\ulcorner \varphi \urcorner = \ulcorner \mathbf{F}(\ulcorner \mathbf{F}(\ulcorner \varphi \urcorner) \urcorner) \urcorner$	DP=: 2

4	$\lceil \varphi \rceil \supset \lceil \mathbf{F}(\lceil \mathbf{F}(\lceil \varphi \rceil) \rceil) \rceil$	T4.15: 3
5	$\lceil \mathbf{F}(\lceil \mathbf{F}(\lceil \varphi \rceil) \rceil) \rceil \supset \lceil \varphi \rceil$	T4.16: 3

Lines 4 and 5 of this proof establish $\vdash \lceil \varphi \rceil \supset \lceil \mathbf{T}(\lceil \varphi \rceil) \rceil$ and $\vdash \lceil \mathbf{T}(\lceil \varphi \rceil) \rceil \supset \lceil \varphi \rceil$. The desired result follows from these two claims by PC. \square

THEOREM 4.38. $\mathbf{F}(\overline{\lceil \varphi \rceil}) \dashv\vdash \varphi$

For the \vdash direction:

1	$\mathbf{F}(\overline{\lceil \varphi \rceil})$	
2	$\mathbf{F}(\lceil \mathbf{F}(\lceil \varphi \rceil) \rceil)$	PP=: 1
3	φ	T4.37: 2

The proof of the \dashv direction is the reverse of the one just given.

THEOREM 4.39. $\mathbf{T}(\lceil \varphi \rceil) \vdash \mathbf{F}(\overline{\lceil \varphi \rceil})$ (VI4A 143, 804; cf. §42)

1	$\mathbf{F}(\lceil \mathbf{F}(\lceil \varphi \rceil) \rceil)$	
2	$\mathbf{F}(\overline{\lceil \varphi \rceil})$	PP=: 1

While Theorem 4.39 holds for any term of the form $\lceil \varphi \rceil$, it does not hold for terms in general, i.e., it is not the case that $\mathbf{T}(A) \vdash \mathbf{F}(\overline{A})$ for any term A . The converse, however, does hold for any term, as is shown by the following theorem:¹⁴³

THEOREM 4.40. $\mathbf{F}(A) \vdash \mathbf{T}(\overline{A})$

1	$\mathbf{F}(A)$	
2	$A = A$	T4.7
3	$\mathbf{T}(\lceil A = A \rceil)$	T4.37: 2
4	$\mathbf{F}(\overline{\lceil A = A \rceil})$	T4.38: 2
5	$A = \overline{\lceil A = A \rceil}$	T4.32: 1, 4
6	$\overline{A} = \overline{\overline{\lceil A = A \rceil}}$	T4.11: 5
7	$\overline{A} = \lceil A = A \rceil$	DP=: 6
8	$\mathbf{T}(\overline{A})$	SC: 3, 7

¹⁴³ Theorem 4.40 is the first theorem stated in this appendix whose proof makes use of ‘mixed’ propositions (i.e., propositions expressing the coincidence of a non-propositional term and either a propositional term or a Boolean compound of propositional terms). As noted above, it is not entirely clear whether Leibniz intended to include such mixed propositions in the language of his calculus (see nn. 29 and 95). If such propositions were to be excluded from the language, we would need either to posit Theorem 4.40 as a principle of the calculus or to find an alternative way of proving it (e.g., by positing Theorem 4.42 as a principle). Once this has been done, all the results established in this appendix are valid for a restricted version of the language which excludes mixed propositions.

We now show that the method of proof by *reductio* is valid in Leibniz's calculus.

THEOREM 4.41 (Proof by *Reductio*). *If $\varphi \vdash \psi$ and $\varphi \vdash \mathbf{F}(\ulcorner \psi \urcorner)$, then $\vdash \mathbf{F}(\ulcorner \varphi \urcorner)$.*

1	φ	
2	ψ	$\varphi \vdash \psi: 1$
3	$\ulcorner \varphi \urcorner \supset \ulcorner \psi \urcorner$	PC: 1-2
4	φ	
5	$\mathbf{F}(\ulcorner \psi \urcorner)$	$\varphi \vdash \mathbf{F}(\ulcorner \psi \urcorner): 4$
6	$\ulcorner \varphi \urcorner \supset \ulcorner \mathbf{F}(\ulcorner \psi \urcorner) \urcorner$	PC: 4-5
7	$\ulcorner \varphi \urcorner \supset \overline{\ulcorner \psi \urcorner}$	PP=: 6
8	$\ulcorner \varphi \urcorner \supset \ulcorner \psi \urcorner \overline{\ulcorner \psi \urcorner}$	T4.22: 3, 7
9	$\mathbf{F}(\ulcorner \varphi \urcorner)$	T4.30: 8

The following theorem is an example of a claim that can be established by *reductio*:

THEOREM 4.42. $\vdash \mathbf{F}(\ulcorner A = \overline{A} \urcorner)$ (§11, §156, §171.7 in conjunction with §171.1)

1	$A = \overline{A}$	
2	$A \supset \overline{A}$	T4.15: 1
3	$\mathbf{F}(\overline{A})$	SC: 1, 2
4	$\mathbf{T}(\overline{A})$	T4.40: 2
5	$\mathbf{F}(\ulcorner A = \overline{A} \urcorner)$	T4.41: 1-3, 1-4

THEOREM 4.43. $\mathbf{T}(AB) \vdash \mathbf{T}(A)$

1	$\mathbf{F}(\ulcorner \mathbf{F}(AB) \urcorner)$	
2	$\mathbf{F}(A)$	
3	$AB \supset A$	T4.21
4	$\mathbf{F}(AB)$	T4.28: 2, 3
5	$\ulcorner \mathbf{F}(A) \urcorner \supset \ulcorner \mathbf{F}(AB) \urcorner$	PC: 2-4
6	$\mathbf{F}(\ulcorner \mathbf{F}(A) \urcorner)$	T4.28: 1, 5

THEOREM 4.44. $A \supset B, \mathbf{T}(A) \vdash \mathbf{T}(B)$ (§55)

1	$A = AB$
2	$\mathbf{T}(A)$

3	T (AB)	SC: 1, 2
4	T (BA)	CM=: 3
5	T (B)	T4.43: 4

Next we show that the method of proof by cases is valid in Leibniz's calculus:

THEOREM 4.45 (Proof by Cases). *If $\varphi \vdash \psi$ and $\mathbf{F}(\ulcorner \varphi \urcorner) \vdash \psi$, then $\vdash \psi$.*

1	φ	
2	ψ	$\varphi \vdash \psi$: 1
3	$\ulcorner \varphi \urcorner \supset \ulcorner \psi \urcorner$	PC: 1-2
4	$\overline{\ulcorner \psi \urcorner} \supset \overline{\ulcorner \varphi \urcorner}$	T4.26: 3
5	F ($\ulcorner \varphi \urcorner$)	
6	ψ	F ($\ulcorner \varphi \urcorner$) $\vdash \psi$: 5
7	$\ulcorner \mathbf{F}(\ulcorner \varphi \urcorner) \urcorner \supset \ulcorner \psi \urcorner$	PC: 5-6
8	$\overline{\ulcorner \varphi \urcorner} \supset \ulcorner \psi \urcorner$	PP=: 7
9	$\overline{\ulcorner \psi \urcorner} \supset \overline{\overline{\ulcorner \varphi \urcorner}}$	T4.26: 8
10	$\overline{\ulcorner \psi \urcorner} \supset \overline{\ulcorner \varphi \urcorner} \overline{\overline{\ulcorner \varphi \urcorner}}$	T4.22: 4, 9
11	F ($\overline{\ulcorner \psi \urcorner}$)	T4.30: 10
12	ψ	T4.38: 11

The following theorem is an example of a claim that can be established by the method of proof by cases:

THEOREM 4.46. $\mathbf{F}(\ulcorner \ulcorner \varphi \urcorner = \ulcorner \psi \urcorner \urcorner) \vdash \mathbf{F}(\ulcorner \varphi \urcorner \ulcorner \psi \urcorner)$

1	F ($\ulcorner \ulcorner \varphi \urcorner = \ulcorner \psi \urcorner \urcorner$)	
2	F ($\ulcorner \varphi \urcorner \ulcorner \psi \urcorner$)	
3	$\ulcorner \varphi \urcorner \ulcorner \psi \urcorner \supset \ulcorner \ulcorner \varphi \urcorner = \ulcorner \psi \urcorner \urcorner$	T4.31: 2
4	T ($\ulcorner \varphi \urcorner \ulcorner \psi \urcorner$)	
5	T ($\ulcorner \varphi \urcorner$)	T4.43: 4
6	T ($\ulcorner \psi \urcorner \ulcorner \varphi \urcorner$)	CM=: 4
7	T ($\ulcorner \psi \urcorner$)	T4.43: 6
8	F ($\overline{\ulcorner \varphi \urcorner}$)	T4.39: 5
9	F ($\overline{\ulcorner \psi \urcorner}$)	T4.39: 7

