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A Reconstruction of Aristotle's Modal Syllogistic

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Ever since Łukasiewicz, it has been *opinio communis* that Aristotle's modal syllogistic is incomprehensible due to its many faults and inconsistencies, and that there is no hope of finding a single consistent formal model for it. The aim of this paper is to disprove these claims by giving such a model.

My main points shall be, first, that Aristotle's syllogistic is a pure term logic that does not recognize an extra syntactic category of individual symbols besides syllogistic terms and, second, that Aristotelian modalities are to be understood as certain relations between terms as described in the theory of the predicables developed in the Topics. Semantics for modal syllogistic is to be based on Aristotelian genus-species trees. The reason that attempts at consistently reconstructing modal syllogistic have failed up to now lies not in the modal syllogistic itself, but in the inappropriate application of modern modal logic and extensional set theory to the modal syllogistic.

After formalizing the underlying predicable-based semantics (Section 1) and having defined the syllogistic propositions by means of its term logical relations (Section 2), this paper will set out to demonstrate in detail that this reconstruction yields all claims on validity, invalidity and inconclusiveness that Aristotle maintains in the modal syllogistic (Section 3 and 4).

1. Predicable-based semantics for modal syllogistic

Ten years ago, R. Smith aptly summed up the state of research on Aristotle's modal syllogistic:

> In recent years, interpreters have expended enormous energy in efforts to find some interpretation of the modal syllogistic that is consistent and nevertheless preserves all (or nearly all) of Aristotle's results; generally, the outcomes of such attempts have been disappointing. I believe this simply confirms that Aristotle's system is incoherent and that no amount of tinkering can rescue it.
>
> Smith *1995*:45

This is the prevailing view on Aristotle's modal syllogistic even today.[2] Although adequate reconstructions of Aristotle's apodeictic syllogistic (An. pr. A 8-12) have been given by Johnson (*1989*), Thomason (*1993*, *1997*) and Brenner (*2000*), it is far from obvious how to extend these interpretations so as to consistently cover the whole of modal syllogistic developed in An. pr. A 8-22. Indeed, the three recent studies offering a comprehensive formal investigation of the whole modal syllogistic accuse Aristotle either of committing a number of logical mistakes

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[2] That Aristotle's modal syllogistic contains errors, hidden ambiguities or inconsistencies which make it impossible to give a single adequate formal reconstruction for it is held, for instance, by Becker (*1933*, pp. 72f), Bocheński (*1956*, p. 101), Łukasiewicz (*1957*, p. 133), Kneale (*1962*, p. 91), Hintikka (*1973*, p. 141), van Rijen (*1989*, p. 198), Striker (*1994*, p. 39), Patterson (*1995*, pp. 23–30, 168–185, 194–198), Nortmann (*1996*, pp. 133, 266–282, 376), Thom (*1996*, pp. 123–149), Brennan (*1997*, pp. 229f), Schmidt (*2000*, pp. 29–40).

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(Nortmann *1996*, Thom *1996*) or of switching between several kinds of modal propositions and thereby failing to develop a coherent modal logic (Schmidt *2000*).[3]

The major aim of this paper is to overcome these difficulties and to provide a single formal model that exactly captures Aristotle's claims on (in)validity and inconclusiveness in the whole modal syllogistic. This model is intended to be not without a certain explanatory value for our understanding of why Aristotle's modal syllogistic looks the way it does; but in the following we shall focus on the logical reconstruction and sketch the explanatory background only in a cursory manner. Whereas the reconstructions mentioned above rest on modern modal predicate logic and set theory,[4] we shall prefer a kind of mereological framework based on the following ideas about the logical form of modal propositions.

Syllogistic propositions (*protaseis*) have a tripartite structure consisting of a copula (*to einai ē mē einai*, An. pr. 24b16-18), a subject term and a predicate term. The copulae of the modal syllogistic differ from each other in quantity, quality and modality. With respect to quantity and quality, these are universal or particular and affirmative or negative, and with respect to modality they are necessary ($\mathbb{N}$), assertoric ($\mathbb{X}$), one-way possible ($\mathbb{M}$) or two-way possible ($\mathbb{Q}$). There is, for instance, a particular negative necessity copula and a universal affirmative assertoric copula. Although there are well-known logical relations between the copulae of modal syllogistic (e.g. contrarity, subalternation, implication, etc.), each of them has to be taken as an independent relation between terms in its own right, and cannot be reduced to other copulae by adding or omitting sentential operators of negation and modality.

Logically, necessary and possible propositions are not obtained by enriching assertoric propositions with modal sentential operators, but by applying a necessity copula (*to ex anagkēs hyparchein ē mē hyparchein*, An. pr. 30a1f) or a possibility copula instead of an assertoric one to two terms. This is not to deny that linguistically, the formulations of necessity propositions can be obtained from the formulations of assertoric propositions in Ancient Greek by adding modal expressions such as *ex anagkēs*. These expressions, however, do not correspond to independent sentential operators, but are to be seen as modifiers of the copula.[5] Aristotelian modality is an integrated and inseparable part of the copula. Copula-modalities are applied to two terms and yield a proposition, with the result that they cannot be iterated as they yield an entity of entirely different type than the two entities they are applied to. The modal operators of modern modal predicate logic, on the other hand, yield things of the same type they are applied to, viz. (open or closed) well-formed formulae, and can therefore be iterated. In Aristotle's modal syllogistic, however, there are no iterable modal sentential operators and, as a consequence, no *de dicto* modalities.

In the assertoric syllogistic, Aristotle only takes the quantity and quality of propositions into account; these only depend on whether a predicative relation holds between certain terms or not. In the modal syllogistic, he also takes into account the modality of propositions, which depends on the way in which a predicative relation holds or fails to hold. Thus, Aristotle needs to distinguish between several kinds of

[3] See Nortmann (*1996*, pp. 133, 266–282, 376), Thom (*1996*, pp. 123–149) and Schmidt (*2000*, pp. 29–40).

[4] The interpretations of Nortmann (*1996*), Schmidt (*2000*) and Brenner (*2000*) are carried out within modern modal predicate logic, whereas Johnson (*1989*), Thomason (*1993*, *1997*) and Thom (*1996*) draw heavily on modern set theory.

[5] See De int. 21b26-30 and Whitaker (*1996*, p. 159).

predicative relations in the modal syllogistic. These kinds of predicative relations, in turn, are investigated within the theory of the predicables developed in the Topics, thereby providing the semantic foundation of the modal syllogistic.

This is the picture of the modal syllogistic put forward by R. Patterson (*1995*, pp. 15–41), which this paper will adopt as a starting point. However, the following formalization will differ from Patterson's semi-formal interpretation in the way this picture is carried out, most notably in giving up the logical distinction between syllogistic terms and zero-order individuals and thereby avoiding Patterson's ambiguity between what he calls 'weak' and 'strong' modalities (see p. 111). Nevertheless, Patterson's idea of a predicable-based modal copula is taken as a guiding principle of this paper:

> [...] what are we to do [...] about determining which syllogisms or conversion principles are rightly taken as valid and primary? The answer is that we can consult the underlying relations among genus, species, accident and proprium for which the modal system is supposed to provide a logical calculus.
>
> Patterson *1995*, pp. 48f

Every (universal affirmative) syllogistic proposition (*protasis*) expresses one of the four predicables: genus with *differentia*, definition, *proprium* or accident (Top. A 4 101b15-19). Aristotle proves this claim by reducing the four predicables to two basic relations (Top. A 8 103b6-19). If *a* is predicated of *b*, then *b* can either be predicated of *a* or not. In the first case, *a* is the definition or *proprium* of *b*, in the latter, it is genus or accident. Moreover, either *a* is predicated essentially of *b* or not. In the first case, *a* is the definition or genus of *b*, in the latter, its *proprium* or accident. Thus, the predicables can be reduced to the two basic relations of essential predication $\mathbf{E}ab$ and plain accidental predication Υab, to be understood in the inclusive sense of 'accidental' as including also essential and necessary predications.[6]

In Top. A 9 103b27-35 Aristotle points out that there are to be distinguished several kinds of essential predication (*to ti esti*) with respect to the ten categories (see also An. pr. A 37). For instance, 'animal' is essentially said of 'man' within the category of substance, whereas 'colour' is essentially said of 'white' or 'whiteness' within the category of quality and 'moving' of 'walking' within the category of, say, activity. Contrary to Patterson (*1995*:41), the distinction between the ten categories, especially between substantial and non-substantial essential predication, is of importance also for the logical purposes of modal syllogistic. This can be seen, for instance, from Theophrastian counterexamples against Aristotle's valid syllogism aaa-1-NXN relying on a non-substantial essential predication in the major premiss (see Alexander in An. Pr. 124.28-30): if 'moving' is predicated essentially (necessarily) of 'walking' and 'walking' is predicated of 'man,' then by aaa-1-NXN we have that 'moving' should be predicated essentially (necessarily) of 'man,' which is false.

With substantial essential predications in the major premiss, on the other hand, these difficulties do not arise. Given, for example, the major premiss that 'animal' is predicated essentially of 'mammal' within the category of substance, the substance term 'mammal' can be predicated only of substantial species such as 'horse' and 'man' of which both the middle term 'mammal' and the major term 'animal' are

6 In this paper, the symbol Υ will be used for transitive universal predication rather than for indeterminate predication, which is not transitive.

predicated essentially (so-called unnatural predications being excluded in the minor premiss, see p. 101). By contrast, in Theophrastus' counterexample – at first glance – nothing prevents the non-substantial middle term 'walking' from being predicated of a substantial species of which neither 'walking' itself nor the major term 'moving' is predicated essentially.

Aristotle's examples of universal affirmative necessities in An. pr. A 1-22 include both substantial essential predication ('animal' is predicated necessarily of 'man' in 30a30, etc.) and non-substantial essential predication ('moving' is predicated necessarily of 'awake' in 38a42f). Therefore we cannot simply rule out non-substantial essential predication in order to preserve the validity of aaa-1-NXN, but have to admit non-substantial as well as substantial essential predication and have to look for another way to handle Theophrastian counterexamples against aaa-1-NXN (see p. 103). For simplicity, we will neglect the distinction between the several non-substantial categories and distinguish only between substantial and non-substantial essential predication, the first being symbolized by $\mathbf{E}ab$ and the latter by $\widetilde{\mathbf{E}}ab$.

Thus, the following formalization will rest on three primitive relations $\boldsymbol{\Upsilon}$, $\mathbf{E}$ and $\widetilde{\mathbf{E}}$ of plain accidental predication, substantial essential predication and non-substantial essential predication. We will distinguish between the level of predicable-semantics formulated in terms of the three primitive relations on the one hand, and between the level of syllogistic propositions such as $\mathbb{X}^{\mathbf{o}}ab$, $\mathbb{N}^{\mathbf{a}}ab$, $\mathbb{Q}^{\mathbf{e}}ab$ etc. on the other hand. The first provides, so to speak, the semantic truth conditions for the latter. For the sake of formal simplicity, syllogistic propositions will be treated as abbreviations of semantic relations within the object language of predicable-semantics.[7]

In the remainder of this section, the primitive relations will be implicitly characterized by definitions and axioms, constituting an axiomatic system referred to as $\mathcal{A}$. The definitions and axioms of $\mathcal{A}$ are not intended to give an exhaustive description of Aristotelian predicable-semantics, but to capture only those aspects of it which are relevant for the formal proofs of modal syllogistic. There will be short comments explaining the formalizations that $\mathcal{A}$ rests on. Naturally enough, these comments cannot give a full and detailed justification of $\mathcal{A}$, but are only to provide a broad understanding of how $\mathcal{A}$ is related to Aristotelian predicable-semantics and how this works.

To start with, a term a will be said to belong to the category of substance (i.e. a is a substance term) if there is a term z that is $\mathbf{E}$-predicated of a:

$$(\mathrm{df}_1) \quad \Sigma a =_{df} \exists z \mathbf{E}za$$

$\mathbf{E}$ has been introduced as the relation of substantial essential predication providing (parts of) the essence or definition of substance terms. Therefore every term that is subject of an $\mathbf{E}$-predication should be expected to be a substance term. On the other hand, not every term that is predicate of an $\mathbf{E}$-predication is a substance term since *differentiae*, which are not substance terms,[8] are also essentially predicated of substance terms (Top. 153a16-18, 154a27f). This is in accordance with Aristotle's examples of substantial universal affirmative necessities, whose subject is always a

7 This strategy is very like resolving, for instance, the modal operators of modern modal logic into non-modal first-order logic by taking them as abbreviations for their truth conditions within possible-world semantics.

8 For the non-substantiality of *differentiae* see Top. 122b16f, 128a26-28, 144a18-22, Met. Δ 1020a33-b2, 1024b4-6, **K** 1068b19, Phys. 226a28.

substance term such as 'swan' or 'snow,' but whose predicate term can be either a substantial genus such as 'animal' (30a30, etc.) or a non-substantial *differentia* (or inseparable accident) such as 'white' (36b11, 38a31f, 38b20). We shall not commit ourselves as to whether 'white' is to be regarded as a *differentia* or as an inseparable accident in these examples; the only thing that is important for present purposes is the fact that non-substance terms can also be predicated necessarily of substance terms. In the same way we leave it undecided whether Aristotle wished to include *propria* among the necessary propositions of modal syllogistic. For terminological ease we will continue referring to the **E**-relation as essential predication, although *propria* and inseparable accidents are not predicated essentially in the strict sense. The following formalization, however, is perfectly compatible with a broader understanding of the **E**-relation including also inseparable accidents and *propria* if one wants these to be embraced in the intended interpretation of **E**.

Secondly, two substance terms *a* and *b* will be referred to as incompatible (**K***ab*) if they are completely disjunct, that means if there is no common term of which both of them are predicated. This **K**-incompatibility relation holds, for instance, between substance terms such as 'man' and 'horse,' and it will be useful for defining negative necessity copulae.

$$(\mathrm{df}_2)\quad \mathbf{K}ab =_{df} \Sigma a \wedge \Sigma b \wedge \neg\exists z(\Upsilon az \wedge \Upsilon bz)$$

Defining possibility copulae will require a symmetric (two-way) contingency relation Πab to be seen as stating that *a* may and may not be predicated of *b*, and vice versa. There are many ways that such a contingency relation could be expressed by the primitive relations of $\mathcal{A}$; at first glance, it is not clear which one should be preferred. However, the following definition turns out to yield a formally useful notion of contingency, requiring that not both *a* and *b* be substance terms, that neither of them be **E**-predicated of the other and that whenever one of them is a substance term, there be a term *z* of which both of them are Υ-predicated:

$$(\mathrm{df}_3)\quad \Pi ab =_{df} \neg(\Sigma a \wedge \Sigma b) \wedge \neg\mathbf{E}ab \wedge \neg\mathbf{E}ba \wedge ((\Sigma a \vee \Sigma b) \supset \exists z(\Upsilon az \wedge \Upsilon bz))$$

The first condition $\neg(\Sigma a \wedge \Sigma b)$ is natural as two-way contingency cannot hold between two substance terms; the second condition $\neg\mathbf{E}ab \wedge \neg\mathbf{E}ba$ is equally natural as contingency cannot obtain between any two terms one of which is essentially predicated of the other within the category of substance (e.g. 'white' and 'swan'). The third condition $(\Sigma a \vee \Sigma b) \supset \exists z(\Upsilon az \wedge \Upsilon bz)$ ensures that if the contingency relation holds between two terms one of which is a substance term, these must not be disjunct, but must 'overlap.' In other words, whenever the contingent possibility relation Π holds between a substantial genus such as 'animal' and a non-substance term such as 'sleeping,' this possibility must be partly 'realized' in that 'sleeping' is predicated of some entity[9] that 'animal' applies to.[10]

[9] According to (df_3), this entity is a term of the same logical type as the genus 'animal.' This is not to rule out that, intuitively, it can be understood as an individual. We shall not commit ourselves as to how exactly the atomic terms of $\mathcal{A}$ are to be understood (e.g. infima species, individual essence, full-blown individual). This would require a detailed study of philosophical questions about atomic terms, which is beyond the scope of this paper. Note, however, that the third condition of (df_3) makes Πab collapse to plain predication Υab if *b* is an atomic substance term. That means, (df_3) does not seem to yield an intuitively convincing notion of contingency for atomic substance terms. We shall not put technical effort into avoiding these consequences as this would not pertain directly to the formal reconstruction of

Finally, there are four abbreviations in (df_{4-7}). $\overline{\Pi}ab$ is an affirmative one-way possibility relation stating that either the contingency relation holds between a and b, or a is $\boldsymbol{\Upsilon}$-predicated of b. $\widehat{\mathbf{E}}ab$ means that a is essentially predicated of b, either within the category of substance or within one of the non-substantial categories. By analogy, $\widehat{\Sigma}a$ means that a is subject of at least one substantial or non-substantial essential predication. $\overline{\mathbf{E}}$ is a slightly weakened version of $\mathbf{E}$, which will be convenient for defining negative one-way possibility.