10	$\overline{\lceil \varphi \rceil} = \overline{\lceil \psi \rceil}$	T4.32: 8, 9
11	$\overline{\overline{\lceil \varphi \rceil}} = \overline{\overline{\lceil \psi \rceil}}$	T4.11: 10
12	$\lceil \varphi \rceil = \lceil \psi \rceil$	DP=: 11
13	$\mathbf{T}(\lceil \lceil \varphi \rceil = \lceil \psi \rceil \rceil)$	T4.37: 12
14	$\mathbf{F}(\overline{\lceil \lceil \varphi \rceil = \lceil \psi \rceil \rceil})$	T4.39: 13
15	$\overline{\lceil \lceil \varphi \rceil = \lceil \psi \rceil \rceil} \supset \overline{\lceil \varphi \rceil \lceil \psi \rceil}$	T4.31: 14
16	$\overline{\lceil \varphi \rceil \lceil \psi \rceil} \supset \overline{\lceil \lceil \varphi \rceil = \lceil \psi \rceil \rceil}$	T4.26: 15
17	$\lceil \varphi \rceil \lceil \psi \rceil \supset \lceil \lceil \varphi \rceil = \lceil \psi \rceil \rceil$	DP=: 16
18	$\lceil \varphi \rceil \lceil \psi \rceil \supset \lceil \lceil \varphi \rceil = \lceil \psi \rceil \rceil$	T4.45: 2–3, 4–17
19	$\mathbf{F}(\lceil \varphi \rceil \lceil \psi \rceil)$	T4.28: 1, 18

We now show that many important laws applying to propositional terms apply more generally to Boolean compounds of these terms. To this end, we introduce the notion of a generalized propositional term.

DEFINITION 4.47. *A propositional term is a term of the form $\lceil \varphi \rceil$, where φ is any proposition. The generalized propositional terms are defined inductively as follows:*

1. *Every propositional term is a generalized propositional term.*
2. *If A and B are generalized propositional terms, then \overline{A} and AB are generalized propositional terms.*

In what follows, we use ‘ L ’, ‘ M ’, ‘ N ’ to stand for generalized propositional terms.

THEOREM 4.48 (Generalized Propositional Privation). *For any generalized propositional term L : $\vdash \overline{L} = \lceil \mathbf{F}(L) \rceil$.*

Proof. The proof proceeds by induction on the structure of generalized propositional terms. If L is a propositional term, the claim is justified by PP.

Now, suppose that L and M are generalized propositional terms for which the claim holds. In other words, suppose that:

$$\begin{aligned} (\text{PP}_L) \quad & \vdash \overline{L} = \lceil \mathbf{F}(L) \rceil \\ (\text{PP}_M) \quad & \vdash \overline{M} = \lceil \mathbf{F}(M) \rceil. \end{aligned}$$

We first show that the claim holds for \overline{L} , i.e., that $\vdash \overline{\overline{L}} = \lceil \mathbf{F}(\overline{L}) \rceil$:

1	$\lceil \mathbf{F}(\overline{L}) \rceil = \lceil \mathbf{F}(\overline{L}) \rceil$	T4.7
2	$\lceil \mathbf{F}(\lceil \mathbf{F}(L) \rceil) \rceil = \lceil \mathbf{F}(\overline{L}) \rceil$	PP _L =: 1
3	$\overline{\lceil \mathbf{F}(L) \rceil} = \lceil \mathbf{F}(\overline{L}) \rceil$	PP=: 2
4	$\overline{\overline{L}} = \lceil \mathbf{F}(\overline{L}) \rceil$	PP _L =: 3

Finally, we show that the claim holds for LM , i.e., $\vdash \overline{LM} = \ulcorner \mathbf{F}(LM) \urcorner$. It follows from PP_L and PP_M , by the reasoning given in the proofs of T4.37 and T4.46, that:

$$\begin{aligned} \text{(T4.37}_L\text{)} \quad & \vdash L = \ulcorner \mathbf{T}(L) \urcorner \\ \text{(T4.37}_M\text{)} \quad & \vdash M = \ulcorner \mathbf{T}(M) \urcorner \\ \text{(T4.46}_{LM}\text{)} \quad & \mathbf{F}(\ulcorner L = M \urcorner) \vdash \mathbf{F}(LM). \end{aligned}$$

Given these claims, the proof proceeds as follows:

1	$\ulcorner \mathbf{T}(LM) \urcorner \supset \ulcorner \mathbf{T}(L) \urcorner$	T4.43, PC
2	$\ulcorner \mathbf{T}(LM) \urcorner \supset L$	T4.37 _L =: 1
3	$\ulcorner \mathbf{T}(ML) \urcorner \supset \ulcorner \mathbf{T}(M) \urcorner$	T4.43, PC
4	$\ulcorner \mathbf{T}(ML) \urcorner \supset M$	T4.37 _M =: 3
5	$\ulcorner \mathbf{T}(LM) \urcorner \supset M$	CM=: 4
6	$\ulcorner \mathbf{T}(LM) \urcorner \supset LM$	T4.22: 2, 5
7	$\overline{LM} \supset \overline{\ulcorner \mathbf{T}(LM) \urcorner}$	T4.26: 6
8	$\overline{LM} \supset \overline{\overline{\ulcorner \mathbf{F}(LM) \urcorner}}$	PP=: 7
9	$\overline{LM} \supset \ulcorner \mathbf{F}(LM) \urcorner$	DP=: 8
10	$L = M$	
11	$\overline{L} = \ulcorner \mathbf{F}(L) \urcorner$	PP _L
12	$\overline{\overline{L}} = \ulcorner \mathbf{F}(LL) \urcorner$	ID=: 11
13	$\overline{\overline{LM}} = \ulcorner \mathbf{F}(LM) \urcorner$	SC: 10, 12
14	$\ulcorner \mathbf{F}(LM) \urcorner \supset \overline{\overline{LM}}$	T4.16: 13
15	$\mathbf{F}(\ulcorner L = M \urcorner)$	
16	$\mathbf{F}(LM)$	T4.46 _{LM} : 15
17	$LM \supset \overline{\ulcorner \mathbf{F}(LM) \urcorner}$	T4.31: 16
18	$\overline{\overline{\ulcorner \mathbf{F}(LM) \urcorner}} \supset \overline{\overline{LM}}$	T4.26: 17
19	$\ulcorner \mathbf{F}(LM) \urcorner \supset \overline{\overline{LM}}$	DP=: 18
20	$\ulcorner \mathbf{F}(LM) \urcorner \supset \overline{\overline{LM}}$	T4.45: 10–14, 15–19
21	$\overline{\overline{LM}} = \ulcorner \mathbf{F}(LM) \urcorner$	T4.19: 9, 20

□

The following two theorems are corollaries of Theorem 4.48 (by the same reasoning given in the proofs of Theorems 4.37 and 4.39):

THEOREM 4.49. *For any generalized propositional term L : $\vdash L = \ulcorner \mathbf{T}(L) \urcorner$.*

THEOREM 4.50. For any generalized propositional term L : $\mathbf{T}(L) \vdash \mathbf{F}(\overline{L})$.

THEOREM 4.51. For any generalized propositional terms L , M , and N :

$$LM \supset N \dashv\vdash L \supset \lceil M \supset N \rceil.$$

Proof. For the \vdash direction:

1	$LM \supset N$					
2	$\mathbf{F}(LM\overline{N})$	T4.25: 1				
3	$L \supset \overline{\overline{MN}}$	LP: 2				
4	$L \supset \lceil \mathbf{F}(M\overline{N}) \rceil$	T4.48=: 3				
5	<table style="border-collapse: collapse; margin-left: 1em;"> <tr> <td style="border-left: 1px solid black; padding-left: 5px;">$\mathbf{F}(M\overline{N})$</td> <td></td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 5px;">$M \supset N$</td> <td style="padding-left: 20px;">T4.25: 5</td> </tr> </table>	$\mathbf{F}(M\overline{N})$		$M \supset N$	T4.25: 5	
$\mathbf{F}(M\overline{N})$						
$M \supset N$	T4.25: 5					
6	$\lceil \mathbf{F}(M\overline{N}) \rceil \supset \lceil M \supset N \rceil$	PC: 5–6				
7	$L \supset \lceil M \supset N \rceil$	T4.18: 4, 7				

For the \dashv direction:

1	$L \supset \lceil M \supset N \rceil$					
2	<table style="border-collapse: collapse; margin-left: 1em;"> <tr> <td style="border-left: 1px solid black; padding-left: 5px;">$M \supset N$</td> <td></td> </tr> <tr> <td style="border-left: 1px solid black; padding-left: 5px;">$\mathbf{F}(M\overline{N})$</td> <td style="padding-left: 20px;">T4.25: 2</td> </tr> </table>	$M \supset N$		$\mathbf{F}(M\overline{N})$	T4.25: 2	
$M \supset N$						
$\mathbf{F}(M\overline{N})$	T4.25: 2					
3	$\lceil M \supset N \rceil \supset \lceil \mathbf{F}(M\overline{N}) \rceil$	PC: 2–3				
4	$L \supset \lceil \mathbf{F}(M\overline{N}) \rceil$	T4.18: 1, 4				
5	$L \supset \overline{\overline{MN}}$	T4.48=: 5				
6	$\mathbf{F}(LM\overline{N})$	LP: 6				
7	$LM \supset N$	T4.25: 7				

□

THEOREM 4.52. For any generalized propositional terms L and M :

$$\mathbf{T}(L), \mathbf{T}(M) \vdash \mathbf{T}(LM).$$

1	$\mathbf{T}(L)$	
2	$\mathbf{T}(M)$	
3	$\mathbf{F}(\overline{L})$	T4.50: 1
4	$\overline{L} \supset \overline{M}$	T4.31: 3
5	$\overline{\overline{M}} \supset \overline{\overline{L}}$	T4.26: 4
6	$M \supset L$	DP=: 5

7	$M \supset M$	T4.17
8	$M \supset LM$	T4.22: 6, 7
9	$\mathbf{T}(LM)$	T4.44: 2, 8

THEOREM 4.53 (Strong Propositional Containment). *For any set of propositions Γ :*

$$\Gamma \cup \{\varphi\} \vdash \psi \quad \text{iff} \quad \Gamma \vdash \ulcorner \varphi \urcorner \supset \ulcorner \psi \urcorner.$$

Proof. Since derivations are finite, it suffices to establish the claim for finite sets of propositions. Let Γ consist of the propositions χ_1, \dots, χ_n . The proof proceeds by induction on n . If $n = 0$, the above biconditional follows by PC. Now, suppose that $n > 0$ and that the biconditional holds for all $m < n$.

We first prove the left-to-right direction of the biconditional. Suppose:

$$\chi_1, \dots, \chi_n, \varphi \vdash \psi.$$

By T4.43, T4.37, and CM, we have $\mathbf{T}(\ulcorner \chi_n \urcorner \supset \ulcorner \varphi \urcorner) \vdash \chi_n$ and $\mathbf{T}(\ulcorner \chi_n \urcorner \supset \ulcorner \varphi \urcorner) \vdash \varphi$. Hence:

$$\chi_1, \dots, \chi_{n-1}, \mathbf{T}(\ulcorner \chi_n \urcorner \supset \ulcorner \varphi \urcorner) \vdash \psi.$$

By the left-to-right direction of the induction hypothesis:

$$\chi_1, \dots, \chi_{n-1} \vdash \ulcorner \mathbf{T}(\ulcorner \chi_n \urcorner \supset \ulcorner \varphi \urcorner) \urcorner \supset \ulcorner \psi \urcorner.$$

By T4.49 and T4.51:

$$\chi_1, \dots, \chi_{n-1} \vdash \ulcorner \chi_n \urcorner \supset \ulcorner \ulcorner \varphi \urcorner \supset \ulcorner \psi \urcorner \urcorner.$$

And so, by the right-to-left direction of the induction hypothesis, we have:

$$\chi_1, \dots, \chi_n \vdash \ulcorner \varphi \urcorner \supset \ulcorner \psi \urcorner.$$

We next prove the right-to-left direction of the biconditional. Suppose:

$$\chi_1, \dots, \chi_n \vdash \ulcorner \varphi \urcorner \supset \ulcorner \psi \urcorner.$$

By the left-to-right direction of the induction hypothesis:

$$\chi_1, \dots, \chi_{n-1} \vdash \ulcorner \chi_n \urcorner \supset \ulcorner \ulcorner \varphi \urcorner \supset \ulcorner \psi \urcorner \urcorner.$$

By T4.51 and T4.49:

$$\chi_1, \dots, \chi_{n-1} \vdash \ulcorner \mathbf{T}(\ulcorner \chi_n \urcorner \supset \ulcorner \varphi \urcorner) \urcorner \supset \ulcorner \psi \urcorner.$$

And so, by the right-to-left direction of the induction hypothesis:

$$\chi_1, \dots, \chi_{n-1}, \mathbf{T}(\ulcorner \chi_n \urcorner \supset \ulcorner \varphi \urcorner) \vdash \psi.$$

But, by T4.37 and T4.52, we have $\chi_n, \varphi \vdash \mathbf{T}(\ulcorner \chi_n \urcorner \supset \ulcorner \varphi \urcorner)$. Hence:

$$\chi_1, \dots, \chi_n, \varphi \vdash \psi. \quad \square$$

The following two theorems are corollaries of Theorem 4.53 (by the same reasoning given in the proofs of Theorems 4.41 and 4.45):

THEOREM 4.54 (Strong Proof by *Reductio*). *For any set of propositions Γ , if $\Gamma \cup \{\varphi\} \vdash \psi$ and $\Gamma \cup \{\varphi\} \vdash \mathbf{F}(\ulcorner \psi \urcorner)$, then $\Gamma \vdash \mathbf{F}(\ulcorner \varphi \urcorner)$.*

THEOREM 4.55 (Strong Proof by Cases). *For any set of propositions Γ , if $\Gamma \cup \{\varphi\} \vdash \psi$ and $\Gamma \cup \{\mathbf{F}(\neg\varphi)\} \vdash \psi$, then $\Gamma \vdash \psi$.*

Theorems 4.54 and 4.55 play an important role in the completeness proof provided in Section 4.6. In the remainder of the present section, we show that all the laws of classical propositional logic are derivable in Leibniz's calculus.

DEFINITION 4.56. *If Θ is a set of terms, $\mathbf{T}(\Theta)$ is the set of propositions:*

$$\{\mathbf{T}(A) : A \in \Theta\}.$$

For any set of terms Θ and term A , we write $\Theta \vdash_{\mathbf{T}} A$ in place of $\mathbf{T}(\Theta) \vdash \mathbf{T}(A)$.

DEFINITION 4.57. *If L and M are generalized propositional terms, we write $\neg L$ for the term \overline{L} , and $(L \& M)$ for the term LM . We write $(L \rightarrow M)$ for the term $\neg(L \& \neg M)$.¹⁴⁴*

THEOREM 4.58. *For any generalized propositional terms L and M :*

$$\vdash (L \rightarrow M) = \neg L \supset M \neg.$$

1	$\mathbf{F}(\overline{LM})$	
2	$L \supset M$	T4.25: 1
3	$\neg \mathbf{F}(\overline{LM}) \supset \neg L \supset M \neg$	PC: 1-2
4	$L \supset M$	
5	$\mathbf{F}(\overline{LM})$	T4.25: 4
6	$\neg L \supset M \neg \supset \neg \mathbf{F}(\overline{LM}) \neg$	PC: 4-5
7	$\neg \mathbf{F}(\overline{LM}) \neg = \neg L \supset M \neg$	T4.19: 3, 6
8	$\overline{\overline{LM}} = \neg L \supset M \neg$	T4.48=: 7

THEOREM 4.59. *For any generalized propositional terms L, M, N :*

- (i) $L \rightarrow M, L \vdash_{\mathbf{T}} M$
- (ii) $\vdash_{\mathbf{T}} L \rightarrow (L \& L)$
- (iii) $\vdash_{\mathbf{T}} (L \& M) \rightarrow L$
- (iv) $\vdash_{\mathbf{T}} (L \rightarrow M) \rightarrow (\neg(M \& N) \rightarrow \neg(N \& L))$.

Proof.

- (i): Given T4.58, it suffices to show that $\mathbf{T}(\neg L \supset M \neg), \mathbf{T}(L) \vdash \mathbf{T}(M)$. This follows from T4.37 and T4.44.
- (ii): Given T4.58, it suffices to show that $\vdash \mathbf{T}(\neg L \supset LL \neg)$. This follows from ID, T4.16, and T4.37.
- (iii): Given T4.58, it suffices to show that $\vdash \mathbf{T}(\neg LM \supset L \neg)$. This follows from T4.21 and T4.37.