(df_4) $\overline{\Pi}ab =_{df} \Pi ab \vee \boldsymbol{\Upsilon}ab$
(df_5) $\widehat{\mathbf{E}}ab =_{df} \mathbf{E}ab \vee \widetilde{\mathbf{E}}ab$
(df_6) $\widehat{\Sigma}a =_{df} \exists z \widehat{\mathbf{E}}za$
(df_7) $\overline{\mathbf{E}}ab =_{df} \mathbf{E}ab \vee (\Sigma a \wedge \boldsymbol{\Upsilon}ab)$

There are five simple axioms governing the primitive and derived relations of $\mathcal{A}$. Firstly, plain accidental predication $\boldsymbol{\Upsilon}$ is reflexive and transitive.[11] Moreover, both substantial and and non-substantial essential predication implies plain accidental predication (from $\widehat{\mathbf{E}}ab$ follows $\boldsymbol{\Upsilon}ab$):

(ax_1) $\boldsymbol{\Upsilon}aa$
(ax_2) $\boldsymbol{\Upsilon}ab \wedge \boldsymbol{\Upsilon}bc \supset \boldsymbol{\Upsilon}ac$
(ax_3) $\widehat{\mathbf{E}}ab \supset \boldsymbol{\Upsilon}ab$

The two remaining axioms are crucial. (ax_4) introduces downward monotonicity (transparency) of $\mathbf{E}$ with respect to $\boldsymbol{\Upsilon}$ in the right argument, which is needed to account for the validity of syllogisms such as aaa-1-NXN:

(ax_4) $\mathbf{E}ab \wedge \boldsymbol{\Upsilon}bc \supset \mathbf{E}ac$

modal syllogistic. According to An. pr. 43a40-43, atomic terms lie beyond the main concerns of the syllogistic anyway. In this paper, we shall adopt the view that in $\mathcal{A}$ there are some atomic terms below infimae species such as 'man' and 'horse.' These atomic terms, however, do not correspond to the class of all individual men or horses in a certain situation of the world, but have a merely technical function in that they ensure the third condition of (df_3) if the contingent possibility Π holds between an infima species and an accident ('man' – 'sleeping').

10 This can be seen as a kind of atemporal version of the species-relative principle of plenitude stated by Hintikka (*1973*, pp. 100f): If 'being cut up' possibly applies to the substantial species 'cloak' and if, as a result, it is possible for this cloak to be cut up, then this possibility need not be realized with respect to this individual cloak, but with respect to another individual cloak of the same species. Or in Aristotle's terms: if some matter (*hylē*) possibly is a man, then this possibility must be already realized not with respect to this very same matter, but with respect to another piece of matter, i.e. there must already exist another individual of the same species 'man' (Met. Θ 8 1049b17-27). The third condition of (df_3) can also be regarded as an implicit characterization of the assertoric universal negative proposition $\forall z(\boldsymbol{\Upsilon}az \supset \neg\boldsymbol{\Upsilon}bz)$. For instance, if 'walking' is possibly predicated of the substance term 'raven' in the minor premiss, the 'instable' universal negative proposition that 'raven' applies to no 'walking' or 'bird' applies to no 'walking' is ruled out and cannot be used as a major premiss. This ensures the validity of notorious syllogisms such as eae-1-XQM by excluding counterexamples such as 'walking-bird-raven,' given by Patterson (*1995*, p. 184). Certainly, even though 'walking' is possibly predicated of 'raven,' there may very well be situations in the world in which no individual raven is walking. However, syllogistic propositions do not have extensional truth conditions bound to classes of individuals in a certain situation of the world. Their truth conditions are given by predicable-based term logic in which 'instable' assertoric universal negative propositions and 'unrealized possibilities' as the above are not admissible.

11 For the reflexivity of $\boldsymbol{\Upsilon}$, see for example An. pr. 68a19f. For transitivity, see for example An. pr. 25b32-35, 44b4f, An. post. 96a12-15, Top. 122a35-37.

Intuitively, (ax_4) requires that substance terms be **ϒ**-predicated of nothing but their subspecies. If *a* is **E**-predicated of *b*, then *b* must be a substance term by (df_1), and whenever a substance term is **ϒ**-predicated of a *c*, this must be one of its substantial subspecies, of which both *a* and *b* are **E**-predicated. To put it in the terminology of Cat. 3a7f,[12] substance terms (substances) are never predicated merely accidentally (*en hypokeimenōi einai*), but always essentially (*kath' hypokeimenou legesthai*).[13] According to this line of reasoning, (ax_4) could be strengthened so that *b* too is required to be **E**-predicated of *c*:

(1) $\mathbf{E}ab \wedge \Upsilon bc \supset (\mathbf{E}ac \wedge \mathbf{E}bc)$

Although (1) seems a valid principle of predicable-semantics as well as (ax_4), and although we will not make use of any structures violating (1) throughout this paper, we shall not take it as an axiom, but shall confine ourselves to the weaker (ax_4) since (1) is not of direct relevance for the formal proofs of the modal syllogistic.[14] Nevertheless, (ax_4) is strong enough to yield several interesting consequences. First, (ax_4) together with (ax_3) implies transitivity of **E**-predication.[15] Moreover, (ax_4) together with (ax_2) ensures downward monotonicity (transparency) of the incompatibility relation **K** with respect to **ϒ**, which is needed to validate syllogisms such as eae-1-NXN:[16]

(2) $\mathbf{K}ab \wedge \Upsilon bc \supset \mathbf{K}ac$

A further consequence of (ax_4) is the fact that substance terms can be **ϒ**-predicated only of substance terms:[17]

(3) $\Sigma a \wedge \Upsilon ab \supset \Sigma b$

Thus, a substance term such as 'man' cannot be **ϒ**-predicated of a non-substance term such as 'moving.' So-called unnatural predications, in which substance terms are predicated of non-substance terms, are excluded in $\mathcal{A}$. Of course, there may be situations in the world in which every individual thing that is moving is human, and 'man' applies to all 'moving' in the sense of the extensional Russell-style universal

12 See also An. post. 83a24f.

13 Plain accidental predication **ϒ** is not intended to include matter-form predications in which substance terms such as 'man' or 'house' are 'predicated' of matter (for example, 'bones' and 'flesh' or 'wood' and 'bricks'). These predications are not essential since the matter may cease to be a man or a house. Matter-form predication is relevant for the investigation of generation and destruction of substantial beings in the Physics and Metaphysics, but it can be disregarded in the modal syllogistic.

14 Unlike (ax_4), (1) implies that substance terms are precisely those terms which are **E**-predicated of themselves: From $\mathbf{E}aa$ follows $\exists z\mathbf{E}za$, that is Σa. Conversely, if Σa holds, there is a *z* such that $\mathbf{E}za$. (ax_1) ensures Υaa, so that $\mathbf{E}za \wedge \Upsilon aa$ yield $\mathbf{E}aa$ by (1). (ax_4), on the other hand, does not exclude $\mathbf{E}za \wedge \Upsilon ab \wedge \neg\mathbf{E}ab$ nor $\mathbf{E}za \wedge \Upsilon aa \wedge \neg\mathbf{E}aa$. By taking ($ax_4$) as an axiom instead of (1), we leave it open for which terms **E** is reflexive.

15 For the transitivity of **E**, see for example Top. 121a26, 122a3-6, 122a31-37, 143a21f, An. post. 91a18-21, Cat. 1b10-15, 3b4f.

16 (ax_4) ensures that $\Sigma a \wedge \Sigma c$ follows from $\Sigma a \wedge \Sigma b \wedge \Upsilon bc$ (see formula (3)); and (ax_2) ensures that $\neg\exists z(\Upsilon az \wedge \Upsilon cz)$ follows from $\neg\exists z(\Upsilon az \wedge \Upsilon bz) \wedge \Upsilon bc$ (for the details see Theorem 6, p. 121). Due to the symmetry of **K**, the downward monotonicity of **K** in the right argument *b* carries over also to the left argument *a*.

17 Assume $\Sigma a \wedge \Upsilon ab$. From Σa it follows by (df_1) that there is a *z* such that $\mathbf{E}za$. This together with Υab gives $\mathbf{E}zb$ by (ax_4), and Σb by (df_1).

propositions of modern predicate logic. However, Aristotle's universal propositions do not have the Russell-structure of extensional (referential) set inclusion, nor are they evaluated relative to 'situations in the world.' Their evaluation criteria are provided by the term logic of Aristotelian predicables in which unnatural predications are not admissible since substance terms are predicated only of their substantial subspecies. Indeed, unnatural predications (predications *per accidens*[18]) are excluded from (syllogistic) demonstrations by Aristotle in the Posterior Analytics A 22:

> Let us assume then, that the predicate is always predicated *simpliciter* (*haplōs*) of that of which it is predicated, and not per *accidens*. For that is the way in which demonstrations demonstrate.
>
> An. post. 83a18-21

Thus, the demonstrations of the modal syllogistic (in particular the proof of aaa-1-NXN) work only under the condition that unnatural premisses be excluded and that substance terms never be Υ-predicated of non-substance terms (van Rijen *1989*, p. 207). This allows us to reject the following counterexample against aaa-1-NXN raised by Theophrastus: 'animal' is predicated essentially (necessarily) of 'man' and 'man' is predicated of 'moving,' and yet 'animal' is not predicated essentially (necessarily) of 'moving' (Alexander in An. pr. 124.24f). Given that the assertoric universal affirmative proposition $\mathbb{X}^{\mathbf{a}}ab$ of the syllogistic coincides with the plain accidental predication Υab of the underlying predicable-semantics (see p. 108), Theophrastus' counterexample relies on the unnatural minor premiss that 'man' is Υ-predicated of 'moving,' which is not admissible.[19]

Rejecting unnatural predication does not imply rejecting any syllogistic proposition with a substantial predicate term and a non-substantial subject term whatsoever – which would surely be false in view of the fact that Aristotle accepts, for instance, particular affirmative, universal negative and particular negative assertoric premisses with the predicate term 'animal' and the subject term 'white'.[20] The exclusion of unnatural predication is confined to the plain accidental Υ-predication on the level of the underlying predicable-semantics and carries over to the level of syllogistic propositions only indirectly. Given that the assertoric particular affirmative proposition $\mathbb{X}^{\mathbf{i}}ab$ states that a and b are not disjunct with respect to Υ ($\exists z(\Upsilon bz \wedge \Upsilon az)$, see p. 108), nothing prevents the subject term b from being a non-substance term while the predicate term a is a substance term. In this case, the proviso on unnatural predication requires z to be a substance term as well as a, but there is no substance term Υ-predicated of a non-substance term. Contrary to

[18] *katēgorein kata sumbebēkos*, see An. post. 81b25-29, 83a14-17, An. pr. 43a33-36.

[19] The unnatural premiss that 'man' is predicated of 'moving' is excluded by the author of An. pr. 34b7-18 because of its 'temporal definedness.' The reference to 'temporal definedness' and the whole passage 34b7-18 have been doubted and considered non-Aristotelian by commentators for many reasons (e.g. Patterson *1995*, pp. 167–174), the most telling reason being that predicating 'man' of 'moving' does not seem to be temporally defined or restricted in any higher degree than predicating 'moving' or 'health' of 'man' or 'animal' (e.g. An. pr. 37b37, 30a29f, 30b5f) or than denying that 'white' applies to some 'man' (An. pr. 35b18f) or than many other of Aristotle's assertoric premisses in the modal syllogistic. Given the above, the proper reason for excluding the premiss seems to be its unnaturalness, but not temporal or tense-logical reasons that are entirely alien to the Prior Analytics. On this view, there is a close connection between *katēgorein haplōs* in An. post. 83a18-21 and *haplōs* in An. pr. 34b8, 34b18.

[20] See for example An. pr. 26b25, 29a9f, 30b33f, 35b18f.

Striker (1994, p. 47f), Aristotle's premiss that 'animal' applies to some 'white' (An. pr. 26b25, 29a9f, 35b18f) is not an instance of unnatural predication. By analogy, the negative assertoric propositions $\mathbb{X}^{\mathbf{e}}ab$ and $\mathbb{X}^{\mathbf{o}}ab$ only deny certain $\boldsymbol{\Upsilon}$-predications (see p. 108), so that there is no risk of unnatural predication even if *a* happens to be a substance term and *b* a non-substance term. Thus, nothing prevents Aristotle from stating that 'animal' applies to no 'white' (30b33f) and that 'animal' does not apply to some 'white' (35b18f) or from converting the premiss that 'moving' applies to no 'animal' (30a33). It is only in the assertoric universal affirmative proposition $\mathbb{X}^{\mathbf{a}}ab$, which will turn out to coincide with $\boldsymbol{\Upsilon}$-predication, that the exclusion of unnatural predication carries over directly to an assertoric syllogistic proposition in $\mathcal{A}$, and Aristotle indeed never uses unnatural $\mathbb{X}^{\mathbf{a}}$-propositions as premisses of counterexamples in the Prior Analytics A 1-22.

Finally, the fifth axiom states that the downward monotonicity of essential predication with respect to $\boldsymbol{\Upsilon}$ introduced for substantial essential predication by (ax_4), holds for non-substantial essential predication $\widetilde{\mathbf{E}}$ too:

$$(ax_5) \quad \widetilde{\mathbf{E}}ab \wedge \boldsymbol{\Upsilon}bc \supset \widetilde{\mathbf{E}}ac$$

For example, if 'colour' is essentially predicated of the non-substance term 'white' (or 'whiteness') within one of the non-substance categories, and 'white' (or 'whiteness') is predicated of a subspecies of 'white' such as, say, 'colonial white,' then 'colour' is also essentially predicated of 'colonial white.' At first glance, (ax_5) may seem to give rise to well-known fallacies since 'white' can be predicated not only of non-substantial species, but also of substance terms such as 'swan:' if 'colour' is essentially predicated of 'white,' and 'white' is predicated of 'swan,' then according to (ax_5), 'colour' should be essentially predicated of 'swan,' which is false as swans cannot be said to be a colour.

Such fallacies can be avoided by taking into account Aristotle's distinction between two kinds of non-substance terms.[21] On the one hand, there are terms such as 'colour' and 'illness' that denote a definable non-substantial entity but cannot be predicated of substance terms (swans cannot be said to be a colour). On the other hand, there are paronymic variants such as 'coloured' and 'ill' that can be predicated of substance terms but do not have a definition within their non-substance category, and therefore cannot be subjects of any essential predication (see Met. Z 6 1031b22-28). The fallacy crucially relies on the term's 'white' being used like 'illness' in the major premiss and like 'ill' in the minor. The premisses of the fallacy do not have the same middle term and must be rejected due to a *quaternio terminorum*. For the same reason, Theophrastus' counterexample against aaa-1-NXN cited on p. 97 must be rejected; in the major premiss, 'moving' is essentially predicated of 'walking' and 'walking' is used like 'illness,' whereas in the minor premiss 'walking' is $\boldsymbol{\Upsilon}$-predicated of 'man' and therefore used like 'ill.'

For the sake of formal simplicity, the distinction between the two uses of non-substance terms will not be made explicit in $\mathcal{A}$ since this does not directly contribute to the logical reconstruction of the modal syllogistic. Nevertheless, fallacies and counterexamples against aaa-1-NXN such as the above are done away with by assuming that the subjects of non-substantial essential predication $\widetilde{\mathbf{E}}$ are

[21] See for instance Top. 109a34-b12, 133b24-31, 147a12f, Cat. 10a27-b11.

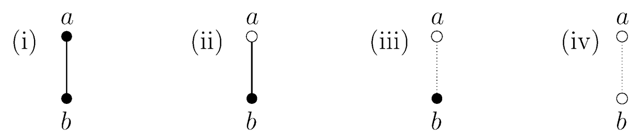
terms such as ‘illness’ which cannot be predicated merely accidentally of substance terms.[22]

This completes the introduction of axioms for $\mathcal{A}$. Together with (df_{1-7}), the axioms (ax_{1-5}) determine which term logical predicable-structures are admissible as models for modal syllogistic and which are not. Such structures can be represented graphically by diagrams using the following ingredients:

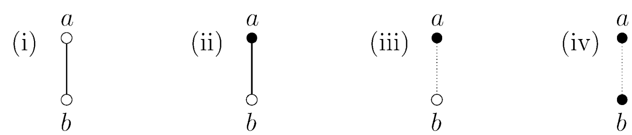
(4)
● substance term, Σa
○ non-substance term, $\neg\Sigma a$
—— substantial essential predication, $\mathbf{E}ab$
.... merely accidental predication, $\boldsymbol{\Upsilon}ab$
- - - non-substantial essential predication, $\widetilde{\mathbf{E}}ab$

In the following diagrams, predicative relations between terms correspond to downward paths: *a* is $\boldsymbol{\Upsilon}$-predicated of *b* if there is any downward path from *a* to *b* or if *a* is identical with *b*. By analogy, *a* is **E**-predicated of *b* if there is a downward path from *a* to *b* consisting only of **E**-lines. There will be no graphical convention indicating which substance terms are **E**-predicated of themselves, but it can be consistently assumed that all substance terms are **E**-predicated of themselves (see note 14). The same holds *mutatis mutandis* for $\widetilde{\mathbf{E}}$-predication, except that $\widetilde{\mathbf{E}}$-lines are assumed to be covered by **E**-lines if the two predications happen to coincide[23], but the following diagrams are understood not to contain any ‘hidden’ $\widetilde{\mathbf{E}}$-predications unless stated otherwise. Given these conventions, the structures in (5) are admissible in $\mathcal{A}$, whereas the structures in (6) are not admissible:

(5)



(6)



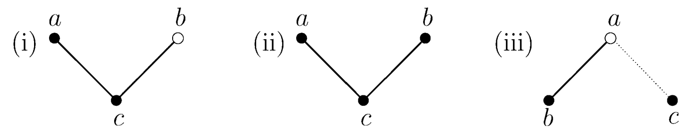
(6i) is not admissible since a non-substance term *b* is subject of an **E**-predication, which is ruled out by (df_1). (6ii) and (6iii) are not admissible as a substance term is $\boldsymbol{\Upsilon}$-predicated of a non-substance term, which contradicts formula (3) (p. 101). (6iv) is not admissible as *a* is predicated of *b* merely accidentally: *a* is a substance term and therefore must be subject of an **E**-predication; since there is no other term

[22] Note, however, that in the formal proofs of invalidity in Section 4 (p. 121ff) there are three structures in which a substance term is subject of an $\widetilde{\mathbf{E}}$-predication (Theorem 63, invalidity of aoo–3–NQM; Theorem 74, inconclusiveness of ae-2-NQ; Theorem 76, inconclusiveness of ea-2-QN). There are several strategies to avoid these structures or make them plausible, but it would require lengthy digression to explore these in more detail.