¹⁴⁴ In employing this notation, we omit parentheses where there is no threat of ambiguity.

(iv): Given T4.58 and T4.37, it suffices to show that $\vdash \ulcorner L \supset M \urcorner \supset \ulcorner \overline{MN} \supset \overline{NL} \urcorner$. By PC, this follows from $L \supset M \vdash \overline{MN} \supset \overline{NL}$. This latter claim follows from T4.26, CM, and T4.13. \square

Given the notation introduced in Definition 4.57, generalized propositional terms can be viewed as formulae of a language of propositional logic in which $\&$ and \neg are the only primitive propositional connectives. Claims (i)–(iv) listed in Theorem 4.59 constitute a complete axiomatization of classical propositional logic.¹⁴⁵ Thus, if L is a generalized propositional term and Θ is a set of such terms, then $\mathbf{T}(\Theta) \vdash \mathbf{T}(L)$ whenever L follows from Θ in classical propositional logic. In this sense, Leibniz's calculus is capable of deriving all the laws of classical propositional logic.

4.5. Auto-Boolean Semantics. In this section, we introduce a special class of Boolean interpretations (see Definition 4.33), which we call auto-Boolean interpretations, and show that Leibniz's calculus is sound with respect to this class. We then show that there exist nonstandard (i.e., non-auto-Boolean) models of Leibniz's calculus in which the laws of bivalence and noncontradiction do not hold for propositional terms.

DEFINITION 4.60. *An auto-Boolean interpretation is a Boolean interpretation (\mathfrak{A}, μ) such that $0 \neq 1$ and:*

$$\mu(\ulcorner A = B \urcorner) = \begin{cases} 1 & \text{if } \mu(A) = \mu(B) \\ 0 & \text{otherwise.} \end{cases}$$

If Γ is a set of propositions and φ is a proposition, we write $\Gamma \models \varphi$ to indicate that, for any auto-Boolean interpretation (\mathfrak{A}, μ) : if $(\mathfrak{A}, \mu) \models \psi$ for all $\psi \in \Gamma$, then $(\mathfrak{A}, \mu) \models \varphi$.

The following theorem lists a few straightforward consequences of Definition 4.60:

THEOREM 4.61. *For any auto-Boolean interpretation (\mathfrak{A}, μ) :*

$$\begin{aligned} (\mathfrak{A}, \mu) \models A \supset B & \quad \text{iff} \quad \mu(A) \leq \mu(B) \\ (\mathfrak{A}, \mu) \models \mathbf{F}(A) & \quad \text{iff} \quad \mu(A) = 0 \\ (\mathfrak{A}, \mu) \models \mathbf{T}(A) & \quad \text{iff} \quad \mu(A) \neq 0 \\ (\mathfrak{A}, \mu) \models \mathbf{F}(\ulcorner A = B \urcorner) & \quad \text{iff} \quad \mu(A) \neq \mu(B) \\ (\mathfrak{A}, \mu) \models \mathbf{T}(\ulcorner A = B \urcorner) & \quad \text{iff} \quad \mu(A) = \mu(B). \end{aligned}$$

THEOREM 4.62. *If $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$.*

Proof. It will suffice to show that all the principles of Leibniz's calculus are valid in any auto-Boolean interpretation. This is obvious for ID, DP, CM, SC, and LP.

For PP, since, for any proposition φ , $\mu(\ulcorner \varphi \urcorner) \in \{0, 1\}$, we have:

$$\begin{aligned} \mu(\overline{\ulcorner \varphi \urcorner}) &= \begin{cases} 1 & \text{if } \mu(\ulcorner \varphi \urcorner) = 0 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } \mu(\ulcorner \varphi \urcorner) = \mu(\ulcorner \varphi \urcorner) \wedge \mu(\ulcorner \varphi \urcorner)' \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } \mu(\ulcorner \varphi \urcorner) = \mu(\ulcorner \varphi \urcorner \overline{\ulcorner \varphi \urcorner}) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

¹⁴⁵ This axiomatization of classical propositional logic is given by Rosser 1953: 55–9.

$$\begin{aligned}
&= \mu(\ulcorner \ulcorner \varphi \urcorner = \ulcorner \varphi \urcorner \overline{\ulcorner \varphi \urcorner} \urcorner) \\
&= \mu(\ulcorner \mathbf{F}(\ulcorner \varphi \urcorner) \urcorner).
\end{aligned}$$

For PC, we note that $\varphi \models \psi$ iff, for every auto-Boolean interpretation (\mathfrak{A}, μ) , either $(\mathfrak{A}, \mu) \models \psi$ or $(\mathfrak{A}, \mu) \not\models \varphi$. This means that either $\mu(\ulcorner \psi \urcorner) = 1$ or $\mu(\ulcorner \varphi \urcorner) = 0$. But since $\mu(\ulcorner \psi \urcorner), \mu(\ulcorner \varphi \urcorner) \in \{0, 1\}$, this latter disjunction holds iff $\mu(\ulcorner \varphi \urcorner) \leq \mu(\ulcorner \psi \urcorner)$, i.e., $(\mathfrak{A}, \mu) \models \ulcorner \varphi \urcorner \supset \ulcorner \psi \urcorner$. Hence, $\varphi \models \psi$ iff $\models \ulcorner \varphi \urcorner \supset \ulcorner \psi \urcorner$. \square

It follows from the last two clauses of Theorem 4.61 that, in every auto-Boolean interpretation (\mathfrak{A}, μ) , the following biconditional holds: $(\mathfrak{A}, \mu) \models \mathbf{T}(\ulcorner \varphi \urcorner)$ iff $(\mathfrak{A}, \mu) \not\models \mathbf{F}(\ulcorner \varphi \urcorner)$. As it turns out, however, neither direction of this biconditional is a consequence of the principles of Leibniz's calculus. In other words, for either direction of the biconditional, there are Boolean models of Leibniz's calculus in which the relevant conditional does not hold. For the left-to-right direction, note that Leibniz's calculus is sound with respect to any Boolean interpretation whose domain contains just one element, i.e., in which $0 = 1$. This is so because, in any such interpretation, every proposition is satisfied since every proposition is of the form $A = B$. But this means that both $\mathbf{T}(\ulcorner \varphi \urcorner)$ and $\mathbf{F}(\ulcorner \varphi \urcorner)$ are satisfied for any proposition φ .

The right-to-left direction of the above biconditional asserts the following law of bivalence: either $(\mathfrak{A}, \mu) \models \mathbf{F}(\ulcorner \varphi \urcorner)$ or $(\mathfrak{A}, \mu) \models \mathbf{T}(\ulcorner \varphi \urcorner)$. In the remainder of this section, we prove that this law is independent of the principles of Leibniz's calculus by constructing a nonstandard model in which it fails to hold.

DEFINITION 4.63. *Let \mathfrak{B}_4 be the four-element Boolean algebra freely generated from the single element a . We write v for the (unique) function mapping the terms of Leibniz's calculus to elements of \mathfrak{B}_4 satisfying:*

- (i) (\mathfrak{B}_4, v) is a Boolean interpretation.
- (ii) If A is a simple term, $v(A) = a$.
- (iii) If $v(A) = v(B)$, then $v(\ulcorner A = B \urcorner) = 1$.
- (iv) If $v(A) \neq v(B)$ and $0 \notin \{v(A), v(B)\}$, then $v(\ulcorner A = B \urcorner) = v(A) \wedge v(B)$.
- (v) If $v(A) \neq v(B)$ and $0 \in \{v(A), v(B)\}$, then $v(\ulcorner A = B \urcorner) = v(A)' \wedge v(B)'$.

(For an illustration of clauses (iii)–(v), see Figure 1.) We write $\Gamma \models_v \varphi$ to indicate that, if $(\mathfrak{B}_4, v) \models \psi$ for all $\psi \in \Gamma$, then $(\mathfrak{B}_4, v) \models \varphi$. We write $\models_v \varphi$ to indicate that $\emptyset \models_v \varphi$.

THEOREM 4.64. *For any proposition φ , the following three claims are equivalent:*

- (i) $v(\ulcorner \varphi \urcorner) = 1$
- (ii) $\models_v \varphi$
- (iii) $\models_v \mathbf{T}(\ulcorner \varphi \urcorner)$.

Proof. It follows straightforwardly from Definition 4.63 that $v(\ulcorner A = B \urcorner) = 1$ iff $v(A) = v(B)$. Hence, (i) and (ii) are equivalent. To show the equivalence of (i) and (iii), we note that (iii) holds iff:

$$v\left(\ulcorner \ulcorner \varphi \urcorner = \ulcorner \varphi \urcorner \overline{\ulcorner \varphi \urcorner} \urcorner\right) \leq v\left(\ulcorner \ulcorner \varphi \urcorner = \ulcorner \varphi \urcorner \overline{\ulcorner \varphi \urcorner} \urcorner\right)'.$$

This inequality is satisfied iff $v\left(\ulcorner \ulcorner \varphi \urcorner = \ulcorner \varphi \urcorner \overline{\ulcorner \varphi \urcorner} \urcorner\right) = 0$. But since $v(\ulcorner \varphi \urcorner \overline{\ulcorner \varphi \urcorner}) = 0$, by Definition 4.63, this last condition holds iff $v(\ulcorner \varphi \urcorner) = 1$. \square

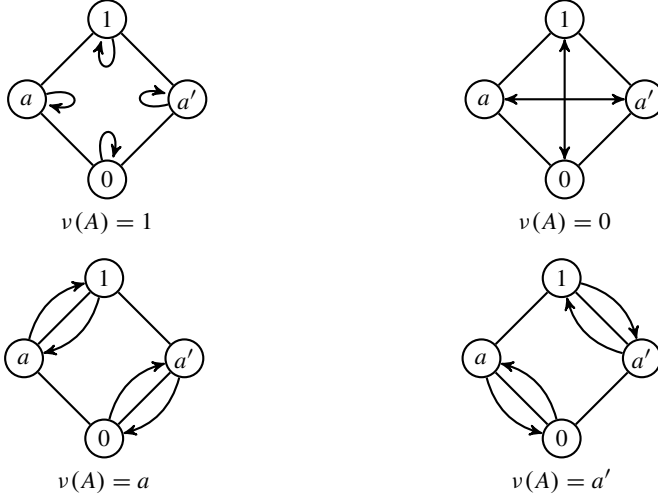


Fig. 1. A graph of the function v illustrating clauses (iii)–(v) of Definition 4.63. The arrows in the diagrams point from the value of $v(B)$ to the value of $v(\ulcorner A = B \urcorner)$, for a fixed value of $v(A)$.

THEOREM 4.65. *If $\Gamma \models_v \ulcorner \varphi \urcorner \supset \ulcorner \psi \urcorner$, then $\Gamma \cup \{\varphi\} \models_v \psi$.*

Proof. Suppose $\models_v \varphi$. By T4.64, $v(\ulcorner \varphi \urcorner) = 1$. Now, suppose $\models_v \ulcorner \varphi \urcorner \supset \ulcorner \psi \urcorner$. It follows that $1 \leq v(\ulcorner \psi \urcorner)$, and so $v(\ulcorner \psi \urcorner) = 1$, which, by T4.64, implies $\models_v \psi$. Hence, $\ulcorner \varphi \urcorner \supset \ulcorner \psi \urcorner \models_v \psi$, which suffices to establish the desired result. \square

DEFINITION 4.66. *Let 0 and 1 be the least and greatest elements of \mathfrak{B}_4 , respectively. For $x \in \{0, 1\}$, we write μ_x for the (unique) function mapping the terms of Leibniz's calculus to elements of \mathfrak{B}_4 satisfying:*

- (i) (\mathfrak{B}_4, μ_x) is an auto-Boolean interpretation.
- (ii) If A is a simple term, $\mu_x(A) = x$.

We write $\Gamma \models_{0-1} \varphi$ to indicate that, for any $x \in \{0, 1\}$, if $(\mathfrak{B}_4, \mu_x) \models \psi$ for all $\psi \in \Gamma$, then $(\mathfrak{B}_4, \mu_x) \models \varphi$. We write $\models_{0-1} \varphi$ to indicate that $\emptyset \models_{0-1} \varphi$.

THEOREM 4.67. *For any term A and $x \in \{0, 1\}$:*

$$\mu_x(A) = \begin{cases} 1 & \text{if } v(A) = 1 \\ 0 & \text{if } v(A) = 0 \\ x & \text{if } v(A) = a \\ x' & \text{if } v(A) = a' \end{cases}$$

Proof. The proof proceeds by induction on the structure of A . If A is a simple term, $v(A) = a$. Thus, since $\mu_x(A) = x$, the claim holds. Now, suppose that the claim holds for A and B . Then:

$$\mu_x(\bar{A}) = \mu_x(A)' = \left(\begin{cases} 1 & \text{if } v(A) = 1 \\ 0 & \text{if } v(A) = 0 \\ x & \text{if } v(A) = a \\ x' & \text{if } v(A) = a' \end{cases} \right)'$$

$$\begin{aligned}
&= \begin{cases} 0 & \text{if } v(A) = 1 \\ 1 & \text{if } v(A) = 0 \\ x' & \text{if } v(A) = a \\ x & \text{if } v(A) = a' \end{cases} = \begin{cases} 0 & \text{if } v(\overline{A}) = 0 \\ 1 & \text{if } v(\overline{A}) = 1 \\ x' & \text{if } v(\overline{A}) = a' \\ x & \text{if } v(\overline{A}) = a \end{cases} \\
\mu_x(AB) = \mu_x(A) \wedge \mu_x(B) &= \left(\begin{cases} 1 & \text{if } v(A) = 1 \\ 0 & \text{if } v(A) = 0 \\ x & \text{if } v(A) = a \\ x' & \text{if } v(A) = a' \end{cases} \right) \wedge \left(\begin{cases} 1 & \text{if } v(B) = 1 \\ 0 & \text{if } v(B) = 0 \\ x & \text{if } v(B) = a \\ x' & \text{if } v(B) = a' \end{cases} \right) \\
&= \begin{cases} 1 & \text{if } v(AB) = 1 \\ 0 & \text{if } v(AB) = 0 \\ x & \text{if } v(AB) = a \\ x' & \text{if } v(AB) = a' \end{cases}
\end{aligned}$$

It only remains to show that the claim holds for the propositional term $\ulcorner A = B \urcorner$. First, suppose $v(\ulcorner A = B \urcorner) = 1$. Then, $v(A) = v(B)$ and so, by the induction hypothesis, $\mu_x(A) = \mu_x(B)$. Hence, since (\mathfrak{B}_4, μ_x) is an auto-Boolean interpretation, $\mu_x(\ulcorner A = B \urcorner) = 1$.

Next, suppose $v(\ulcorner A = B \urcorner) = 0$. Then, $v(A) = v(B)'$ and so, by the induction hypothesis, $\mu_x(A) = \mu_x(B)'$. Hence, since (\mathfrak{B}_4, μ_x) is an auto-Boolean interpretation, $\mu_x(A) \neq \mu_x(B)$ and so $\mu_x(\ulcorner A = B \urcorner) = 0$.

Next, suppose $v(\ulcorner A = B \urcorner) = a$. Then, without loss of generality, we may suppose that one of the following two cases holds: (i) $v(A) = a$ and $v(B) = 1$; or (ii) $v(A) = a'$ and $v(B) = 0$. In case (i), by the induction hypothesis, $\mu_x(A) = x$ and $\mu_x(B) = 1$. Thus, since $x \in \{0, 1\}$ and (\mathfrak{B}_4, μ_x) is an auto-Boolean interpretation, $\mu_x(\ulcorner A = B \urcorner) = x$. In case (ii), by the induction hypothesis, $\mu_x(A) = x'$ and $\mu_x(B) = 0$. Thus, since $x \in \{0, 1\}$ and (\mathfrak{B}_4, μ_x) is an auto-Boolean interpretation, $\mu_x(\ulcorner A = B \urcorner) = x$.