[23] See Theorem 63 and 74, pp. 134ff.

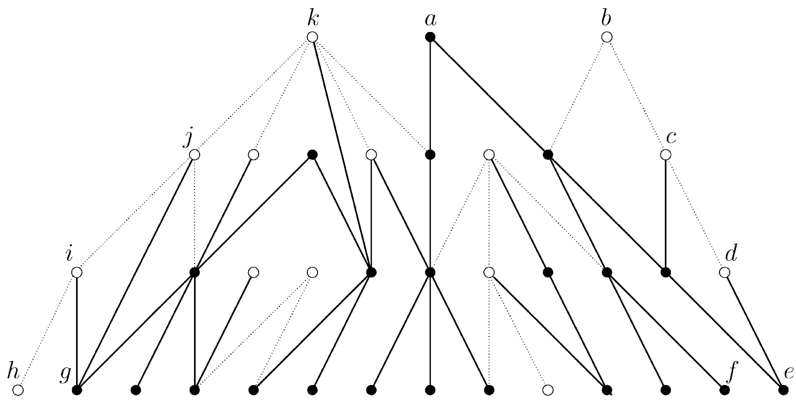
E-predicated of a in (6iv), a must be **E**-predicated of itself; but $\mathbf{E}aa$ and $\boldsymbol{\Upsilon}ab$ yield $\mathbf{E}ab$ by (ax_4) so that a must not be predicated of b merely accidentally, but essentially. Moreover, both (7i) and (7ii) are admissible in $\mathcal{A}$,[24] and also (7iii) is admissible since it is not ruled out that a non-substance term a, while being predicated essentially (necessarily) of a substance term b (that means, being its *differentia* or inseparable accident), is predicated merely accidentally of another term c (see Top. 144b6):[25]

(7)



Finally, the interdependencies between $\boldsymbol{\Upsilon}$- and **E**-predication are illustrated by a more complex admissible predicable-structure in which a substantial genus-*differentia*-species tree is intertwined with several merely accidental $\boldsymbol{\Upsilon}$-predications:

(8)



$\boldsymbol{\Upsilon}kg$	$\mathbf{K}ef$	$\neg\mathbf{K}cg$	$\neg\boldsymbol{\Pi}jg$	$\neg\boldsymbol{\Pi}hg$	$\neg\boldsymbol{\Pi}ke$	$\neg\boldsymbol{\Pi}dg$	$\neg\boldsymbol{\Pi}cf$
$\mathbf{E}af$	$\neg\mathbf{K}hg$	$\boldsymbol{\Pi}kg$	$\boldsymbol{\Pi}ka$[26]	$\boldsymbol{\Pi}ca$	$\boldsymbol{\Pi}kd$	$\neg\boldsymbol{\Pi}ec$	$\boldsymbol{\Pi}bf$
$\mathbf{K}ag$	$\neg\mathbf{K}af$	$\boldsymbol{\Pi}kh$	$\neg\boldsymbol{\Pi}ja$	$\neg\boldsymbol{\Pi}ea$	$\boldsymbol{\Pi}bh$	$\boldsymbol{\Pi}eb$	

24 (7i) is a perfectly plausible structure with b being a non-substantial *differentia* (or inseparable accident) of a species c whose genus is a. On the contrary, (7ii) does not seem to be a plausible structure as b is a substance term and therefore cannot be *differentia* (see note 8) or inseparable accident, but must be a substantial genus of c. Thus, c has two genera a and b neither of which is predicated of the other, which is ruled out by Aristotle e.g. in Top. 121b29f, 122b1-4, 144a12f. This could be avoided by an additional tree-axiom $\Sigma a \wedge \Sigma b \wedge \mathbf{E}ac \wedge \mathbf{E}bc \supset (\mathbf{E}ab \vee \mathbf{E}ba)$. This tree-property, however, is not necessary for the formal proofs of the modal syllogistic in this paper. Therefore it will not be added to $\mathcal{A}$, although it can be consistently assumed to be valid throughout this paper since we will not make use of structures such as (7ii) violating this tree-property.

25 For instance, the inconclusiveness of ae-2-NQ and ea-2-QN (38a26-b23) requires counterexamples the middle term of which is predicated necessarily of one of the outer terms while being predicated two-way contingently of the other.

26 The contingent two-way possibility relation Π defined in (df_3) (p. 99) holds between k and a whereas it fails to hold between j and a, as a is a substance term and $\exists z(\boldsymbol{\Upsilon}az \wedge \boldsymbol{\Upsilon}jz)$ is false, which makes the last conjunct of (df_3) false. On the other hand, Πkb is true although $\exists z(\boldsymbol{\Upsilon}kz \wedge \boldsymbol{\Upsilon}bz)$ is false, as neither k nor b is a substance term, which makes the last conjunct true.

2. Modal copulae

Having introduced the underlying semantics, it is time to show how the syllogistic copulae are interpreted within it. Aristotle's semantic definition of the assertoric universal affirmative copula $\mathbb{X}^{\mathbf{a}}$ is given by the so-called *dictum de omni*:

> We say that one term is predicated of all of another whenever nothing [of the subject] can be taken of which the other is not said.
>
> An. pr. 24b28-30

Usually, the *dictum de omni* is taken to express the familiar Russell-style extensional set inclusion of modern first-order logic:

(9) $\forall z(B(z) \supset A(z))$

According to this formalization, $\mathbb{X}^{\mathbf{a}}ab$ means that the predicate term is predicated of every individual of which the subject term is predicated. The domain of quantification is semantically and syntactically of different type than the two argument terms A and B. These are first-order sets of individuals, whereas the quantifier ranges over zero-order individuals. However, Aristotle does not say anywhere that the quantification 'nothing' in 24b28-30 refers to individuals:

> It is by our partisan view about the nature of predication that we are persuaded to interpret this 'nothing' (μηδέν) in the sense of 'no individual.'
>
> Mignucci *1996*, p. 4

Doing away with this partisan view, the 'nothing' in 24b28-30 can also refer to terms instead of individuals – and at a closer look it even must do so. To mention just some pieces of evidence in favour of the term logical *dictum de omni*: Apart from 24b28-30, there are only two further occurrences of the construction 'nothing + can be taken (*esti labein*)' in the Analytics (An. post. 74a7f, 80a28-30); in both of them, the 'nothing' clearly refers to terms, not to individuals.[27] Second, in ecthetic proofs, Aristotle sets out an entity which belongs to the domain of quantification. In the apodeictic syllogistic (An. pr. 30a6-14), ecthetic proofs proceed by using this entity as subject and predicate argument of syllogistic premisses. Thus, the entity set out of the domain of quantification must be of the same type as the argument terms of syllogistic propositions and cannot be an individual of different logical type than terms. Boolean comprehension principles of set theory which would allow a type-shift from individuals to terms (sets) by ensuring that for every collection of individuals there is a term (set) predicated exactly of these individuals, are not available within Aristotelian genus-species term logic. Thus, it would be impossible to set out a term in ecthetic proofs if the quantification of syllogistic copulae ranged over individuals of other type than terms.

Third, in An. pr. A 41 (49b14-32), Aristotle distinguishes two non-equivalent readings of the assertoric universal affirmation $\mathbb{X}^{\mathbf{a}}ab$. The proper reading is: 'a is predicated universally of everything of which b is universally predicated' and the

[27] Moreover, in An. pr. 27b20, 28b25, 45b5, the accusative-object of the infinitive *labein* is explicitly specified as 'term' (*horos*).

secondary reading is: '*a* is predicated universally of everything of which *b* is at least particularly predicated.' This distinction would be completely pointless if the quantification 'everything' referred to individuals, since the distinction between particular and universal predication collapses with individuals as subjects. Fourth, only the term logical *dictum de omni* of necessity propositions can account for the obvious perfectness of aaa-1-NXN as explained by Aristotle:

> For since *a* applies or does not apply necessarily to all *b*, and *c* is one of the *b*s, clearly one or other of these will be necessary for *c* too.
>
> An. pr. 30a21-23

The minor term *c* is among the *b*s, and the major term *a* is predicated necessarily of every term (not individual) among the *b*s. Indeed, this is everything Aristotle needs in order to establish the perfect validity of aaa-1-NXN. The minor term *c* is seized directly by the term-quantification of the major premiss. On the other hand, if the quantification of the major premiss ranged over individuals, aaa-1-NXN would need a much more extended argumentation, of which there is no evidence in Aristotle.

We conclude that in Aristotle's syllogistic the domain of quantification consists of entities of the same type as syllogistic terms and that there is no logical distinction between individuals on the one hand and syllogistic terms on the other hand. It is in this sense that we claim that the syllogistic is a pure term logic in which there is no room for individuals. This is not to say that Aristotle cannot occasionally also use proper names such as Mikkalos or Aristomenes (An. pr. 47b22-30) in the syllogistic; all we claim is that these are treated as syllogistic terms by Aristotle and that he does not recognize an extra category of singular terms as opposed to general terms:

> The tendency to distinguish sharply between singular and general terms depends mainly on the fact that they are thought to be the bearer of different logical relations.
>
> Mignucci *1996*, p. 11

This, however, is not the case in Aristotle; the very same predicative relation which holds between species and genus holds also between individuals and their species.[28] Given that the quantification of syllogistic propositions ranges over entities of the same type as their argument terms, the semantic structure of the universal affirmative proposition $\mathbb{X}^{\mathbf{a}}ab$ as given by the *dictum de omni* cannot be correctly described by (9), and we have to take the following structure instead (Stekeler-Weithofer *1986*, p. 76, Mignucci *1996*):

(10) $\forall z(\Upsilon bz \supset \Upsilon az)$

It is in this non-referential, 'mereological' interpretation of syllogistic propositions that $\mathcal{A}$ decidedly differs from Patterson's (*1995*) approach and from the most important recent formalizations of modal syllogistic, in which the logical distinction between terms (predicates) and individuals is brought in either by modern predicate

[28] Cat. 1b12-15, see also An. pr. 43a30-32, Cat. 3a38f, Top. 122b21, 144b2f, Ackrill 1963, p. 76.

logic (Nortmann *1996*, Schmidt *2000*, Brenner *2000*) or by set theory (Johnson *1989*, Thomason *1993*, *1997*, Thom *1996*). In what follows it will be shown that the insistence on the pure term logical structure of modal propositions is not just a pedantic quibble, but that it has important logical consequences (see p. 111).

As a first consequence, the subject term b itself can be taken as instantiation of the general quantifier in (10), with the result that (10) is equivalent to Υab by transitivity and reflexivity of Υ. In the term logical framework of $\mathcal{A}$, the quantified formula (10) can be replaced by a direct relation Υab between the subject term b and the predicate term a within the same object language, whereas the Russell-formalization (9) cannot be replaced by a direct relation between A and B within the same first-order language. Once we have settled on the term logical interpretation of syllogistic propositions, the definitions of the remaining assertoric copulae are straightforward:

$$(11) \quad \begin{aligned} \mathbb{X}^{\mathbf{a}}ab &=_{df} \Upsilon ab \\ \mathbb{X}^{\mathbf{e}}ab &=_{df} \forall z(\Upsilon bz \supset \neg\Upsilon az) \\ \mathbb{X}^{\mathbf{i}}ab &=_{df} \exists z(\Upsilon bz \wedge \Upsilon az) \\ \mathbb{X}^{\mathbf{o}}ab &=_{df} \neg\Upsilon ab \end{aligned}$$

Note that these definitions together with reflexivity (ax_1) and transitivity (ax_2) of Υ yield an adequate formalization of the assertoric syllogistic without stating any additional existence presuppositions. In particular, the subalternations from $\mathbb{X}^{\mathbf{a}}ab$ to $\mathbb{X}^{\mathbf{i}}ab$ and from $\mathbb{X}^{\mathbf{e}}ab$ to $\mathbb{X}^{\mathbf{o}}ab$ are valid in $\mathcal{A}$ since the reflexivity of Υ ensures that for every term b there is at least one thing, viz. b itself, of which b is Υ-predicated. The fact that existence presuppositions can be dispensed with is a specific advantage of the term logical framework. In referential first-order logic, on the other hand, nothing prevents there from being no individual of which a certain predicate B is predicated, and there must be introduced additional existence presuppositions in order to ensure the subalternation from $\forall z(B(z) \supset A(z))$ to $\exists z(B(z) \wedge A(z))$.

Turning to necessity propositions, the universal affirmative necessity $\mathbb{N}^{\mathbf{a}}ab$ is defined as stating that the predicate term a is predicated $\widehat{\mathbf{E}}$-essentially – either in the sense of substantial or of non-substantial essential predication – of every term that the subject term b is Υ-predicated of:

$$(12) \quad \forall z(\Upsilon bz \supset \widehat{\mathbf{E}}az)$$

This is equivalent to $\widehat{\mathbf{E}}ab$;[29] it makes no difference whether a is predicated essentially of everything of which b is predicated, or whether a is predicated essentially of b itself. The universal affirmative necessity copula $\mathbb{N}^{\mathbf{a}}$ obtains exactly between those terms related by substantial or non-substantial essential predication. This is in accordance with Aristotle's usage of universal affirmative necessities in the premisses of counterexamples in An. pr. A 1-22. On the one hand, there are substantial essential predications the subject of which is a substance term such as 'horse,' 'man,' 'swan' and 'snow,' the predicate being either a substantial genus such

[29] $\widehat{\mathbf{E}}ab$ follows from (12) by (ax_1). In view of (df_5), the converse follows from the fact that $\mathbf{E}ab$ implies $\forall z(\Upsilon bz \supset \mathbf{E}az)$ by (ax_4) and that $\widetilde{\mathbf{E}}ab$ implies $\forall z(\Upsilon bz \supset \widetilde{\mathbf{E}}az)$ by (ax_5). For the equivalence of $\widehat{\mathbf{E}}ab$ and (12), see also Alexander in An. pr. 126.25-27.

as 'animal' (e.g. An. pr. 30a30, 30b33f, 31b5, 31b41) or a non-substance term such as 'white' (e.g. An. pr. 36b11, 38a31f, 38b20), to be seen as a *differentia* or inseparable accident of a substantial species (see p. 99). On the other hand, there is only one (nevertheless important) universal affirmative necessity-premiss the subject of which is a non-substance term and which therefore is to be regarded as a non-substantial essential predication; 'moving' is predicated necessarily of 'awake' (38a42f) in order to prove the notoriously problematic inconclusiveness of ea-2-QN.[30]

By analogy, the particular affirmative necessity $\mathbb{N}^{\mathbf{i}}ab$ could be defined as

(13) $\exists z(\boldsymbol{\Upsilon}bz \wedge \widehat{\mathbf{E}}az)$

However, using this definition, $\mathbb{N}^{\mathbf{i}}ab$ fails to convert to $\mathbb{N}^{\mathbf{i}}ba$; in order to avoid this problem we follow Thom and Brenner[31] in weakening (13) by disjunction as follows:

(14) $\exists z((\boldsymbol{\Upsilon}bz \wedge \widehat{\mathbf{E}}az) \vee (\boldsymbol{\Upsilon}az \wedge \widehat{\mathbf{E}}bz))$

There are no restrictions as to whether *a* or *b* has to be a substance term in (14).[32] Note that $\mathbb{N}^{\mathbf{i}}ab$ follows from the particular assertoric affirmative proposition $\mathbb{X}^{\mathbf{i}}ab$ if either *a* or *b* is a substance term that is **E**-predicated of itself.[33] Given that in fact all substance terms are **E**-predicated of themselves (see note 14) and that, for example, $\mathbb{X}^{\mathbf{i}}$ holds between 'moving' and the substance term 'animal,' $\mathbb{N}^{\mathbf{i}}$ also obtains between these terms. It might seem implausible that 'moving' applies necessarily to some 'animal' just because it applies assertorically to some 'animal;' but if we are prepared to accept that 'animal' applies necessarily to some 'moving' (for example, to 'man'),[34] then Aristotle's doctrine of $\mathbb{N}^{\mathbf{i}}$-conversion forces us to accept the converted counterpart too.[35] Thus, the only way to have an $\mathbb{X}^{\mathbf{i}}$-proposition which is not necessarily true is that neither *a* nor *b* be substance terms (for which **E** is reflexive). Indeed, there are only two cases in the modal syllogistic where Aristotle needs an $\mathbb{X}^{\mathbf{i}}$-proposition that is not an $\mathbb{N}^{\mathbf{i}}$-proposition, and in both of them neither the subject nor the predicate is a substance term: 'moving' applies to some 'white' (30b5f) and 'awake' applies to some 'two-footed' (31b27-33).[36]

[30] See the non-substantial essential minor premiss in the proof of the inconclusiveness of ea-2-QN in Theorem 76 (p. 138).

[31] See Thom (*1991*, p. 146; *1996*, p. 146) and Brenner (*2000*, p. 336).

[32] Both *a* and *b* may be substance terms, or both may be non-substance terms (e.g. 'white'–'inanimate' 36b15), or the predicate term *a* may be a substance term while *b* is a non-substance term (see note 34), or the subject term *b* may be a substance term while *a* is a non-substance term (e.g. 'two-footed'-'animal' 31b28f or 'white'–'man' 36b14).

[33] $\mathbf{E}bb$ or $\mathbf{E}aa$ together with $\exists z(\boldsymbol{\Upsilon}bz \wedge \boldsymbol{\Upsilon}az)$ implies $\exists z(\mathbf{E}bz \wedge \boldsymbol{\Upsilon}az)$ or $\exists z(\boldsymbol{\Upsilon}bz \wedge \mathbf{E}az)$ respectively by (ax_4).