By parallel reasoning, if $v(\ulcorner A = B \urcorner) = a'$, then $\mu_x(\ulcorner A = B \urcorner) = x'$. Thus, the claim holds for $\ulcorner A = B \urcorner$. This completes the proof. \square

THEOREM 4.68. *If $\vdash \mathbf{F}(\ulcorner \varphi \urcorner)$, then $\models_v \mathbf{F}(\ulcorner \varphi \urcorner)$.*

Proof. Suppose $\vdash \mathbf{F}(\ulcorner A = B \urcorner)$. It follows by T4.62 that $\models_{0-1} \mathbf{F}(\ulcorner A = B \urcorner)$. Hence, for every $x \in \{0, 1\}$, $\mu_x(A) \neq \mu_x(B)$. Thus, by T4.67, $v(A) = v(B)'$. By clauses (iv) and (v) of Definition 4.63, $v(\ulcorner A = B \urcorner) = 0$, which implies that $\models_v \mathbf{F}(\ulcorner A = B \urcorner)$. \square

THEOREM 4.69. *If $\Gamma \vdash \varphi$, then $\Gamma \models_v \varphi$.*

Proof. Suppose $\Gamma \vdash \varphi$. Then, since proofs are finite, there are propositions $\psi_1, \dots, \psi_n \in \Gamma$ such that $\psi_1, \dots, \psi_n \vdash \varphi$. So, by strong PC (T4.53) and disquotation (T4.37) we have:

$$\vdash \mathbf{T}(\ulcorner \ulcorner \psi_1 \urcorner \supset \ulcorner \ulcorner \psi_2 \urcorner \supset \dots \supset \ulcorner \ulcorner \psi_n \urcorner \supset \ulcorner \varphi \urcorner \dots \urcorner \urcorner).$$

Since, for any term A , $\mathbf{T}(A)$ is the proposition $\mathbf{F}(\ulcorner \mathbf{F}(A) \urcorner)$, it follows by T4.68 that:

$$\models_v \mathbf{T}(\ulcorner \ulcorner \psi_1 \urcorner \supset \ulcorner \ulcorner \psi_2 \urcorner \supset \dots \supset \ulcorner \ulcorner \psi_n \urcorner \supset \ulcorner \varphi \urcorner \dots \urcorner \urcorner).$$

By T4.64:

$$\models_v \ulcorner \psi_1 \urcorner \supset \ulcorner \ulcorner \psi_2 \urcorner \supset \dots \supset \ulcorner \ulcorner \psi_n \urcorner \supset \ulcorner \varphi \urcorner \dots \urcorner \urcorner.$$

And so, by T4.65, $\psi_1, \dots, \psi_n \models_v \varphi$, from which it follows that $\Gamma \models_v \varphi$. \square

Theorem 4.69 states that the Boolean interpretation (\mathfrak{B}_4, ν) is a model of Leibniz's calculus. Note, however, that, for any simple term A , neither $\models_{\nu} \mathbf{T}(\ulcorner A = A\overline{A} \urcorner)$ nor $\models_{\nu} \mathbf{F}(\ulcorner A = A\overline{A} \urcorner)$. Hence, the law of bivalence for propositional terms does not hold in (\mathfrak{B}_4, ν) .

4.6. The Auto-Boolean Completeness of Leibniz's Calculus. In this section, we prove that Leibniz's calculus is complete with respect to the class of auto-Boolean interpretations introduced in Definition 4.60.

DEFINITION 4.70. A proposition φ is provably false if $\vdash \mathbf{F}(\ulcorner \varphi \urcorner)$. A set of propositions Γ is inconsistent if $\Gamma \vdash \varphi$ for some provably false φ . Γ is consistent if it is not inconsistent.

THEOREM 4.71. If $\Gamma \vdash \varphi$ and $\Gamma \vdash \mathbf{F}(\ulcorner \varphi \urcorner)$, then Γ is inconsistent.

Proof. Let ψ be a proposition of the form $A = \overline{A}$. By T4.42, ψ is provably false. But since $\Gamma \cup \{\mathbf{F}(\ulcorner \psi \urcorner)\} \vdash \varphi$ and $\Gamma \cup \{\mathbf{F}(\ulcorner \psi \urcorner)\} \vdash \mathbf{F}(\ulcorner \varphi \urcorner)$, it follows by strong *reductio* (T4.54) that $\Gamma \vdash \mathbf{T}(\ulcorner \psi \urcorner)$. Hence, by disquotation (T4.37), $\Gamma \vdash \psi$. \square

THEOREM 4.72. $\Gamma \vdash \varphi$ iff $\Gamma \cup \{\mathbf{F}(\ulcorner \varphi \urcorner)\}$ is inconsistent.

Proof. Suppose $\Gamma \vdash \varphi$. Then $\Gamma \cup \{\mathbf{F}(\ulcorner \varphi \urcorner)\} \vdash \varphi$ and $\Gamma \cup \{\mathbf{F}(\ulcorner \varphi \urcorner)\} \vdash \mathbf{F}(\ulcorner \varphi \urcorner)$. Hence, by T4.71, $\Gamma \cup \{\mathbf{F}(\ulcorner \varphi \urcorner)\}$ is inconsistent.

Now, suppose that $\Gamma \cup \{\mathbf{F}(\ulcorner \varphi \urcorner)\}$ is inconsistent. Then $\Gamma \cup \{\mathbf{F}(\ulcorner \varphi \urcorner)\} \vdash \psi$, for some provably false ψ . Since $\vdash \mathbf{F}(\ulcorner \psi \urcorner)$, we have $\Gamma \cup \{\mathbf{F}(\ulcorner \varphi \urcorner)\} \vdash \mathbf{F}(\ulcorner \psi \urcorner)$. Thus it follows by strong *reductio* (T4.54) that $\Gamma \vdash \mathbf{T}(\ulcorner \varphi \urcorner)$. Hence, by disquotation (T4.37), $\Gamma \vdash \varphi$. \square

DEFINITION 4.73. A set of propositions Γ is unsatisfiable if there is no auto-Boolean interpretation (\mathfrak{A}, μ) such that $(\mathfrak{A}, \mu) \models \varphi$ for all $\varphi \in \Gamma$. Γ is satisfiable if it is not unsatisfiable.

THEOREM 4.74. $\Gamma \models \varphi$ iff $\Gamma \cup \{\mathbf{F}(\ulcorner \varphi \urcorner)\}$ is unsatisfiable.

Proof. Suppose that $\Gamma \models \varphi$ and that (\mathfrak{A}, μ) is an auto-Boolean interpretation such that $(\mathfrak{A}, \mu) \models \psi$ for all $\psi \in \Gamma$. Then $(\mathfrak{A}, \mu) \models \varphi$, and so, by T4.61, $\mu(\ulcorner \varphi \urcorner) = 1$. But since $0 \neq 1$ (see Definition 4.60), $\mu(\ulcorner \varphi \urcorner) \neq 0$. Thus, by T4.61, $(\mathfrak{A}, \mu) \not\models \mathbf{F}(\ulcorner \varphi \urcorner)$. Hence, $\Gamma \cup \{\mathbf{F}(\ulcorner \varphi \urcorner)\}$ is unsatisfiable.

Now, suppose that $\Gamma \cup \{\mathbf{F}(\ulcorner \varphi \urcorner)\}$ is unsatisfiable, and that (\mathfrak{A}, μ) is an auto-Boolean interpretation such that $(\mathfrak{A}, \mu) \models \psi$ for all $\psi \in \Gamma$. Then $(\mathfrak{A}, \mu) \not\models \mathbf{F}(\ulcorner \varphi \urcorner)$, and so, by T4.61, $\mu(\ulcorner \varphi \urcorner) \neq 0$. Since $\mu(\ulcorner \varphi \urcorner) \in \{0, 1\}$, it follows that $\mu(\ulcorner \varphi \urcorner) = 1$, and so, by T4.61, $(\mathfrak{A}, \mu) \models \varphi$. Hence, $\Gamma \models \varphi$. \square

Given Theorems 4.72 and 4.74, it suffices to prove completeness (i.e., that $\Gamma \vdash \varphi$ if $\Gamma \models \varphi$) to show that every consistent set of propositions is satisfiable.

DEFINITION 4.75. A set of propositions Γ is maximally consistent if it is both consistent and not a proper subset of any consistent set of propositions.

THEOREM 4.76. If Γ is consistent, then either $\Gamma \cup \{\varphi\}$ or $\Gamma \cup \{\mathbf{F}(\ulcorner \varphi \urcorner)\}$ is consistent.