[34] Aristotle often gives a very similar example stating that 'animal' applies necessarily to some 'white' (30b6, 32a2, 36b6f, 36b14).

[35] As a consequence, Aristotle's example stating that 'white' applies necessarily to some 'man' (36b14) can be considered true in $\mathcal{A}$ by taking 'man' to be an **E**-reflexive substance term, regardless of whether there are subspecies of 'man' of which 'white' is predicated essentially. It suffices that 'white' accidentally applies to an (improper) subspecies of 'man' while 'man' is predicated essentially of it(self).

[36] We follow Alexander (134.28-31) in considering the terms 'moving'–'animal'–'white' in 30b5f a counterexample against eio-1-XNN and against aii-1-XNN at the same time. In the case of aii-1-XNN, 'moving' applies to all 'animal' and 'animal' applies necessarily to some 'white.' By aii-1-XXX, 'moving' must apply assertorically to some 'white,' but in order to refute aii-1-XNN 'moving' must not apply necessarily to some 'white.' In 31b27-33, the invalidity of aii-3-XNN and iai-3-NXN is shown by a similar counterexample, except that 'moving' is replaced by 'awake' and 'white' by 'two-footed.'

We have seen that the universal affirmative necessity $\mathbb{N}^{\mathbf{a}}ab$ is not obtained from the corresponding assertoric proposition $\mathbb{X}^{\mathbf{a}}ab$ by adding modal sentential operators, but by replacing the $\mathbf{\Upsilon}$-copula of accidental predication by the $\widehat{\mathbf{E}}$-copula of essential predication. In the same way, the universal negative necessity $\mathbb{N}^{\mathbf{e}}ab$ is not obtained from $\mathbb{N}^{\mathbf{a}}ab$ by adding any negation operators, but by replacing the $\widehat{\mathbf{E}}$-relation in (12) by the incompatibility relation **K**:

(15) $\forall z(\mathbf{\Upsilon}bz \supset \mathbf{K}az)$

Again, (15) turns out to be equivalent to **K***ab*.[37] As a result, both argument terms of $\mathbb{N}^{\mathbf{e}}ab$ are required to be substance terms by the definition of **K** in (df_2). This is in accordance only with four[38] of the six universal negative necessity premisses[39] given by Aristotle in An. pr. A 1-22. The two exceptions, however, occur in a superfluous counterexample which can easily be replaced by a counterexample fulfilling the requirement that both arguments of $\mathbb{N}^{\mathbf{e}}$ be substance terms.[40] This allows us to accept **K***ab* as a good approximation to Aristotle's universal negative necessity without running into any logical difficulties. Whereas the universal affirmative necessity $\mathbb{N}^{\mathbf{a}}$ coincides with $\widehat{\mathbf{E}}$, which allows for non-substantial as well as substantial essential predication, the universal negative necessity $\mathbb{N}^{\mathbf{e}}$ coincides with **K**, which is confined to the category of substance. This is in accordance with the fact that Aristotle makes use of both substantial and non-substantial $\mathbb{N}^{\mathbf{a}}$-propositions in his counterexamples (see p. 98), while he never uses non-substantial $\mathbb{N}^{\mathbf{e}}$-propositions (e.g. 'health' necessarily applies to no 'illness') in An. pr. A 1-22.

The equivalence of $\forall z(\mathbf{\Upsilon}bz \supset \widehat{\mathbf{E}}az)$ and $\widehat{\mathbf{E}}ab$ and the equivalence of $\forall z(\mathbf{\Upsilon}bz \supset \mathbf{K}az)$ and **K***ab* is a specific characteristic of the term logic $\mathcal{A}$. For neither of the universal necessity propositions does it make any difference whether they are thought of as stating a direct relation between the predicate term and the subject term, or as stating this relation between the predicate term *a* and everything of which the subject term *b* is predicated; for due to the reflexivity of $\mathbf{\Upsilon}$, *b* itself will be among the 'things' (viz. terms) of which it is predicated. Referential extensional logic, on the other hand, distinguishes zero-order individuals logically from first-order predicates. Whence

[37] This follows from (ax_1) and formula (2) on p. 101.

[38] These four examples are: 'animal' necessarily applies to no 'snow' and to no 'pitch' (36a30f); 'sleeping horse' and 'waking horse' necessarily apply to no 'man' (40a37f). The necessity of the latter two examples is based on the fact that 'horse' necessarily applies to no 'man,' which satisfies the above requirement. Term conjunctions such as 'sleeping horse' are not expressible in $\mathcal{A}$ and we shall not commit ourselves as to their exact status, whether they are to be regarded as substance terms or not. Anyway, term conjunctions are not of logical relevance for the modal syllogistic. The only examples in An. pr. A 1-22 making use of term conjunctions are 'sleeping' – 'sleeping horse' – 'man' and 'sleeping' – 'waking horse' – 'man' in 40a37f, designed to show the inconclusiveness of ae–3–QN. These examples can be replaced by 'white' – 'snow' – 'animal' and 'white' – 'pitch' – 'animal' used in 35a24, 35b10 and 36a30 to show the inconclusiveness of ae-1-QX, ao-1-QX and ae-1-QN.

[39] We take into account only those cases of universal 'necessarily-not-applying' stated as premisses of counterexamples. Aristotle speaks of universal 'not being possible to apply' also to characterize the relation between the major and the minor term in one of the two counterexamples showing the inconclusiveness of modal premiss-pairs (e.g. 33b3-17, 35a24, 35b19, 36b7). But what Aristotle needs in these cases is only the negation of the one-way possibility $\mathbb{M}^{\mathbf{i}}ac$ (see p. 136); $\neg\mathbb{M}^{\mathbf{i}}ac$ is weaker than $\mathbb{N}^{\mathbf{e}}ac$ in $\mathcal{A}$ and can be also true even if not both of the arguments are substance terms (see the $\mathbb{M}^{\mathbf{i}}$-definition in (27), p. 115; Theorem 10, p. 122, and Theorem 69, p. 136).

[40] The exceptions – 'white' necessarily applies to no 'raven' and to no 'pitch' (36b9f) – occur in the two counterexamples showing the inconclusiveness of ie-1-QN. This, however, follows from the inconclusiveness of ae-1-QN already shown in 36a30 by a counterexample fulfilling the requirement that both arguments of $\mathbb{N}^{\mathbf{e}}$ be substance terms (see note 38).

there is a far-reaching difference between the two cases, since first-order predicates are a far cry from the zero-level individuals of which they are predicated and therefore the subject term *B* cannot be among the 'things' (viz. individuals) of which it is predicated. Translating $\forall z(\boldsymbol{\Upsilon} bz \supset \widehat{\mathbf{E}}az)$ and $\forall z(\boldsymbol{\Upsilon} bz \supset \mathbf{K}az)$ into referential logic yields a *de re* reading or, as Patterson (*1995*) calls it,[41] a 'weak' modal copula: The predicate term is asserted to stand in a certain relation to the individuals the subject term is predicated of without stating anything about the relation of the predicate term to the subject term itself. For example, in a situation of the world where only men are in this room, there will be a *de re* necessity that 'horse' applies to no individual 'being in this room'. Given the extensional Russell-style interpretation (9) of $\mathbb{X}^{\mathbf{a}}$-propositions, this *de-re* reading of the universal negative necessity validates syllogisms such as eae-1-NXN, but fails to validate $\mathbb{N}^{\mathbf{e}}$-conversion.[42]

On the other hand, the referential counterpart to $\widehat{\mathbf{E}}ab$, $\mathbf{K}ab$ and to Patterson's 'strong' modal copulae stating a direct relation between the predicate and the subject term is a *de dicto* reading such as: it is *de dicto* necessary that 'being in this room' applies to no individual 'being in that room.' This reading validates $\mathbb{N}^{\mathbf{e}}$-conversion, but fails to validate eae-1-NXN with a Russell-style minor premiss.[43] Consequently, referential Russell-style approaches to modal syllogistic distinguishing between a 'strong' *de dicto* reading and a 'weak' *de re* reading of modal propositions, for example Becker (*1933*, p. 46) and Patterson (*1995*, pp. 23–30), cannot account for eae-1-NXN and $\mathbb{N}^{\mathbf{e}}$-conversion with a uniform reading of modal propositions, but are compelled to accuse Aristotle of having confused two readings of modal propositions by inadvertently switching from one to the other.

However, there is no distinction between *de re* and *de dicto* readings in Aristotle's modal syllogistic; for there are no modal sentential operators which could be applied to *dicta*, and no zero-level individuals which could serve as the *res* of *de re* modalities. In Aristotelian term logic, the $\mathbb{N}^{\mathbf{e}}$-reading $\forall z(\boldsymbol{\Upsilon} bz \supset \mathbf{K}az)$, which obviously validates eae-1-NXN, coincides with the $\mathbb{N}^{\mathbf{e}}$-reading $\mathbf{K}ab$, which guarantees $\mathbb{N}^{\mathbf{e}}$-conversion by the symmetry of $\mathbf{K}$. There is no need to put a lot of effort into avoiding the *de re–de dicto* ambiguity[44]; it simply vanishes once we have settled on a pure term logic and given up the logical distinction between first-order predicates and zero-level individuals.

To complete the introduction of necessity copulae, the following would be obvious as a definition of the particular negative $\mathbb{N}^{\mathbf{o}}ab$:

(16) $\exists z(\boldsymbol{\Upsilon} bz \wedge \mathbf{K}az)$

[41] Brennan (*1997*, p. 230) has pointed out that, contrary to what Patterson (*1995*, pp. 30–37) claims, his distinction between 'weak' and 'strong' modal copulae, though being based on essentialist predicable-semantics, fails to bear any relevant differences to the traditional distinction between *de re* and *de dicto* modalities. The reason for this, as we shall argue, lies in Patterson's using referential logic instead of term logic.

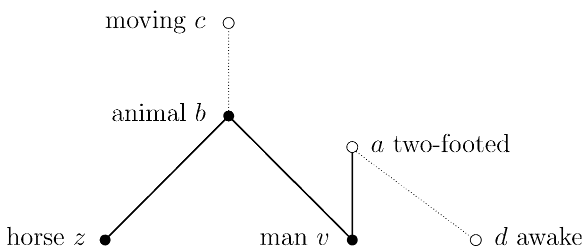
[42] Given that 'horse' is *de re* incompatible to every individual 'being in this room,' it does not follow that 'being in this room' is *de re* incompatible to every individual being a 'horse.' On the other hand, given the Russell-style minor premiss that every individual 'being in this house' is an individual 'being in this room,' it follows that 'horse' is *de re* incompatible to every individual 'being in this house.'

[43] Given that it is *de dicto* necessary that 'being in this room' applies to no individual 'being in that room,' it follows that it is *de dicto* necessary that 'being in that room' applies to no individual 'being in this room.' On the other hand, given the Russell-style minor premiss that every individual 'being in this house' is an individual 'being in that room,' it does not follow that it is *de dicto* necessary that 'being in this room' applies to no individual 'being in this house' (provided both rooms belong to the same house).

[44] See, for example, the complex logical tools employed by Nortmann (*1996*, p. 122), in order to validate $\mathbb{N}^{\mathbf{e}}$-conversion.

However, using this definition, $\mathbb{N}^{\mathbf{o}}ab$ is upward monotonic (transparent) with respect to Υ in the subject argument b and downward monotonic with respect to Υ in the predicate argument a, so that (16) fails to account for the invalidity of oao–3–NXN and aoo-2-XNN.[45] The problem is that, according to (16), the predicate term a of $\mathbb{N}^{\mathbf{o}}ab$ must be a substance term by (df_2), whereas in Aristotle's counterexample against oao–3–NXN, the predicate of the major premiss is the *differentia* 'two-footed,' which is not a substance term (see note 8) although essentially predicated of substance terms:[46] 'two-footed' necessarily does not apply to some 'animal' (for example, to 'horse'), 'moving' applies to all 'animal,' and yet it is not the case that 'two-footed' necessarily does not apply to some 'moving' (An. pr. 32a4f).

(17)



According to Alexander (in An. pr. 144.2f), the same $\mathbb{N}^{\mathbf{o}}$-premiss can also be used for refuting aoo-2-XNN: 'two-footed' applies to all 'awake,' 'two-footed' necessarily does not apply to some 'animal,' and yet it is not the case that 'awake' necessarily does not apply to some 'animal.'[47] Thus, in order to refute oao–3–NXN and aoo-2-XNN in $\mathcal{A}$ we have to take into account structures such as the above, allowing for $\mathbb{N}^{\mathbf{o}}$-premisses the predicate a of which is a non-substantial *differentia* while the subject b essentially applies to a species z incompatible with all substantial subspecies of the

[45] Upward monotonicity $\Upsilon cb \wedge \exists z(\Upsilon bz \wedge \mathbf{K}az) \supset \exists z(\Upsilon cz \wedge \mathbf{K}az)$ follows from (ax_2), downward monotonicity $\Upsilon ac \wedge \exists z(\Upsilon bz \wedge \mathbf{K}az) \supset \exists z(\Upsilon bz \wedge \mathbf{K}cz)$ from formula (2) (p. 101).

[46] There are two more examples of $\mathbb{N}^{\mathbf{o}}$-premisses with a non-substantial predicate term in An. pr. A 1-22: 'white' necessarily does not apply to some 'man' and to some 'inanimate' (36b15). These examples, however, can be disregarded as they serve to show the inconclusiveness of io-1-QN, which follows from the inconclusiveness of ae-1-QN already shown in 36a30 by a counterexample with a substantial middle term (see note 38). In the remaining examples of $\mathbb{N}^{\mathbf{o}}$-premisses in An. pr. A 1-22 the predicate is a substance term ('animal'-'white' 36b6f, 36b14f, perhaps also 31a15-17, for which see note 47).

[47] The terms Aristotle seems to offer for refuting aoo-2-XNN in 31a15-17 by referring back to the refutation of aee-2-NXN in 30b33-38 are 'animal'–'man'–'white,' with 'animal' being the middle term and 'white' the minor term. It is, however, unclear how these terms are to yield a refutation of aoo-2-XNN. If 'animal' is taken as middle term and 'white' as minor term, the major premiss ('animal' applies to all 'man') is necessarily true, which yields a necessary conclusion by aoo-2-NNN. These difficulties seem to have been already known to the earliest commentators before Alexander (in An. pr. 144.4f), as in some manuscripts Alexander read the addendum 'with only one term being interchanged.' Still, it remains problematic how to obtain a refutation of aoo-2-XNN by changing the order of the above terms or by replacing one of them by another (see Alex. in An. pr. 144.4-22). We shall not go into details here, but shall take over Alexander's terms (in An. pr. 144.2f) in order to construct a refutation of aoo-2-XNN in $\mathcal{A}$.

differentia a (for example, the subspecies v[48]). This can be achieved by the following formula, to be added to (16) by disjunction:[49]

(18) $\exists zv(\widehat{\mathbf{E}}bz \wedge \widehat{\mathbf{E}}av \wedge \forall u(\mathbf{\Upsilon}au \wedge \widehat{\Sigma}u \supset \mathbf{K}zu))$

Now, the necessity copulae can be defined as follows:

$$\begin{array}{llc} (19) & \mathbb{N}^{\mathbf{a}}ab =_{df} & \widehat{\mathbf{E}}ab \\ & \mathbb{N}^{\mathbf{e}}ab =_{df} & \mathbf{K}ab \\ & \mathbb{N}^{\mathbf{i}}ab =_{df} & \exists z((\mathbf{\Upsilon}bz \wedge \widehat{\mathbf{E}}az) \vee (\mathbf{\Upsilon}az \wedge \widehat{\mathbf{E}}bz)) \\ & \mathbb{N}^{\mathbf{o}}ab =_{df} & \exists z(\mathbf{\Upsilon}bz \wedge \mathbf{K}az) \vee \exists zv(\widehat{\mathbf{E}}bz \wedge \widehat{\mathbf{E}}av \wedge \forall u(\mathbf{\Upsilon}au \wedge \widehat{\Sigma}u \supset \mathbf{K}zu)) \end{array}$$

Having defined assertoric and necessity copulae, one-way and two-way possibility copulae remain to be considered. For the definition of two-way copulae, we have introduced the two-way contingency relation Πab in (df_3). Since there is no difference between affirmative and negative two-way possibility, both universal two-way copulae $\mathbb{Q}^{\mathbf{a}}$ and $\mathbb{Q}^{\mathbf{e}}$ can be defined using the proven structure of universal propositions as follows:

(20) $\forall z(\mathbf{\Upsilon}bz \supset \Pi az)$

Unlike the universal $\mathbb{X}^{\mathbf{a}}$-copula in (10) and the $\mathbb{N}^{\mathbf{a}}$-copula in (12), the $\mathbb{Q}^{\mathbf{a}}$-copula in (20) cannot be equivalently simplified to Πab. Anyway, both particular copulae $\mathbb{Q}^{\mathbf{i}}$ and $\mathbb{Q}^{\mathbf{o}}$ should read as follows:

(21) $\exists z(\mathbf{\Upsilon}bz \wedge \Pi az)$

The only difficulty with this definition being that $\mathbb{Q}^{\mathbf{i/o}}ab$ does not convert to $\mathbb{Q}^{\mathbf{i/o}}ba$, it should, again, be extended by the converted counterpart. This time, it is favourable to add the converted counterpart by conjunction:

(22) $\exists z(\mathbf{\Upsilon}bz \wedge \Pi az) \wedge \exists z(\mathbf{\Upsilon}az \wedge \Pi bz)$

Surprisingly, this formula turns out to be equivalent to Πab,[50] which leads to the following definition of two-way possibility copulae:

48 Note that z need not be incompatible with non-substance terms such as d, of which a is predicated merely accidentally.

49 The resulting disjunctive definition of $\mathbb{N}^{\mathbf{o}}ab$ in (19) bears a close structural resemblance to Brenner's (*2000*, p. 336) formalization of the particular negative necessity:

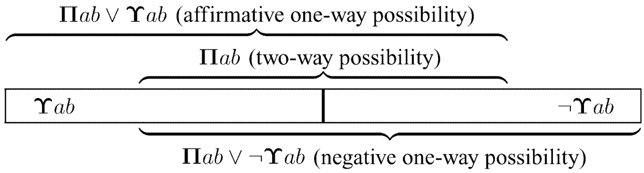
$$\exists x(Bx \wedge \forall y(Ay \supset \Box x \neq y)) \quad \vee \quad \exists x(\Box Bx \wedge \neg Ax \wedge \forall y(\Box Ay \supset \Box x \neq y)).$$

50 From Πab, we have $\mathbf{\Upsilon}bb \wedge \Pi ab \wedge \mathbf{\Upsilon}aa \wedge \Pi ba$ by (ax_1) and the symmetry of Π, which yields (22). For the converse, let there be a z such that $\mathbf{\Upsilon}bz \wedge \Pi az$ and a z_1 such that $\mathbf{\Upsilon}az_1 \wedge \Pi bz_1$. First, this implies $\neg(\Sigma a \wedge \Sigma b)$, as otherwise we would have Σz by formula (3) (p. 101), which contradicts Πaz, Σa and (df_3). Second, this implies $\neg\mathbf{E}ab$ since otherwise (ax_4) would give $\mathbf{E}az$, which contradicts Πaz and (df_3). Third, we have $\neg\mathbf{E}ba$ since otherwise (ax_4) would give $\mathbf{E}bz_1$, which contradicts Πbz_1 and (df_3). Fourth, we have $(\Sigma a \vee \Sigma b) \supset \exists z(\mathbf{\Upsilon}az \wedge \mathbf{\Upsilon}bz)$ as in the case of, say, Σb (the case of Σa is analogous), the formula Πbz_1 gives the existence of a z_2 such that $\mathbf{\Upsilon}bz_2 \wedge \mathbf{\Upsilon}z_1z_2$, which gives $\mathbf{\Upsilon}bz_2 \wedge \mathbf{\Upsilon}az_2$ by $\mathbf{\Upsilon}az_1$ and (ax_2). These four consequences together establish Πab.

(23) $$\mathbb{Q}^{\mathbf{a}}ab \;=\; \mathbb{Q}^{\mathbf{e}}ab \;=_{df}\; \forall z(\boldsymbol{\Upsilon} bz \supset \Pi az)$$
$$\mathbb{Q}^{\mathbf{i}}ab \;=\; \mathbb{Q}^{\mathbf{o}}ab \;=_{df}\; \Pi ab$$

Within one-way possibility, negative copulae are not equivalent to their affirmative counterparts and there is to be distinguished a negative one-way possibility relation from an affirmative one. Both relations can be obtained in a straightforward way by considering the following diagram:

(24)



Between any two terms *a* and *b*, the $\boldsymbol{\Upsilon}$ relation either holds or it fails to hold. In either case, the two-way possibility relation Π between them may hold or not. Now, affirmative one-way possibility includes the whole sphere of two-way possibility and all cases in which $\boldsymbol{\Upsilon}$ does hold; that is, $\Pi ab \vee \boldsymbol{\Upsilon} ab$ or, using the abbreviation introduced in (df_4), $\overline{\Pi}ab$. This gives the following definition of the affirmative one-way possibility propositions $\mathbb{M}^{\mathbf{a}}ab$ and $\mathbb{M}^{\mathbf{i}}$ab:

(25) a. $\forall z(\boldsymbol{\Upsilon} bz \supset \overline{\Pi}az)$
b. $\exists z(\boldsymbol{\Upsilon} bz \wedge \overline{\Pi}az)$

Negative one-way possibility, on the other hand, includes two-way possibility and all cases in which $\boldsymbol{\Upsilon}$ does not hold, that is, $\Pi ab \vee \neg\boldsymbol{\Upsilon} ab$. This disjunction is equivalent to the formula $\neg\mathbf{E}ab \wedge \neg(\Sigma a \wedge \boldsymbol{\Upsilon} ab)$ or, using the abbreviation introduced in (df_7), $\neg\overline{\mathbf{E}}ab$.[51] Thus, we have the following characterization of the negative one-way possibility propositions $\mathbb{M}^{\mathrm{e}}ab$ and $\mathbb{M}^{\mathrm{o}}ab$:

(26) a. $\forall z(\boldsymbol{\Upsilon} bz \supset \neg\overline{\mathbf{E}}az)$
b. $\exists z(\boldsymbol{\Upsilon} bz \wedge \neg\overline{\mathbf{E}}az)$

These definitions call for improvement only in that (26a) needs to be extended disjunctively by its converted counterpart to ensure $\mathbb{M}^{\mathrm{e}}$-convertibility. As (26b) can be equivalently simplified to $\neg\overline{\mathbf{E}}ab$,[52] the final definitions of the one-way possibility copulae are:

[51] $\Pi ab \vee \neg\boldsymbol{\Upsilon} ab$ implies $\neg\mathbf{E}ab$ by (df_3) and (ax_3). Moreover, $\Pi ab \vee \neg\boldsymbol{\Upsilon} ab$ implies $\neg(\Sigma a \wedge \boldsymbol{\Upsilon} ab)$, as otherwise we would have Σb by formula (3) (p. 101), which contradicts Πab, Σa and (df_3). Thus, $\Pi ab \vee \neg\boldsymbol{\Upsilon} ab$ implies $\neg\overline{\mathbf{E}}ab$. For the converse, assume $\neg\Pi ab \wedge \boldsymbol{\Upsilon} ab$. Due to $\boldsymbol{\Upsilon} ab$ and (ax_1), we have $\exists z(\boldsymbol{\Upsilon} az \wedge \boldsymbol{\Upsilon} bz)$ and therefore $(\Sigma a \vee \Sigma b) \supset \exists z(\boldsymbol{\Upsilon} az \wedge \boldsymbol{\Upsilon} bz)$. Thus, it follows from $\neg\Pi ab$ and (df_3) that either $\Sigma a \wedge \Sigma b$ holds, or $\mathbf{E}ab$ or $\mathbf{E}ba$. In the first case, we have $\Sigma a \wedge \boldsymbol{\Upsilon} ab$; in the second, we have $\mathbf{E}ab$; and in the third case, we have $\Sigma a \wedge \boldsymbol{\Upsilon} ab$ by (df_1). Thus, $\neg(\Pi ab \vee \neg\boldsymbol{\Upsilon} ab)$ implies $\overline{\mathbf{E}}ab$.

[52] (26b) follows from $\neg\overline{\mathbf{E}}ab$ by (ax_1). For the converse, assume $\boldsymbol{\Upsilon} bz \wedge \neg\overline{\mathbf{E}}az$ and $\overline{\mathbf{E}}ab$, that is $\mathbf{E}ab$ or $\Sigma a \wedge \boldsymbol{\Upsilon} ab$. In the first case, we have $\mathbf{E}az$ by (ax_4), contradicting $\neg\overline{\mathbf{E}}az$. In the latter case, we have $\boldsymbol{\Upsilon} az$ by (ax_2) and consequently $\Sigma a \wedge \boldsymbol{\Upsilon} az$, contradicting $\neg\overline{\mathbf{E}}az$.

$$\begin{array}{llll}(27) & \mathbb{M}^{\mathbf{a}}ab & =_{df} & \forall z(\Upsilon bz \supset \overline{\Pi}az) \\ & \mathbb{M}^{\mathbf{e}}ab & =_{df} & \forall z(\Upsilon bz \supset \neg\overline{\mathbf{E}}az) \vee \forall z(\Upsilon az \supset \neg\overline{\mathbf{E}}bz) \\ & \mathbb{M}^{\mathbf{i}}ab & =_{df} & \exists z(\Upsilon bz \wedge \overline{\Pi}az) \\ & \mathbb{M}^{\mathbf{o}}ab & =_{df} & \neg\overline{\mathbf{E}}ab\end{array}$$

This completes the introduction of syllogistic copulae in $\mathcal{A}$. It is shown in Section 4.1 (p. 121ff) that these formalizations yield the well-known relations between modal copulae such as conversion, subalternation and inter-modal implication ($\mathbb{N}$ implies $\mathbb{X}$, $\mathbb{X}$ implies $\mathbb{M}$, $\mathbb{Q}$ implies $\mathbb{M}$).

Note that in $\mathcal{A}$ there is no separate doubly modalized or ampliated possibility such as is drawn upon by many commentators to account for the validity of QQQ-syllogisms. Take for instance the perfect syllogism aaa-1-QQQ: if b possibly applies to all c then, in order to obtain the conclusion that a possibly applies to all c, the major premiss should state that a possibly applies to everything to which b applies *possibly*, not only that a possibly applies to everything to which b applies *accidentally* in the sense of Υ-predication. That means that instead of (20), the major premiss apparently calls for a structure such as:

$$(28) \quad \forall z(\Pi bz \supset \Pi az)$$

Aristotle distinguishes the two two-way possibilities (28) and (20) in An. pr. 32b23-37, but never returns to this distinction and fails to make it clear where he uses which two-way possibility. In $\mathcal{A}$, there is no need to accuse Aristotle of a tacit ambiguity of two-way possibility propositions since (28) follows from (20) (see Theorem 9, p. 122). (28) is just another (weakened) aspect of the very same two-way possibility (20), and once Aristotle has made it explicit, there is no further need to distinguish between two different readings of two-way possibility.

To conclude this section, the axioms, definitions and syllogistic copulae of $\mathcal{A}$ are summarized below. Formally, $\mathcal{A}$ is a theory in familiar first-order logic with three binary non-logical relations:

$$\begin{array}{lll}(29) & \Upsilon ab & \text{(plain accidental predication, } b \text{ the subject term)} \\ & \mathbf{E}ab & \text{(substantial essential predication)} \\ & \widetilde{\mathbf{E}}ab & \text{(non-substantial essential predication)}\end{array}$$

(df$_1$) $\Sigma a =_{df} \exists z\mathbf{E}za$
(df$_2$) $\mathbf{K}ab =_{df} \Sigma a \wedge \Sigma b \wedge \neg\exists z(\Upsilon az \wedge \Upsilon bz)$
(df$_3$) $\Pi ab =_{df} \neg(\Sigma a \wedge \Sigma b) \wedge \neg\mathbf{E}ab \wedge \neg\mathbf{E}ba \wedge ((\Sigma a \vee \Sigma b) \supset \exists z(\Upsilon az \wedge \Upsilon bz))$
(df$_4$) $\overline{\Pi}ab =_{df} \Pi ab \vee \Upsilon ab$
(df$_5$) $\widehat{\mathbf{E}}ab =_{df} \mathbf{E}ab \vee \widetilde{\mathbf{E}}ab$
(df$_6$) $\widehat{\Sigma}a =_{df} \exists z\widehat{\mathbf{E}}za$
(df$_7$) $\overline{\mathbf{E}}ab =_{df} \mathbf{E}ab \vee (\Sigma a \wedge \Upsilon ab)$

(ax$_1$) Υaa
(ax$_2$) $\Upsilon ab \wedge \Upsilon bc \supset \Upsilon ac$
(ax$_3$) $\widehat{\mathbf{E}}ab \supset \Upsilon ab$
(ax$_4$) $\mathbf{E}ab \wedge \Upsilon bc \supset \mathbf{E}ac$
(ax$_5$) $\widetilde{\mathbf{E}}ab \wedge \Upsilon bc \supset \widetilde{\mathbf{E}}ac$

$\mathbb{X}^{\mathbf{a}}ab$	$\mathbf{\Upsilon}ab$
$\mathbb{X}^{\mathbf{e}}ab$	$\forall z(\mathbf{\Upsilon}bz \supset \neg\mathbf{\Upsilon}az)$
$\mathbb{X}^{\mathbf{i}}ab$	$\exists z(\mathbf{\Upsilon}bz \wedge \mathbf{\Upsilon}az)$
$\mathbb{X}^{\mathbf{o}}ab$	$\neg\mathbf{\Upsilon}ab$
$\mathbb{N}^{\mathbf{a}}ab$	$\widehat{\mathbf{E}}ab$
$\mathbb{N}^{\mathbf{e}}ab$	$\mathbf{K}ab$
$\mathbb{N}^{\mathbf{i}}ab$	$\exists z((\mathbf{\Upsilon}bz \wedge \widehat{\mathbf{E}}az) \vee (\mathbf{\Upsilon}az \wedge \widehat{\mathbf{E}}bz))$
$\mathbb{N}^{\mathbf{o}}ab$	$\exists z(\mathbf{\Upsilon}bz \wedge \mathbf{K}az) \vee \exists zv(\widehat{\mathbf{E}}bz \wedge \widehat{\mathbf{E}}av \wedge \forall u(\mathbf{\Upsilon}au \wedge \widehat{\mathbf{\Sigma}}u \supset \mathbf{K}zu))$
$\mathbb{M}^{\mathbf{a}}ab$	$\forall z(\mathbf{\Upsilon}bz \supset \overline{\mathbf{\Pi}}az)$
$\mathbb{M}^{\mathbf{e}}ab$	$\forall z(\mathbf{\Upsilon}bz \supset \neg\overline{\mathbf{E}}az) \vee \forall z(\mathbf{\Upsilon}az \supset \neg\overline{\mathbf{E}}bz)$
$\mathbb{M}^{\mathbf{i}}ab$	$\exists z(\mathbf{\Upsilon}bz \wedge \overline{\mathbf{\Pi}}az)$
$\mathbb{M}^{\mathbf{o}}ab$	$\neg\overline{\mathbf{E}}ab$
$\mathbb{Q}^{\mathbf{a}/\mathbf{e}}ab$	$\forall z(\mathbf{\Upsilon}bz \supset \mathbf{\Pi}az)$
$\mathbb{Q}^{\mathbf{i}/\mathbf{o}}ab$	$\mathbf{\Pi}ab$

Syllogistic Copulae in $\mathcal{A}$

3. The syllogisms

This section presents a summary of Aristotle's validity and invalidity claims, largely following Smith (*1989*) in determining exactly which (in)validity claims Aristotle makes in the Prior Analytics. In the tables below, validity is indicated by boldface numerals and invalidity by italics. There are two kinds of invalidity claim in the modal syllogistic. Either a certain syllogism is declared invalid, which means certain premisses do not yield a certain conclusion but may yield another conclusion, or a premiss-pair is declared inconclusive, which means it does not yield any syllogistic conclusion whatsoever. For a more precise characterization of inconclusiveness, see p. 136. In the following tables, inconclusiveness, which is stronger than simple invalidity, is indicated by $\not\vdash$. The number in square brackets refers to the theorem that gives the proof of the respective (in)validity or inconclusiveness within $\mathcal{A}$. The Bekker pages refer to the beginning of the passage where Aristotle states the respective (in)validity or inconclusiveness in the Prior Analytics. If there is no Bekker page given, Aristotle does not make any claims about the syllogism in question.

By comparing the tables below with the corresponding proofs in the next section, one can see that $\mathcal{A}$ captures exactly all Aristotelian claims on validity, invalidity and inconclusiveness in the modal syllogistic. Thus, $\mathcal{A}$ provides a single consistent model for the whole modal syllogistic – which has been the aim of this paper. Given that Aristotelian copula-modalities are complex relations between terms such as suggested on p. 116, the modal syllogistic is a straightforward extension of the assertoric syllogistic. Assertoric syllogistic is determined by (ax_1) and (ax_2), that is by a transitive and reflexive $\boldsymbol{\Upsilon}$-relation of plain accidental predication. The modal syllogistic is obtained from this unsophisticated system by simply adding two relations of substantial and non-substantial essential predication $\mathbf{E}$ and $\widetilde{\mathbf{E}}$ which imply $\boldsymbol{\Upsilon}$ (i.e. (ax_3)) and which are downward monotonic (transparent) in the subject argument with respect to $\boldsymbol{\Upsilon}$ (i.e. (ax_4) and (ax_5)). Just as the assertoric syllogistic is, so to speak, an elementary consequence of (ax_2), that means Barbara XXX transferred to the underlying level of predicable semantics, the modal syllogistic is a simple and coherent consequence of (ax_4) and (ax_5), that is Barbara NXN transferred to the level of predicable semantics.

We are far from claiming that $\mathcal{A}$ is the one and only adequate formal model for Aristotle's modal syllogistic. What this paper should show is that, contrary to the prevailing *opinio communis*, Aristotle's modal syllogistic, despite wide-ranging differences to modern modal logics, can be understood as a perfectly coherent logic of modality in its own right. We hope that $\mathcal{A}$ can give a fresh impetus to further research to develop improved models that give a more convincing interpretation of the modal syllogistic than suggested in the previous sections and that enable us to understand the modal syllogistic as a well-integrated part of Aristotle's logical semantics of predication and modality.