Proof. Suppose that neither $\Gamma \cup \{\varphi\}$ nor $\Gamma \cup \{\mathbf{F}(\ulcorner \varphi \urcorner)\}$ is consistent. Then, $\Gamma \cup \{\varphi\} \vdash \psi$ and $\Gamma \cup \{\mathbf{F}(\ulcorner \varphi \urcorner)\} \vdash \chi$ for some ψ and χ such that $\vdash \mathbf{F}(\ulcorner \psi \urcorner)$ and $\vdash \mathbf{F}(\ulcorner \chi \urcorner)$. By explosion (T4.31), it follows that $\vdash \ulcorner \psi \urcorner \supset \ulcorner \chi \urcorner$, and so, by PC, $\psi \vdash \chi$. Hence, $\Gamma \cup \{\varphi\} \vdash \chi$, and so, using strong proof by cases (T4.55), we have $\Gamma \vdash \chi$. But since χ is provably false, Γ is inconsistent. \square

THEOREM 4.77. *Every consistent set of propositions is a subset of a maximally consistent set of propositions.*

Proof. Let Γ be a consistent set of propositions, and let $\varphi_1, \varphi_2, \dots$ be an enumeration of all the propositions of Leibniz's calculus. We construct an increasing sequence of sets of propositions as follows:

$$\Gamma_0 = \Gamma$$

$$\Gamma_n = \begin{cases} \Gamma_{n-1} \cup \{\varphi_n\} & \text{if } \Gamma_{n-1} \cup \{\varphi_n\} \text{ is consistent} \\ \Gamma_{n-1} \cup \{\mathbf{F}(\ulcorner \varphi_n \urcorner)\} & \text{otherwise} \end{cases} \quad (n \geq 1).$$

By T4.76, each of the Γ_n 's is consistent. Let $\Gamma_\infty = \bigcup_n \Gamma_n$. Then Γ_∞ is consistent; for otherwise, $\Gamma_\infty \vdash \varphi$ for some provably false φ . But since proofs are finite, this implies that $\Gamma_n \vdash \varphi$, for some finite n , contradicting the fact that each Γ_n is consistent. Moreover, Γ_∞ is maximally consistent. For if $\varphi_n \notin \Gamma_\infty$, it follows by the construction of Γ_∞ that $\mathbf{F}(\ulcorner \varphi_n \urcorner) \in \Gamma_\infty$. But then $\Gamma_\infty \cup \{\varphi_n\}$ is inconsistent by T4.71. \square

THEOREM 4.78. *If Γ is maximally consistent, then either $\varphi \in \Gamma$ or $\mathbf{F}(\ulcorner \varphi \urcorner) \in \Gamma$, but not both.*

Proof. If both $\varphi \in \Gamma$ and $\mathbf{F}(\ulcorner \varphi \urcorner) \in \Gamma$, then, by T4.71, Γ is inconsistent. If neither $\varphi \in \Gamma$ nor $\mathbf{F}(\ulcorner \varphi \urcorner) \in \Gamma$, then, by T4.76, one of these two propositions can be added to Γ without loss of consistency, in which case Γ is not maximally consistent. \square

THEOREM 4.79. *If Γ is maximally consistent and $\Gamma \vdash \varphi$, then $\varphi \in \Gamma$.*

Proof. Suppose $\Gamma \vdash \varphi$ and $\varphi \notin \Gamma$. Then, by T4.78, $\mathbf{F}(\ulcorner \varphi \urcorner) \in \Gamma$, and so $\Gamma \vdash \mathbf{F}(\ulcorner \varphi \urcorner)$. Hence, by T4.71, Γ is inconsistent. \square

In the proofs to follow, we make frequent use of T4.78 and T4.79 without explicit reference to these theorems.

In order to prove completeness, we will show that every maximally consistent set is satisfiable. To construct the requisite model, we introduce the following definitions:

DEFINITION 4.80. *Let Γ be a set of propositions. A term A is Γ -true if $\mathbf{T}(A) \in \Gamma$. A set of terms Φ is Γ -true if every $A \in \Phi$ is Γ -true.*

DEFINITION 4.81. *For any set of terms Φ , $\text{cl}(\Phi)$ is the smallest set containing Φ that is closed under composition, i.e., if $A, B \in \text{cl}(\Phi)$, then $AB \in \text{cl}(\Phi)$.*

THEOREM 4.82. *If Γ is maximally consistent and A is Γ -true, then $\text{cl}(\{A\})$ is Γ -true.*

Proof. Since $\Gamma \vdash \mathbf{T}(A)$ it follows by T4.10 that $\Gamma \vdash \mathbf{T}(A^n)$, for all $n \geq 1$. Thus, by T4.79, $\mathbf{T}(A^n) \in \Gamma$, for all $n \geq 1$. But since $\text{cl}(\{A\}) = \{A^n : n \geq 1\}$, it follows that $\text{cl}(\{A\})$ is Γ -true. \square

THEOREM 4.83. *If Γ is maximally consistent, either $\text{cl}(\{A\})$ or $\text{cl}(\{\overline{A}\})$ is Γ -true.*

Proof. By T4.78, either $\mathbf{T}(A) \in \Gamma$ or $\mathbf{F}(A) \in \Gamma$. But, by T4.40, $\mathbf{F}(A) \vdash \mathbf{T}(\overline{A})$. Hence, by T4.79, either A or \overline{A} (or both) are Γ -true. The desired result follows by T4.82. \square

DEFINITION 4.84. *Let Γ be a set of propositions. A set of terms Φ is a Γ -filter if:*

- (i) Φ is closed under composition; and
- (ii) if $A \in \Phi$ and $A \supset B \in \Gamma$, then $B \in \Phi$.

For any set of terms Φ , $\Gamma(\Phi)$ is the smallest Γ -filter containing Φ .

DEFINITION 4.85. A true Γ -filter is a Γ -filter that is Γ -true. A maximally true Γ -filter is a true Γ -filter that is not a proper subset of any true Γ -filter.

THEOREM 4.86. Let Γ be maximally consistent. If Φ is both Γ -true and closed under composition, then $\Gamma(\Phi)$ is Γ -true.

Proof. Let

$$\Phi' = \bigcup_{A \in \Phi} \{B : A \supset B \in \Gamma\}.$$

We show that Φ' is a true Γ -filter containing Φ . By T4.17, $A \supset A \in \Gamma$, and so $\Phi \subset \Phi'$.

Suppose $A, B \in \Phi'$. Then, for some $C, D \in \Phi$, $C \supset A \in \Gamma$ and $D \supset B \in \Gamma$. Thus, given T4.14, we have $CD \supset AB \in \Gamma$. But since Φ is closed under composition, $CD \in \Phi$. Hence, $AB \in \Phi'$, and so Φ' is closed under composition.

Next, suppose $A \in \Phi'$ and $A \supset B \in \Gamma$. Then, for some $C \in \Phi$, $C \supset A \in \Gamma$. Thus, by T4.18, we have $C \supset B \in \Gamma$. Hence, $B \in \Phi'$.

Lastly, suppose $A \in \Phi'$. Then, for some $B \in \Phi$, $B \supset A \in \Gamma$. But since Φ is Γ -true, $\mathbf{T}(B) \in \Gamma$, and so it follows by T4.44 that $\mathbf{T}(A) \in \Gamma$. Hence, Φ' is Γ -true.

Since Φ' is a Γ -filter containing Φ , $\Gamma(\Phi) \subset \Phi'$. Hence, since Φ' is Γ -true, $\Gamma(\Phi)$ is Γ -true. \square

THEOREM 4.87. Let Γ be maximally consistent. If Φ is a true Γ -filter, then either $\text{cl}(\Phi \cup \{A\})$ or $\text{cl}(\Phi \cup \{\bar{A}\})$ is Γ -true.

Proof. If Φ is the empty set, the desired result follows by T4.83.

Now, let Φ be nonempty and suppose that $\text{cl}(\Phi \cup \{A\})$ is not Γ -true. Then there exists $B \in \text{cl}(\Phi \cup \{A\})$ such that $\mathbf{T}(B) \notin \Gamma$. It follows by the definition of $\text{cl}(\Phi \cup \{A\})$ that B is a term of the form $B_1 B_2 \cdots B_n$, where $B_i \in \Phi \cup \{A\}$ for $1 \leq i \leq n$. At least one of the B_i 's must be the term A . For otherwise each $B_i \in \Phi$ and so, since Φ is closed under composition, $B \in \Phi$, contradicting the assumption that Φ is Γ -true.

There are two cases to consider: first, that all of the B_i 's are the term A ; and, second, that some but not all of the B_i 's are the term A . We will show that, in either case, $\text{cl}(\Phi \cup \{\bar{A}\})$ is Γ -true. In the first case, B is the term A^n . So, by T4.10, $B = A \in \Gamma$. But since $\mathbf{T}(B) \notin \Gamma$, it follows by T4.78 that $\mathbf{F}(B) \in \Gamma$. Hence, $\mathbf{F}(A) \in \Gamma$. Now, choose any $C \in \Phi$. By explosion (T4.31), contraposition (T4.26), and DP, we have $\mathbf{F}(A) \vdash C \supset \bar{A}$, and so $C \supset \bar{A} \in \Gamma$. But since Φ is a Γ -filter, it follows that $\bar{A} \in \Phi$. Hence, $\text{cl}(\Phi \cup \{\bar{A}\}) = \text{cl}(\Phi) = \Phi$, which is Γ -true.