Table 1

1st figure	XXX	NNN	NXN	XNN
AaB BaC AaC	**25b37**	**29b36 [19]**	**30a17 [18]**	*30a23 [51]*
AeB BaC AeC	**25b40**	**29b36 [19]**	**30a17 [18]**	*30a32 [53]*
AaB BiC AiC	**26a23**	**29b36 [19]**	**30a37 [18]**	*30b2 [52]*
AeB BiC AoC	**26a25**	**29b36 [19]**	**30b1 [18]**	*30b5 [53]*
2nd figure				
BeA BaC AeC	**27a5**	**29b36 [21]**	**30b9 [20]**	*30b18 [53]*
BaA BeC AeC	**27a9**	**29b36 [21]**	*30b20 [54]*	**30b14 [20]**
BeA BiC AoC	**27a32**	**29b36 [21]**	**31a5 [20]**	*[53]*
BaA BoC AoC	**27a36**	**30a6 [22]**	*31a10 [54]*	*31a15 [56]*
3rd figure				
AaB CaB AiC	**28a17**	**29b36 [25]**	**31a24 [24]**	**31a31 [24]**
AeB CaB AoC	**28a26**	**29b36 [25]**	**31a33 [24]**	*31a37 [55]*
AiB CaB AiC	**28b7**	**29b36 [25]**	*31b31 [52]*	**31b12 [23]**
AaB CiB AiC	**28b11**	**29b36 [25]**	**31b19 [23]**	*31b20 [52]*
AoB CaB AoC	**28b15**	**30a7 [26]**	*32a4 [57]*	*31b40 [55]*
AeB CiB AoC	**28b31**	**29b36 [25]**	**31b33 [23]**	*32a1 [53]*

Table 2

1st figure	QQQ	QXQ	XQM	NQM	QNQ	NQX	QNX
AaB BaC AaC	**32b38 [27]**	**33b33 [28]**	**34a34 [29]**	**35b38 [30]**	**36a2 [32]**	*35b38 [58]*	*36a2 [59]*
AeB BaC AeC	**33a1 [27]**	**33b36 [28]**	**34b19 [29]**	**36a7 [30]**	**36a17 [32]**	**36a7 [31]**	*36a17 [60]*
AaB BeC AaC	**33a5 [27]**	*35a20*$^{\nvdash}$ *[70]*	**35a3 [29]**	**36a25 [30]**	*36a27*$^{\nvdash}$ *[69]*	*[58]*	$^{\nvdash}$ *[69]*
AeB BeC AeC	**33a12 [27]**	*35a20*$^{\nvdash}$ *[70]*	**35a11 [29]**	**[30]**	*36a28*$^{\nvdash}$ *[69]*	**[31]**	*36a28*$^{\nvdash}$ *[69]*
AaB BiC AiC	**33a23 [27]**	**35a30 [28]**	**35a35 [29]**	**36a40 [30]**	**35b23 [32]**	*36a40 [58]*	*35b23 [61]*
AeB BiC AoC	**33a25 [27]**	**35a30 [28]**	**35a35 [29]**	**36a34 [30]**	**36a39 [32]**	**36a34 [31]**	*36a39 [60]*
AiB BaC AiC	*33a34*$^{\nvdash}$ *[68]*	*35b11*$^{\nvdash}$ *[71]*	*35b11*$^{\nvdash}$ *[73]*	*36b3*$^{\nvdash}$ *[72]*	*36b10*$^{\nvdash}$ *[71]*	*36b3*$^{\nvdash}$ *[72]*	*36b10*$^{\nvdash}$ *[71]*
AoB BaC AoC	*33a34*$^{\nvdash}$ *[68]*	*35b11*$^{\nvdash}$ *[71]*	*35b11*$^{\nvdash}$ *[73]*	*36b3*$^{\nvdash}$ *[72]*	*36b10*$^{\nvdash}$ *[71]*	*36b3*$^{\nvdash}$ *[72]*	*36b10*$^{\nvdash}$ *[71]*
AiB BeC AoC	*33a34*$^{\nvdash}$ *[68]*	*35b11*$^{\nvdash}$ *[70]*	*35b11*$^{\nvdash}$ *[73]*	*36b3*$^{\nvdash}$ *[72]*	*36b8*$^{\nvdash}$ *[69]*	*36b3*$^{\nvdash}$ *[72]*	*36b8*$^{\nvdash}$ *[69]*
AoB BeC AoC	*33a34*$^{\nvdash}$ *[68]*	*35b11*$^{\nvdash}$ *[70]*	*35b11*$^{\nvdash}$ *[73]*	*36b3*$^{\nvdash}$ *[72]*	*36b8*$^{\nvdash}$ *[69]*	*36b3*$^{\nvdash}$ *[72]*	*36b8*$^{\nvdash}$ *[69]*
AaB BoC AiC	**33a27 [27]**	*35b8*$^{\nvdash}$ *[70]*	**35b5 [29]**	**35b28 [30]**	*35b23*$^{\nvdash}$ *[69]*	*35b28 [58]*	$^{\nvdash}$ *[69]*
AeB BoC AoC	**[27]**	*35b8*$^{\nvdash}$ *[70]*	**35b5 [29]**	**35b30 [30]**	*35b23*$^{\nvdash}$ *[69]*	**35b30 [31]**	$^{\nvdash}$ *[69]*

Table 3

2nd figure	QQQ	QXM	XQM	NQM	QNM	NQX	QNX
BeA BaC AeC	*37a32*$^{\nvdash}$ *[68]*	*37b19*$^{\nvdash}$ *[77]*	**37b24 [34]**	**38a16 [38]**	*38a26*$^{\nvdash}$ *[76]*	**38a16 [35]**	*38a26*$^{\nvdash}$ *[76]*
BaA BeC AeC	*37b10*$^{\nvdash}$ *[68]*	**37b29 [33]**	*37b19*$^{\nvdash}$ *[75]*	*38b4*$^{\nvdash}$ *[74]*	**38a25 [39]**	*38b4*$^{\nvdash}$ *[74]*	**38a25 [36]**
BaA BaC AaC	*37b10*$^{\nvdash}$ *[68]*	*37b35*$^{\nvdash}$ *[77]*	*37b35*$^{\nvdash}$ *[75]*	*38b13*$^{\nvdash}$ *[74]*	*38b13*$^{\nvdash}$ *[76]*	*38b13*$^{\nvdash}$ *[74]*	*38b13*$^{\nvdash}$ *[76]*
BeA BeC AeC	*37b10*$^{\nvdash}$ *[68]*	**37b29 [33]**	**37b29 [34]**	**38b6 [38]**	**38b6 [39]**	**38b6 [35]**	**38b6 [36]**
BeA BiC AoC	*37b13*$^{\nvdash}$ *[68]*	*37b40*$^{\nvdash}$ *[77]*	**38a3 [34]**	**38b25 [38]**	$^{\nvdash}$ *[76]*	**38b25 [35]**	$^{\nvdash}$ *[76]*
BaA BoC AoC	*37b13*$^{\nvdash}$ *[68]*	*38a8*$^{\nvdash}$ *[78]*	*37b40*$^{\nvdash}$ *[75]*	*38b27*$^{\nvdash}$ *[74]*	**[39]**	*38b27*$^{\nvdash}$ *[74]*	**[37]**
BoA BaC AoC	*37b13*$^{\nvdash}$ *[68]*	*37b19*$^{\nvdash}$ *[77]*	*38a8*$^{\nvdash}$ *[80]*	$^{\nvdash}$ *[79]*	*38b27*$^{\nvdash}$ *[76]*	$^{\nvdash}$ *[79]*	*38b27*$^{\nvdash}$ *[76]*
BiA BeC AoC	*37b13*$^{\nvdash}$ *[68]*	$^{\nvdash}$ *[70]*	*37b40*$^{\nvdash}$ *[75]*	$^{\nvdash}$ *[74]*	$^{\nvdash}$ *[69]*	$^{\nvdash}$ *[74]*	$^{\nvdash}$ *[69]*
BeA BoC AoC	*37b13*$^{\nvdash}$ *[68]*	*38a8*$^{\nvdash}$ *[78]*	**38a4 [34]**	**38b31 [38]**	**[39]**	**38b31 [35]**	**[37]**
BaA BiC AiC	*37b13*$^{\nvdash}$ *[68]*	*37b40*$^{\nvdash}$ *[77]*	*37b40*$^{\nvdash}$ *[75]*	*38b29*$^{\nvdash}$ *[74]*	*38b29*$^{\nvdash}$ *[76]*	*38b29*$^{\nvdash}$ *[74]*	*38b29*$^{\nvdash}$ *[76]*

Table 4

3rd figure	QQQ	QXQ	QXM	XQM	NQM	QNM	NQX	QNX
AaB CaB AiC	**39a14 [40]**	**39b16 [41]**	**[42]**	**39b10 [45]**	**40a11 [48]**	**40a16 [47]**	*40a11 [65]*	*40a16 [66]*
AeB CaB AoC	**39a19 [40]**	**39b17 [41]**	**[42]**	**39b17 [45]**	**40a25 [48]**	**40a18 [47]**	**40a25 [46]**	*40a18 [67]*
AaB CeB AiC	**[40]**	$\nvdash$ *[70]*	$\nvdash$ *[70]*	**39b22 [45]**	**40a33 [48]**	*40a35*$^{\nvdash}$ *[69]*	*40a33 [65]*	*40a35*$^{\nvdash}$ *[69]*
AeB CeB AoC	**39a23 [40]**	$\nvdash$ *[70]*	$\nvdash$ *[70]*	**39b23 [45]**	**[48]**	$\nvdash$ *[69]*	**[46]**	$\nvdash$ *[69]*
AiB CaB AiC	**39a35 [40]**	*[62]*	**39b26 [43]**	**39b26 [45]**	**40a39 [48]**	**40a39 [47]**	*40a39 [65]*	*40a39 [66]*
AaB CiB AiC	**39a31 [40]**	**39b26 [41]**	**[42]**	**39b26 [45]**	**40a39 [48]**	**40a39 [47]**	*40a39 [65]*	*40a39 [61]*
AaB CoB AoC	**[40]**	$\nvdash$ *[70]*	$\nvdash$ *[70]*	*[64]*	*[63]*	$\nvdash$ *[69]*	*[64]*	$\nvdash$ *[69]*
AoB CaB AoC	**39a36 [40]**	*[62]*	**39b31 [43]**	**[44]**	**[50]**	**[47]**	**[49]**	*[67]*
AeB CiB AoC	**39a36 [40]**	**39b27 [41]**	**[42]**	**39b27 [45]**	**40b3 [48]**	**40b2 [47]**	**40b3 [46]**	*40b2 [67]*
AiB CeB AiC	**[40]**	$\nvdash$ *[70]*	$\nvdash$ *[70]*	**39b27 [45]**	**40b8 [48]**	*40b8*$^{\nvdash}$ *[69]*	*40b8 [65]*	*40b8*$^{\nvdash}$ *[69]*
AoB CeB AoC	**39a38 [40]**	$\nvdash$ *[70]*	$\nvdash$ *[70]*	**[44]**	**[50]**	$\nvdash$ *[69]*	**[49]**	$\nvdash$ *[69]*
AeB CoB AoC	**39a38 [40]**	$\nvdash$ *[70]*	$\nvdash$ *[70]*	**[45]**	**[48]**	$\nvdash$ *[69]*	**[46]**	$\nvdash$ *[69]*

4. Proofs

4.1. Preliminaries

Theorem 1: $\vdash_{\mathcal{A}} \Sigma a \wedge \Upsilon ab \supset \Sigma b$

Proof. Assume $\Sigma a \wedge \Upsilon ab$. From Σa it follows by (df_1) that there is a z such that $\mathbf{E}za$. This together with Υab gives $\mathbf{E}zb$ by (ax_4), and Σb by (df_1). □

Theorem 2: $\vdash_{\mathcal{A}} \overline{\mathbf{E}}ab \supset (\Sigma b \wedge \Upsilon ab)$

Proof. Assume $\overline{\mathbf{E}}ab$. By (df_7), we have either $\mathbf{E}ab$ or $\Sigma a \wedge \Upsilon ab$. In the first case, we have $\Sigma b \wedge \Upsilon ab$ by (df_1) and (ax_3). In the latter case, we have Σb by Theorem 1. □

Theorem 3: $\vdash_{\mathcal{A}} \neg(\mathbf{K}ab \wedge \Upsilon ab)$

Proof. By (df_2), $\mathbf{K}ab$ implies $\neg\exists z(\Upsilon az \wedge \Upsilon bz)$, whereas Υab, by (ax_1), implies $\Upsilon ab \wedge \Upsilon bb$, and therefore $\exists z(\Upsilon az \wedge \Upsilon bz)$. □

Theorem 4: $\vdash_{\mathcal{A}} \neg\mathbf{K}aa$

Proof. Follows from (ax_1) and Theorem 3. □

Theorem 5: $\vdash_{\mathcal{A}} \mathbf{K}ab \leftrightarrow \mathbf{K}ba$

Proof. The definition of $\mathbf{K}$ in (df_2) is symmetric. □

Theorem 6:

(6.1) $\vdash_{\mathcal{A}} \mathbf{K}ab \wedge \Upsilon bc \supset \mathbf{K}ac$

(6.2) $\vdash_{\mathcal{A}} \mathbf{K}ab \wedge \Upsilon ac \supset \mathbf{K}cb$

Proof. For 6.1, assume $\mathbf{K}ab \wedge \Upsilon bc$, that means $\Sigma a \wedge \Sigma b \wedge \neg\exists z(\Upsilon az \wedge \Upsilon bz) \wedge \Upsilon bc$. Σb together with Υbc gives Σc by Theorem 1. Moreover, we have $\neg\exists z(\Upsilon az \wedge \Upsilon cz)$; otherwise Υcz together with Υbc would give Υbz by (ax_2), and consequently $\Upsilon az \wedge \Upsilon bz$, which contradicts $\neg\exists z(\Upsilon az \wedge \Upsilon bz)$. Thus, we have $\Sigma a \wedge \Sigma c \wedge \neg\exists z(\Upsilon az \wedge \Upsilon cz)$, that is $\mathbf{K}ac$. 6.2 follows from 6.1 by Theorem 5. □

Theorem 7: $\vdash_{\mathcal{A}} \neg(\mathbf{K}ab \wedge \Pi ab)$

Proof. $\mathbf{K}ab$ implies $\Sigma a \wedge \Sigma b$ by (df_2), whereas Πab implies $\neg(\Sigma a \wedge \Sigma b)$ by (df_3). □

Theorem 8: $\forall z(\Upsilon bz \supset \Pi az) \vdash_{\mathcal{A}} \neg\Sigma a$

Proof. Suppose $\forall z(\Upsilon bz \supset \Pi az)$ and Σa. By (ax_1) we have Πab, from which $(\Sigma a \vee \Sigma b) \supset \exists z(\Upsilon az \wedge \Upsilon bz)$ by (df_3). This, together with Σa, gives a z such that $\Upsilon az \wedge \Upsilon bz$ and, by Theorem 1, also Σz. But at the same time Υbz together with $\forall z(\Upsilon bz \supset \Pi az)$ gives Πaz, which contradicts $\Sigma a \wedge \Sigma z$ and (df_3). □

Theorem 9: $\forall z(\Upsilon bz \supset \Pi az) \vdash_{\mathcal{A}} \forall z(\Pi bz \supset \Pi az)$

Proof. Suppose $\forall z(\Upsilon bz \supset \Pi az)$ and Πbz. We need to show Πaz. By Theorem 8, we have $\neg\Sigma a$, from which $\neg(\Sigma a \wedge \Sigma z)$ and $\neg\mathbf{E}za$ by (df_1). Moreover, we have $\neg\mathbf{E}az$. Otherwise, (df_1) would give Σz, and Πbz, by (df_3), would give $(\Sigma b \vee \Sigma z) \supset \exists u(\Upsilon bu \wedge \Upsilon zu)$ and $\Upsilon bu \wedge \Upsilon zu$. Υzu together with $\mathbf{E}az$ gives $\mathbf{E}au$ by (ax_4). However, Υbu and $\forall z(\Upsilon bz \supset \Pi az)$ yield Πau, which contradicts $\mathbf{E}au$ and (df_3). Thus, we have $\neg(\Sigma a \wedge \Sigma z) \wedge \neg\mathbf{E}az \wedge \neg\mathbf{E}za$ and in order to establish Πaz, it remains to show that $(\Sigma a \vee \Sigma z) \supset \exists u(\Upsilon au \wedge \Upsilon zu)$. So assume $\Sigma a \vee \Sigma z$. From $\neg\Sigma a$ we have Σz, which, together with Πbz, gives an u such that $\Upsilon bu \wedge \Upsilon zu$ by (df_3). Υzu and Σz yield Σu by Theorem 1. Moreover, Υbu and $\forall z(\Upsilon bz \supset \Pi az)$ give Πau. This, together with Σu, gives a w such that $\Upsilon aw \wedge \Upsilon uw$ by (df_3). Now, Υuw and Υzu give Υzw by (ax_2), from which $\Upsilon aw \wedge \Upsilon zw$ and $\exists u(\Upsilon au \wedge \Upsilon zu)$. □

Theorem 10: $\exists z(\Upsilon bz \wedge \overline{\Pi}az) \vdash_{\mathcal{A}} \neg\mathbf{K}ab$

Proof. Suppose $\Upsilon bz \wedge \overline{\Pi}az$ and $\mathbf{K}ab$. Due to $\overline{\Pi}az$ we have either Πaz or Υaz by (df_4). The first case is excluded by Theorem 7. In the latter case, we have $\exists z(\Upsilon bz \wedge \Upsilon az)$, which contradicts $\mathbf{K}ab$ and (df_2). □

Theorem 11: $\exists z(\Upsilon bz \wedge \overline{\Pi}az) \vdash_{\mathcal{A}} \exists z(\Upsilon az \wedge \overline{\Pi}bz)$

Proof. Assume $\Upsilon bz \wedge \overline{\Pi}az$ and $\neg\exists z(\Upsilon az \wedge \overline{\Pi}bz)$, that means $\forall z(\Upsilon az \supset \neg\overline{\Pi}bz)$. This gives $\neg\Pi ba$ and $\neg\exists z(\Upsilon az \wedge \Upsilon bz)$ by (ax_1) and (df_4). $\neg\exists z(\Upsilon az \wedge \Upsilon bz)$, together with Υbz and $\overline{\Pi}az$, yields Πaz by (df_4). $\neg\Pi ba$ implies either $\mathbf{E}ab \vee \mathbf{E}ba$ or $(\Sigma a \wedge \Sigma b) \vee ((\Sigma a \vee \Sigma b) \wedge \neg\ \exists z(\Upsilon az \wedge \Upsilon bz))$ by (df_3). The first case contradicts $\neg\exists z(\Upsilon az \wedge \Upsilon bz)$ by (ax_1) and (ax_3). In the latter case, we have $\Sigma a \vee \Sigma b$, which together with Υbz yields $\Sigma a \vee \Sigma z$ by Theorem 1. By (df_3), $\Sigma a \vee \Sigma z$ and Πaz give an u such that $\Upsilon au \wedge \Upsilon zu$. This, together with Υbz, yields $\Upsilon au \wedge \Upsilon bu$ by (ax_2), which contradicts $\neg\exists z(\Upsilon az \wedge \Upsilon bz)$. □

Theorem 12: The following conversions hold:

(12.1)	$\mathbb{X}^{\mathbf{i}}ab \vdash_{\mathcal{A}} \mathbb{X}^{\mathbf{i}}ba$	(12.7)	$\mathbb{M}^{\mathbf{e}}ab \vdash_{\mathcal{A}} \mathbb{M}^{\mathbf{e}}ba$
(12.2)	$\mathbb{X}^{\mathbf{e}}ab \vdash_{\mathcal{A}} \mathbb{X}^{\mathbf{e}}ba$	(12.8)	$\mathbb{M}^{\mathbf{i}}ab \vdash_{\mathcal{A}} \mathbb{M}^{\mathbf{i}}ba$
(12.3)	$\mathbb{N}^{\mathbf{i}}ab \vdash_{\mathcal{A}} \mathbb{N}^{\mathbf{i}}ba$	(12.9)	$\mathbb{X}^{\mathbf{a}}ab \vdash_{\mathcal{A}} \mathbb{X}^{\mathbf{i}}ba$
(12.4)	$\mathbb{N}^{\mathbf{e}}ab \vdash_{\mathcal{A}} \mathbb{N}^{\mathbf{e}}ba$	(12.10)	$\mathbb{N}^{\mathbf{a}}ab \vdash_{\mathcal{A}} \mathbb{N}^{\mathbf{i}}ba$
(12.5)	$\mathbb{Q}^{\mathbf{i}}ab \vdash_{\mathcal{A}} \mathbb{Q}^{\mathbf{i}}ba$	(12.11)	$\mathbb{M}^{\mathbf{a}}ab \vdash_{\mathcal{A}} \mathbb{M}^{\mathbf{i}}ba$
(12.6)	$\mathbb{Q}^{\mathbf{o}}ab \vdash_{\mathcal{A}} \mathbb{Q}^{\mathbf{o}}ba$	(12.12)	$\mathbb{Q}^{\mathbf{a}}ab \vdash_{\mathcal{A}} \mathbb{Q}^{\mathbf{i}}ba$

Proof. 12.1-12.7 follow from the symmetric definition of the copulae in question. 12.8 is Theorem 11. Finally, 12.9-12.12 follow from (ax_1) and the convertibility of the particular affirmative copulae. □

Theorem 13: For all four modalities $\mathbb{H} \in \{\mathbb{X}, \mathbb{N}, \mathbb{M}, \mathbb{Q}\}$, the two subalternations hold:

(13.1) $\mathbb{H}^{\mathbf{a}}ab \vdash_{\mathcal{A}} \mathbb{H}^{\mathbf{i}}ab$ (13.2) $\mathbb{H}^{\mathbf{e}}ab \vdash_{\mathcal{A}} \mathbb{H}^{\mathbf{o}}ab$

Proof. For all four modalities, we have $\mathbb{H}^{\mathbf{a}}ab \vdash_{\mathcal{A}} \mathbb{H}^{\mathbf{i}}ba$ and $\mathbb{H}^{\mathbf{i}}ab \vdash_{\mathcal{A}} \mathbb{H}^{\mathbf{i}}ba$ by Theorem 12, which gives 13.1. 13.2 is identical with 13.1 in the case of $\mathbb{Q}$ and follows from (ax_1) in the case of $\mathbb{X}$ and $\mathbb{N}$. It remains to show $\mathbb{M}^{\mathbf{e}}ab \vdash_{\mathcal{A}} \mathbb{M}^{\mathbf{o}}ab$. So assume $\mathbb{M}^{\mathbf{e}}ab$. We have to consider two cases. First, $\forall z(\Upsilon bz \supset \neg\overline{\mathbf{E}}az)$ gives $\neg\overline{\mathbf{E}}ab$, that is $\mathbb{M}^{\mathbf{o}}ab$, by ($ax_1$). Second, assume $\forall z(\Upsilon az \supset \neg\overline{\mathbf{E}}bz)$ and $\neg\mathbb{M}^{\mathbf{o}}ab$, that is $\overline{\mathbf{E}}ab$. This gives $\Sigma b \wedge \Upsilon ab$ by Theorem 2. Υab together with $\forall z(\Upsilon az \supset \neg\overline{\mathbf{E}}bz)$ gives $\neg\overline{\mathbf{E}}bb$. From this we have $\neg(\Sigma b \wedge \Upsilon bb)$ by (df_7), which contradicts Σb and (ax_1). □

Theorem 14: For all $* \in \{\mathbf{a}, \mathbf{e}, \mathbf{i}, \mathbf{o}\}$: $\mathbb{N}^{*}ab \vdash_{\mathcal{A}} \mathbb{X}^{*}ab$

Proof. For the **a**- and **i**-copula, the claim follows from (ax_3). As to the **e**-copula, $\mathbf{K}ab$ implies $\forall z(\Upsilon bz \supset \neg\Upsilon az)$ by (df_2). For the **o**-copula, there are to be considered two cases. First, assume $\Upsilon bz \wedge \mathbf{K}az$ and Υab. By (ax_2), we have Υaz, which contradicts $\mathbf{K}az$ and Theorem 3. Second, assume $\widehat{\mathbf{E}}bz \wedge \widehat{\mathbf{E}}av \wedge \forall u(\Upsilon au \wedge \widehat{\Sigma}u \supset \mathbf{K}zu)$ and Υab. By (ax_3) and (df_6) we have Υbz and $\widehat{\Sigma}z$. By (ax_2), we have Υaz and by $\forall u(\Upsilon au \wedge \widehat{\Sigma}u \supset \mathbf{K}zu)$, we have $\mathbf{K}zz$, which contradicts Theorem 4. □

Theorem 15: For all $* \in \{\mathbf{a}, \mathbf{e}, \mathbf{i}, \mathbf{o}\}$: $\mathbb{X}^{*}ab \vdash_{\mathcal{A}} \mathbb{M}^{*}ab$

Proof. For the **a**-copula, assume Υab. Then we have $\forall z(\Upsilon bz \supset \Upsilon az)$ by (ax_2), which gives $\forall z(\Upsilon bz \supset \overline{\Pi}az)$ by (df_4). For the **e**-copula, $\forall z(\Upsilon bz \supset \neg\overline{\mathbf{E}}az)$ follows from $\forall z(\Upsilon bz \supset \neg\Upsilon az)$ by Theorem 2. For the **i**-copula, $\exists z(\Upsilon bz \wedge \overline{\Pi}az)$ follows from $\exists z(\Upsilon bz \wedge \Upsilon az)$ by (df_4). For the **o**-copula, $\neg\overline{\mathbf{E}}ab$ follows from $\neg\Upsilon ab$ by Theorem 2. □

Theorem 16: For all $* \in \{\mathbf{a}, \mathbf{e}, \mathbf{i}, \mathbf{o}\}$: $\mathbb{Q}^{*}ab \vdash_{\mathcal{A}} \mathbb{M}^{*}ab$

Proof. For the **a**-copula, $\forall z(\Upsilon bz \supset \overline{\Pi}az)$ follows from $\forall z(\Upsilon bz \supset \Pi az)$ by (df_4). For the **i**-copula, Πab gives $\Upsilon bb \wedge \Pi ab$ by (ax_1), which yields $\exists z(\Upsilon bz \wedge \overline{\Pi}az)$ by (df_4). For the **e**-copula, assume $\forall z(\Upsilon bz \supset \Pi az)$ and $\neg\mathbb{M}^{\mathrm{e}}$ab. This implies that there is a z such that $\Upsilon bz \wedge \overline{\mathbf{E}}az$. Υbz together with $\forall z(\Upsilon bz \supset \Pi az)$ gives Πaz. From $\overline{\mathbf{E}}az$ we have either $\mathbf{E}az$ or Σa by (df_7), but the first case contradicts Πaz and (df_3), and the latter case contradicts $\forall z(\Upsilon bz \supset \Pi az)$ and Theorem 8. For the **o**-copula, assume Πab and $\neg\mathbb{M}^{\mathrm{o}}ab$, that is $\overline{\mathbf{E}}ab$. This gives either $\mathbf{E}ab$ or $\Sigma a \wedge \Upsilon ab$ by (df_7). The first case contradicts Πab and (df_3). In the latter case, we have Σb by Theorem 1, and therefore $\Sigma a \wedge \Sigma b$, which contradicts Πab and (df_3). □

Theorem 17: For all four modalities $\mathbb{H} \in \{\mathbb{X}, \mathbb{N}, \mathbb{M}, \mathbb{Q}\}$:

(17.1) $\mathbb{H}^{\mathbf{a}}ab \vdash_{\mathcal{A}} \mathbb{M}^{\mathbf{i}}ab$ (17.3) $\mathbb{H}^{\mathbf{e}}ab \vdash_{\mathcal{A}} \mathbb{M}^{\mathbf{o}}ab$

(17.2) $\mathbb{H}^{\mathbf{i}}ab \vdash_{\mathcal{A}} \mathbb{M}^{\mathbf{i}}ab$ (17.4) $\mathbb{H}^{\mathbf{o}}ab \vdash_{\mathcal{A}} \mathbb{M}^{\mathbf{o}}ab$

Proof. 17.2 and 17.4 follow from the Theorems 14, 15 and 16. 17.1 and 17.3 follow from 17.2 and 17.4 by Theorem 13. □

4.2. Proofs of validity

Theorem 18: The following syllogisms are valid:

(18.1) (aaa–1–NXN) $\widehat{\mathbf{E}}ab \wedge \boldsymbol{\Upsilon}bc \vdash_{\mathcal{A}} \widehat{\mathbf{E}}ac$
(18.2) (eae–1–NXN) $\mathbf{K}ab \wedge \boldsymbol{\Upsilon}bc \vdash_{\mathcal{A}} \mathbf{K}ac$
(18.3) (aii–1–NXN) $\widehat{\mathbf{E}}ab \wedge \exists z(\boldsymbol{\Upsilon}cz \wedge \boldsymbol{\Upsilon}bz) \vdash_{\mathcal{A}}$
$\exists z((\boldsymbol{\Upsilon}cz \wedge \widehat{\mathbf{E}}az) \vee (\boldsymbol{\Upsilon}az \wedge \widehat{\mathbf{E}}cz))$
(18.4) (eio–1–NXN) $\mathbf{K}ab \wedge \exists z(\boldsymbol{\Upsilon}cz \wedge \boldsymbol{\Upsilon}bz) \vdash_{\mathcal{A}}$
$\exists z(\boldsymbol{\Upsilon}cz \wedge \mathbf{K}az) \vee \exists zv(\widehat{\mathbf{E}}cz \wedge \widehat{\mathbf{E}}av \wedge \forall u(\boldsymbol{\Upsilon}au \wedge \widehat{\Sigma}u \supset \mathbf{K}zu))$

Proof. 18.1: From $\widehat{\mathbf{E}}ab$ we have either $\mathbf{E}ab$ or $\widetilde{\mathbf{E}}ab$. This together with $\boldsymbol{\Upsilon}bc$ gives $\mathbf{E}ac$ or $\widetilde{\mathbf{E}}ac$ by (ax_4) and (ax_5) respectively, from which $\widehat{\mathbf{E}}ac$ by (df_5). 18.2 is Theorem 6.1. For 18.3, assume $\widehat{\mathbf{E}}ab$ and $\boldsymbol{\Upsilon}cz \wedge \boldsymbol{\Upsilon}bz$. This gives $\widehat{\mathbf{E}}az$ by Theorem 18.1, which yields $\boldsymbol{\Upsilon}cz \wedge \widehat{\mathbf{E}}az$. 18.4: $\mathbf{K}ab$ and $\boldsymbol{\Upsilon}cz \wedge \boldsymbol{\Upsilon}bz$ give $\boldsymbol{\Upsilon}cz \wedge \mathbf{K}az$ by Theorem 6.1. □

Theorem 19: The following syllogisms are valid:

(19.1) (aaa–1–NNN) (19.3) (aii–1–NNN)
(19.2) (eae–1–NNN) (19.4) (eio–1–NNN)

Proof. Follows from Theorem 18 by Theorem 14. □

Theorem 20: The following syllogisms are valid:

(20.1) (eae–2–NXN) $\mathbf{K}ba \wedge \boldsymbol{\Upsilon}bc \vdash_{\mathcal{A}} \mathbf{K}ac$
(20.2) (aee–2–XNN) $\boldsymbol{\Upsilon}ba \wedge \mathbf{K}bc \vdash_{\mathcal{A}} \mathbf{K}ac$
(20.3) (eio–2–NXN) $\mathbf{K}ba \wedge \exists z(\boldsymbol{\Upsilon}cz \wedge \boldsymbol{\Upsilon}bz) \vdash_{\mathcal{A}}$
$\exists z(\boldsymbol{\Upsilon}cz \wedge \mathbf{K}az) \vee \exists zv(\widehat{\mathbf{E}}cz \wedge \widehat{\mathbf{E}}av \wedge \forall u(\boldsymbol{\Upsilon}au \wedge \widehat{\Sigma}u \supset \mathbf{K}zu))$

Proof. 20.1 follows from eae-1-NXN (Theorem 18.2) by $\mathbb{N}^{\mathbf{e}}$-conversion (Theorem 12.4) in the major premiss. 20.2: By $\mathbb{N}^{\mathbf{e}}$-conversion (Theorem 12.4) in the minor premiss, the syllogism eae–1–NXN (Theorem 18.2) yields $\mathbb{N}^{\mathbf{e}}ca$, which gives $\mathbb{N}^{\mathbf{e}}ac$ by Theorem 12.4. 20.3 follows from eio-1-NXN (Theorem 18.4) by $\mathbb{N}^{\mathbf{e}}$-conversion (Theorem 12.4) in the major premiss. □

Theorem 21: The following syllogisms are valid:

(21.1) (eae–2–NNN) (21.3) (eio–2–NNN)
(21.2) (aee–2–NNN)

Proof. Follows from Theorem 20 by Theorem 14. □

Theorem 22: The following syllogism is valid: (aoo–2–NNN)
$\widehat{\mathbf{E}}ba \wedge (\exists z(\boldsymbol{\Upsilon}cz \wedge \mathbf{K}bz) \vee \exists zv(\widehat{\mathbf{E}}cz \wedge \widehat{\mathbf{E}}bv \wedge \forall u(\boldsymbol{\Upsilon}bu \wedge \widehat{\Sigma}u \supset \mathbf{K}zu))) \vdash_{\mathcal{A}}$
$\exists z(\boldsymbol{\Upsilon}cz \wedge \mathbf{K}az) \vee \exists zv(\widehat{\mathbf{E}}cz \wedge \widehat{\mathbf{E}}av \wedge \forall u(\boldsymbol{\Upsilon}au \wedge \widehat{\Sigma}u \supset \mathbf{K}zu))$

Proof. First, assume $\mathbf{\Upsilon} cz \wedge \mathbf{K}bz$ and $\widehat{\mathbf{E}}ba$. This gives $\mathbf{\Upsilon} cz \wedge \mathbf{K}az$ by (ax$_3$) and Theorem 6. Second, assume $\widehat{\mathbf{E}}cz \wedge \widehat{\mathbf{E}}bv \wedge \forall u(\mathbf{\Upsilon} bu \wedge \widehat{\Sigma}u \supset \mathbf{K}zu)$ and $\widehat{\mathbf{E}}ba$. This gives $\widehat{\Sigma}a$ by (df$_6$), and $\mathbf{\Upsilon} ba$ by (ax$_3$). Now, $\forall u(\mathbf{\Upsilon} bu \wedge \widehat{\Sigma}u \supset \mathbf{K}zu)$ yields $\mathbf{K}za$, from which $\mathbf{K}az$ by Theorem 5. $\widehat{\mathbf{E}}cz$ gives $\mathbf{\Upsilon} cz$ by (ax$_3$), which yields $\mathbf{\Upsilon} cz \wedge \mathbf{K}az$. □

Theorem 23: The following syllogisms are valid:

(23.1) (iai–3–XNN) $\exists z(\mathbf{\Upsilon} bz \wedge \mathbf{\Upsilon} az) \wedge \widehat{\mathbf{E}}cb \vdash_{\mathcal{A}}$
$\exists z((\mathbf{\Upsilon} cz \wedge \widehat{\mathbf{E}}az) \vee (\mathbf{\Upsilon} az \wedge \widehat{\mathbf{E}}cz))$

(23.2) (aii–3–NXN) $\widehat{\mathbf{E}}ab \wedge \exists z(\mathbf{\Upsilon} bz \wedge \mathbf{\Upsilon} cz) \vdash_{\mathcal{A}}$
$\exists z((\mathbf{\Upsilon} cz \wedge \widehat{\mathbf{E}}az) \vee (\mathbf{\Upsilon} az \wedge \widehat{\mathbf{E}}cz))$