In the second case, in which some but not all of the B_i 's are the term A , it follows by CM and T4.10 that $\vdash B = CA$ for some $C \in \Phi$. Thus, since $\mathbf{F}(B) \in \Gamma$, we have $\mathbf{F}(CA) \in \Gamma$. So, by LP, $C \supset \bar{A} \in \Gamma$. But since Φ is a Γ -filter, it follows that $\bar{A} \in \Phi$. Hence, $\text{cl}(\Phi \cup \{\bar{A}\}) = \text{cl}(\Phi) = \Phi$, which is Γ -true.

We have thus shown that, if $\text{cl}(\Phi \cup \{A\})$ is not Γ -true, $\text{cl}(\Phi \cup \{\bar{A}\})$ is Γ -true. \square

THEOREM 4.88. Let Γ be maximally consistent. If Φ is a maximally true Γ -filter, then either $A \in \Phi$ or $\bar{A} \in \Phi$, but not both.

Proof. Suppose for contradiction that both $A \in \Phi$ and $\bar{A} \in \Phi$. Then, since Φ is closed under composition, $A\bar{A} \in \Phi$. But since Γ is consistent, it follows by T4.29 and T4.71 that $\mathbf{T}(A\bar{A}) \notin \Gamma$. Hence, Φ is not a true Γ -filter.

Now, suppose for contradiction that neither $A \in \Phi$ nor $\bar{A} \in \Phi$. By T4.87, either $\text{cl}(\Phi \cup \{A\})$ or $\text{cl}(\Phi \cup \{\bar{A}\})$ is Γ -true, and so, by T4.86, either $\Gamma(\text{cl}(\Phi \cup \{A\}))$ or $\Gamma(\text{cl}(\Phi \cup \{\bar{A}\}))$

is a true Γ -filter. But since Φ is a proper subset of both of these sets, Φ is not a maximally true Γ -filter. \square

THEOREM 4.89. (Extension Theorem) *If Γ is maximally consistent, then every true Γ -filter is a subset of a maximally true Γ -filter.*

Proof. Let A_1, A_2, \dots be an enumeration of all terms of Leibniz's calculus, and let Φ be a true Γ -filter. We construct an increasing sequence of Γ -filters as follows:

$$\Phi_0 = \Phi$$

$$\Phi_n = \begin{cases} \Gamma(\text{cl}(\Phi_{n-1} \cup \{A_n\})) & \text{if } \text{cl}(\Phi_{n-1} \cup \{A_n\}) \text{ is } \Gamma\text{-true} \\ \Gamma(\text{cl}(\Phi_{n-1} \cup \{\overline{A_n}\})) & \text{otherwise} \end{cases} \quad (n \geq 1).$$

By T4.86 and T4.87, it follows that each Φ_n is a true Γ -filter. Let $\Phi_\infty = \bigcup_n \Phi_n$. Since the union of an increasing sequence of true Γ -filters is a true Γ -filter, Φ_∞ is a true Γ -filter. Moreover, Φ_∞ is a maximally true Γ -filter since every Γ -filter that is a proper superset of Φ_∞ contains terms A and \overline{A} , and no such set is Γ -true. \square

THEOREM 4.90. *If Γ is maximally consistent and A is Γ -true, then A is a member of a maximally true Γ -filter.*

Proof. If A is Γ -true, then, by T4.82 and T4.86, $\Gamma(\text{cl}(\{A\}))$ is a true Γ -filter. But since $A \in \Gamma(\text{cl}(\{A\}))$, the desired result follows from the extension theorem (T4.89). \square

We are now in a position to construct the canonical model required for the proof of the completeness theorem:

DEFINITION 4.91. *For any set of propositions Γ , we write Ω_Γ for the set of all maximally true Γ -filters; \mathfrak{A}_Γ for the power set algebra on Ω_Γ ; and μ_Γ for the function mapping the terms of Leibniz's calculus to elements of \mathfrak{A}_Γ as follows:*

$$\mu_\Gamma(A) = \{\Phi \in \Omega_\Gamma : A \in \Phi\}.$$

THEOREM 4.92. *If Γ is maximally consistent, then $\mu_\Gamma(A) = \mu_\Gamma(B)$ iff $A = B \in \Gamma$.*

Proof. First, suppose $A = B \in \Gamma$. By T4.15 and T4.16, it follows that $A \supset B \in \Gamma$ and $B \supset A \in \Gamma$. Hence, for every Γ -filter Φ , $A \in \Phi$ iff $B \in \Phi$, and so $\mu_\Gamma(A) = \mu_\Gamma(B)$.

Now, suppose $A = B \notin \Gamma$. By T4.19, either $A \supset B \notin \Gamma$ or $B \supset A \notin \Gamma$. Suppose, without loss of generality, that $A \supset B \notin \Gamma$. By T4.25, $\mathbf{F}(A\overline{B}) \notin \Gamma$, and so $\mathbf{T}(A\overline{B}) \in \Gamma$, i.e., $A\overline{B}$ is Γ -true. Thus, by T4.90, there is a maximally true Γ -filter Φ such that $A\overline{B} \in \Phi$. This implies, by T4.21 and T4.20, that $A \in \Phi$ and $\overline{B} \in \Phi$. But, by T4.88, if $\overline{B} \in \Phi$, then $B \notin \Phi$. Hence, $\mu_\Gamma(A) \neq \mu_\Gamma(B)$. \square

THEOREM 4.93. *Every maximally consistent set of propositions is satisfiable.*

Proof. Let Γ be a maximally consistent set of propositions. We will show that $(\mathfrak{A}_\Gamma, \mu_\Gamma)$ is an auto-Boolean interpretation that satisfies every proposition in Γ .

To show that $(\mathfrak{A}_\Gamma, \mu_\Gamma)$ is an auto-Boolean interpretation (as specified in Definition 4.60), we must first show that \mathfrak{A}_Γ is a Boolean algebra in which $0 \neq 1$, i.e., that Ω_Γ is nonempty. By T4.90, it suffices to show that there is at least one Γ -true term. Since, by ID and T4.37, $\mathbf{T}(\ulcorner A = AA \urcorner) \in \Gamma$, $\ulcorner A = AA \urcorner$ is such a term.

It remains to verify the following three clauses:

- (i) $\mu_\Gamma(\overline{A}) = \mu_\Gamma(A)'$
- (ii) $\mu_\Gamma(AB) = \mu_\Gamma(A) \wedge \mu_\Gamma(B)$
- (iii) $\mu_\Gamma(\ulcorner A = B \urcorner) = \begin{cases} 1 & \text{if } \mu_\Gamma(A) = \mu_\Gamma(B) \\ 0 & \text{otherwise.} \end{cases}$

(i) follows from T4.88. (ii) follows from the fact that, by T4.21 and T4.20, for any maximally true Γ -filter Φ , $AB \in \Phi$ iff both $A \in \Phi$ and $B \in \Phi$. In order to verify (iii), given T4.92, it will suffice to show:

$$\mu_\Gamma(\ulcorner A = B \urcorner) = \begin{cases} 1 & \text{if } A = B \in \Gamma \\ 0 & \text{otherwise.} \end{cases}$$

First, suppose $A = B \in \Gamma$. By T4.38, $\mathbf{F}(\overline{\ulcorner A = B \urcorner}) \in \Gamma$. By T4.78, it follows that $\mathbf{T}(\ulcorner A = B \urcorner) \notin \Gamma$. But then $\ulcorner A = B \urcorner$ is not in any maximally true Γ -filter, i.e., $\mu_\Gamma(\ulcorner A = B \urcorner) = 0$. By T4.92 and DP, $\mu_\Gamma(\ulcorner A = B \urcorner) = \mu_\Gamma(\overline{\overline{\ulcorner A = B \urcorner}})$, and so, by (i):

$$\mu_\Gamma(\ulcorner A = B \urcorner) = \mu_\Gamma(\overline{\overline{\ulcorner A = B \urcorner}}) = \mu_\Gamma(\ulcorner A = B \urcorner)' = 0' = 1.$$

Now, suppose $A = B \notin \Gamma$. By T4.37, $\mathbf{T}(\ulcorner A = B \urcorner) \notin \Gamma$, i.e., $\ulcorner A = B \urcorner$ is not Γ -true. But this means that $\mu_\Gamma(\ulcorner A = B \urcorner) = 0$.

This completes the proof that $(\mathfrak{A}_\Gamma, \mu_\Gamma)$ is an auto-Boolean interpretation. It only remains to show that $(\mathfrak{A}_\Gamma, \mu_\Gamma) \models \varphi$, for every $\varphi \in \Gamma$. This follows directly from T4.92 and the definition of satisfaction. \square

Given Theorems 4.77 and 4.93, every consistent set of propositions is satisfiable. This fact, in conjunction with Theorems 4.72 and 4.74, implies the completeness theorem:

THEOREM 4.94 (Completeness). *If $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.*

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