(23.3) (eio–3–NXN) $\mathbf{K}ab \wedge \exists z(\mathbf{\Upsilon} bz \wedge \mathbf{\Upsilon} cz) \vdash_{\mathcal{A}}$
$\exists z(\mathbf{\Upsilon} cz \wedge \mathbf{K}az) \vee \exists zv(\widehat{\mathbf{E}}cz \wedge \widehat{\mathbf{E}}av \wedge \forall u(\mathbf{\Upsilon} au \wedge \widehat{\Sigma}u \supset \mathbf{K}zu))$

Proof. 23.1: By $\mathbb{X}^{\mathbf{i}}$-conversion (Theorem 12.1) in the major premiss, the syllogism aii–1–NXN (Theorem 18.3) gives $\mathbb{N}^{\mathbf{i}}ca$, which yields $\mathbb{N}^{\mathbf{i}}ac$ by $\mathbb{N}^{\mathbf{i}}$-conversion (Theorem 12.3). 23.2 follows from aii–1–NXN (Theorem 18.3) by $\mathbb{X}^{\mathbf{i}}$-conversion (Theorem 12.1) in the minor premiss. 23.3 follows from eio–1–NXN (Theorem 18.4) by $\mathbb{X}^{\mathbf{i}}$-conversion (Theorem 12.1) in the minor premiss. □

Theorem 24: The following syllogisms are valid:

(24.1) (aai–3–XNN)
(24.2) (aai–3–NXN)
(24.3) (eao–3–NXN)

Proof. Follows from Theorem 23 by Theorem 13. □

Theorem 25: The following syllogisms are valid:

(25.1) (aai–3–NNN)
(25.2) (eao–3–NNN)
(25.3) (iai–3–NNN)
(25.4) (aii–3–NNN)
(25.5) (eio–3–NNN)

Proof. Follows from Theorem 23 and Theorem 24 by Theorem 14. □

Theorem 26: The following syllogism is valid: (oao–3–NNN)
$(\exists z(\mathbf{\Upsilon} bz \wedge \mathbf{K}az) \vee \exists zv(\widehat{\mathbf{E}}bz \wedge \widehat{\mathbf{E}}av \wedge \forall u(\mathbf{\Upsilon} au \wedge \widehat{\Sigma}u \supset \mathbf{K}zu))) \wedge \widehat{\mathbf{E}}cb \vdash_{\mathcal{A}}$
$\exists z(\mathbf{\Upsilon} cz \wedge \mathbf{K}az) \vee \exists zv(\widehat{\mathbf{E}}cz \wedge \widehat{\mathbf{E}}av \wedge \forall u(\mathbf{\Upsilon} au \wedge \widehat{\Sigma}u \supset \mathbf{K}zu))$

Proof. First, assume $\mathbf{\Upsilon} bz \wedge \mathbf{K}az$ and $\widehat{\mathbf{E}}cb$. This gives $\mathbf{\Upsilon} cz \wedge \mathbf{K}az$ by (ax$_3$) and (ax$_2$). Second, assume $\widehat{\mathbf{E}}bz \wedge \widehat{\mathbf{E}}av \wedge \forall u(\mathbf{\Upsilon} au \wedge \widehat{\Sigma}u \supset \mathbf{K}zu)$ and $\widehat{\mathbf{E}}cb$. This, together with $\widehat{\mathbf{E}}bz$, gives $\widehat{\mathbf{E}}cz$ by (ax$_4$), (ax$_3$) and (ax$_5$), which yields $\widehat{\mathbf{E}}cz \wedge \widehat{\mathbf{E}}av \wedge \forall u(\mathbf{\Upsilon} au \wedge \widehat{\Sigma}u \supset \mathbf{K}zu)$. □

Theorem 27: The following syllogisms are valid:

(27.1) (aaa, eae, aea, eee–1–QQQ)
$\forall z(\mathbf{\Upsilon} bz \supset \Pi az) \wedge \forall z(\mathbf{\Upsilon} cz \supset \Pi bz) \vdash_{\mathcal{A}} \forall z(\mathbf{\Upsilon} cz \supset \Pi az)$

(27.2) (aii, eio, aoi, eoo–1–QQQ)
$\forall z(\Upsilon bz \supset \Pi az) \wedge \Pi bc \vdash_{\mathcal{A}} \Pi ac$

Proof. In both syllogisms, the major premiss $\forall z(\Upsilon bz \supset \Pi az)$ implies $\forall z(\Pi bz \supset \Pi az)$ by Theorem 9, which, together with the minor premiss, yields the conclusion. □

Theorem 28: The following syllogisms are valid:

(28.1) (aaa, eae–1–QXQ) $\forall z(\Upsilon bz \supset \Pi az) \wedge \Upsilon bc \vdash_{\mathcal{A}} \forall z(\Upsilon cz \supset \Pi az)$
(28.2) (aii, eio–1–QXQ) $\forall z(\Upsilon bz \supset \Pi az) \wedge \exists z(\Upsilon cz \wedge \Upsilon bz) \vdash_{\mathcal{A}} \Pi ac$

Proof. 28.1 follows from (ax_2). 28.2: Assume $\Upsilon cz \wedge \Upsilon bz$. This, together with the major premiss, yields Πaz. It is to be shown that Πac. First, we have $\neg(\Sigma a \wedge \Sigma c)$, as otherwise Υcz gives $\Sigma a \wedge \Sigma z$ by Theorem 1, which contradicts Πaz and (df_3). Second, we have $\neg\mathbf{E}ac$, as otherwise Υcz yields $\mathbf{E}az$ by (ax_4), which contradicts Πaz and (df_3). Third, we have $\neg\mathbf{E}ca$, as otherwise we would have Σa by (df_1), which contradicts $\forall z(\Upsilon bz \supset \Pi az)$ and Theorem 8. Thus, we have $\neg(\Sigma a \wedge \Sigma c) \wedge \neg\mathbf{E}ac \wedge \neg\mathbf{E}ca$ and in order to establish Πac, it remains to show $(\Sigma a \vee \Sigma c) \supset \exists z(\Upsilon az \wedge \Upsilon cz)$. So assume $\Sigma a \vee \Sigma c$. This, together with Υcz, yields $\Sigma a \vee \Sigma z$ by Theorem 1, which, together with Πaz, gives $\exists u(\Upsilon au \wedge \Upsilon zu)$ by (df_3). Finally, $\exists u(\Upsilon au \wedge \Upsilon zu)$ and Υcz yield $\exists u(\Upsilon au \wedge \Upsilon cu)$ by (ax_2). □

Theorem 29: The following syllogisms are valid:

(29.1) (aaa, aea–1–XQM) $\Upsilon ab \wedge \forall z(\Upsilon cz \supset \Pi bz) \vdash_{\mathcal{A}} \forall z(\Upsilon cz \supset \overline{\Pi}az)$
(29.2) (eae, eee–1–XQM) $\forall z(\Upsilon bz \supset \neg\Upsilon az) \wedge \forall z(\Upsilon cz \supset \Pi bz) \vdash_{\mathcal{A}}$
$\forall z(\Upsilon cz \supset \neg\overline{\mathbf{E}}az) \vee \forall z(\Upsilon az \supset \neg\overline{\mathbf{E}}cz)$
(29.3) (aii, aoi–1–XQM) $\Upsilon ab \wedge \Pi bc \vdash_{\mathcal{A}} \exists z(\Upsilon cz \wedge \overline{\Pi}az)$
(29.4) (eio, eoo–1–XQM) $\forall z(\Upsilon bz \supset \neg\Upsilon az) \wedge \Pi bc \vdash_{\mathcal{A}} \neg\overline{\mathbf{E}}ac$

Proof. 29.1: Assume Υcz. It is to be shown that $\overline{\Pi}az$. In the case of Υaz, we have $\overline{\Pi}az$ by (df_4). Thus, assume $\neg\Upsilon az$. This gives $\neg\mathbf{E}az$ by (ax_3). Moreover, we have $\neg\Sigma a$, as otherwise Υab would give Σb by Theorem 1, which contradicts $\forall z(\Upsilon cz \supset \Pi bz)$ and Theorem 8. As a consequence, we have $\neg\mathbf{E}za$, as otherwise we would have Σa by (df_1). Thus, we have $\neg(\Sigma a \wedge \Sigma z) \wedge \neg\mathbf{E}az \wedge \neg\mathbf{E}za$ and in order to establish Πaz and $\overline{\Pi}az$, it remains to show $(\Sigma a \vee \Sigma z) \supset \exists u(\Upsilon au \wedge \Upsilon zu)$. So assume $\Sigma a \vee \Sigma z$. Due to $\neg\Sigma a$, we have Σz. Υcz and the minor premiss yield Πbz, which gives $(\Sigma b \vee \Sigma z) \supset \exists u(\Upsilon bu \wedge \Upsilon zu)$ by (df_3). Now, Σz yields $\exists u(\Upsilon bu \wedge \Upsilon zu)$, which, together with the major premiss Υab, gives $\exists u(\Upsilon au \wedge \Upsilon zu)$ by (ax_2).

29.2: Negating the conclusion gives a z with $\Upsilon cz \wedge \overline{\mathbf{E}}az$. $\overline{\mathbf{E}}az$ gives $\Sigma z \wedge \Upsilon az$ by Theorem 2. Υcz together with the minor premiss yields Πbz. This gives $(\Sigma b \vee \Sigma z) \supset \exists u(\Upsilon bu \wedge \Upsilon zu)$ by (df_3), which, together with Σz, gives an u such that $\Upsilon bu \wedge \Upsilon zu$. Now, Υzu and Υaz yield Υau by (ax_2), which contradicts Υbu and the major premiss $\forall z(\Upsilon bz \supset \neg\Upsilon az)$.

29.3: By (df_4) and the symmetry of Π, the premisses yield $\Upsilon ab \wedge \overline{\Pi}cb$ and $\exists z(\Upsilon az \wedge \overline{\Pi}cz)$. This gives $\exists z(\Upsilon cz \wedge \overline{\Pi}az)$ by Theorem 11.

29.4: Negating the conclusion gives $\overline{\mathbf{E}}ac$ and $\Sigma c \wedge \Upsilon ac$ by Theorem 2. Πbc and (df_3) yield $(\Sigma b \vee \Sigma c) \supset \exists u(\Upsilon bu \wedge \Upsilon cu)$, which, together with Σc, gives $\Upsilon bu \wedge \Upsilon cu$. Now, Υcu and Υac yield Υau by (ax_2), which contradicts Υbu and the major premiss $\forall z(\Upsilon bz \supset \neg \Upsilon az)$. □

Theorem 30: The following syllogisms are valid:

(30.1)	(aaa, aea–1–NQM)	(30.3)	(aii, aoi–1–NQM)
(30.2)	(eae, eee–1–NQM)[53]	(30.4)	(eio, eoo–1–NQM)

Proof. Follows from Theorem 29 by Theorem 14. □

Theorem 31: The following syllogisms are valid:

(31.1) (eae, eee–1–NQX) $\mathbf{K}ab \wedge \forall z(\Upsilon cz \supset \Pi bz) \vdash_{\mathcal{A}} \forall z(\Upsilon cz \supset \neg \Upsilon az)$
(31.2) (eio, eoo–1–NQX) $\mathbf{K}ab \wedge \Pi bc \vdash_{\mathcal{A}} \neg \Upsilon ac$

Proof. 31.1: Negating the conclusion gives a z with $\Upsilon cz \wedge \Upsilon az$. This, together with the major premiss $\mathbf{K}ab$, yields $\mathbf{K}bz$ by Theorem 6, while together with the minor premiss it yields Πbz, which contradicts Theorem 7.

31.2: Negating the conclusion gives Υac, which together with $\mathbf{K}ab$ yields $\mathbf{K}cb$ by Theorem 6, which contradicts Πbc and Theorem 7. □

Theorem 32: The following syllogisms are valid:

(32.1) (aaa, eae–1–QNQ) (32.2) (aii, eio–1–QNQ)

Proof. Follows from Theorem 28 by Theorem 14. □

Theorem 33: The following syllogism is valid: (aee, eee–2–QXM)
$\forall z(\Upsilon az \supset \Pi bz) \wedge \forall z(\Upsilon cz \supset \neg \Upsilon bz) \vdash_{\mathcal{A}} \forall z(\Upsilon cz \supset \neg \overline{\mathbf{E}}az) \vee \forall z(\Upsilon az \supset \neg \overline{\mathbf{E}}cz)$

Proof. By $\mathbb{X}^{\mathbf{e}}$-conversion (Theorem 12.2) in the minor premiss the syllogism eae-1-XQM (Theorem 29.2) gives $\mathbb{M}^{\mathrm{e}}ca$, which gives $\mathbb{M}^{\mathrm{e}}ac$ by $\mathbb{M}^{\mathrm{e}}$-conversion (Theorem 12.7). □

[53] The premisses of eae-1-NQM are $\mathbf{K}ab$ and $\forall z(\Upsilon cz \supset \Pi bz)$. The first implies Σb by (df_2) and the latter implies $\neg\Sigma b$ by Theorem 8. Thus, these premisses turn out to be inconsistent in $\mathcal{A}$, which makes eae-1-NQM trivially valid. The same problem occurs with the premiss-pairs ea-2-NQ and ae-2-QN. This kind of trivial validity can be avoided by weakening (df_2) to $(\Sigma a \vee \Sigma b) \wedge \neg\exists z(\Upsilon az \wedge \Upsilon bz)$. With this modification, the mentioned premiss-pairs become consistent while all results about (in)validity and inconclusiveness of syllogisms are preserved, except for aoo-2-QNX and oao-3-NQX, which become invalid. This, however, is acceptable since Aristotle does not claim anything about aoo-2-QNX in An. pr. and since it is highly dubious whether he claims the validity of oao-3-NQX. Ross (*1949*, p. 369, 285f) takes Aristotle to claim the validity of oao-3-NQX in An. pr. 40b3-8, whereas Becker (*1933*, Table 3) does not read this difficult passage as stating the validity of oao-3-NQX and Smith (*1989*, p. 235) remains undecided as to this contentious question.

Theorem 34: The following syllogisms are valid:

(34.1) (eae, eee–2–XQM) $\forall z(\Upsilon az \supset \neg\Upsilon bz) \wedge \forall z(\Upsilon cz \supset \Pi bz) \vdash_{\mathcal{A}}$
$\forall z(\Upsilon cz \supset \neg\overline{\mathbf{E}}az) \vee \forall z(\Upsilon az \supset \neg\overline{\mathbf{E}}cz)$
(34.2) (eio, eoo–2–XQM) $\forall z(\Upsilon az \supset \neg\Upsilon bz) \wedge \Pi bc \vdash_{\mathcal{A}} \neg\overline{\mathbf{E}}ac$

Proof. Follows from eae-1-XQM (Theorem 29.2) and eio-1-XQM (Theorem 29.4) by $\mathbb{X}^{\mathbf{e}}$-conversion (Theorem 12.2) in the major premiss. □

Theorem 35: The following syllogisms are valid:

(35.1) (eae, eee–2–NQX) $\mathbf{K}ba \wedge \forall z(\Upsilon cz \supset \Pi bz) \vdash_{\mathcal{A}} \forall z(\Upsilon cz \supset \neg\Upsilon az)$
(35.2) (eio, eoo–2–NQX) $\mathbf{K}ba \wedge \Pi bc \vdash_{\mathcal{A}} \neg\Upsilon ac$

Proof. Follows from eae-1-NQX (Theorem 31.1) and eio-1-NQX (Theorem 31.2) by $\mathbb{N}^{\mathbf{e}}$-conversion (Theorem 12.4) in the major premiss. □

Theorem 36: The following syllogism is valid: (aee, eae–2–QNX)
$\forall z(\Upsilon az \supset \Pi bz) \wedge \mathbf{K}bc \vdash_{\mathcal{A}} \forall z(\Upsilon cz \supset \neg\Upsilon az)$

Proof. By $\mathbb{N}^{\mathbf{e}}$-conversion (Theorem 12.4) in the minor premiss the syllogism eae-1-NQX (Theorem 31.1) gives $\mathbb{X}^{\mathbf{e}}ca$, which yields $\mathbb{X}^{\mathbf{e}}ac$ by $\mathbb{X}^{\mathbf{e}}$-conversion (Theorem 12.2). □

Theorem 37: The following syllogism is valid: (aoo, eoo–2–QNX)
$\forall z(\Upsilon az \supset \Pi bz) \wedge (\exists z(\Upsilon cz \wedge \mathbf{K}bz) \vee$
$\exists zv(\widehat{\mathbf{E}}cz \wedge \widehat{\mathbf{E}}bv \wedge \forall u(\Upsilon bu \wedge \widehat{\Sigma}u \supset \mathbf{K}zu))) \vdash_{\mathcal{A}} \neg\Upsilon ac$

Proof. First, assume $\Upsilon cz \wedge \mathbf{K}bz$ and Υac, which gives Υaz by (ax_2). This, together with the major premiss, yields Πbz, which contradicts $\mathbf{K}bz$ and Theorem 7. Second, assume $\widehat{\mathbf{E}}cz \wedge \widehat{\mathbf{E}}bv \wedge \forall u(\Upsilon bu \wedge \widehat{\Sigma}u \supset \mathbf{K}zu)$ and Υac. $\widehat{\mathbf{E}}cz$ gives Υcz by (ax_3). Moreover, $\widehat{\mathbf{E}}bv$ gives $\Upsilon bv \wedge \widehat{\Sigma}v$ by (ax_3) and (df_6). This, together with $\forall u\, (\Upsilon bu \wedge \widehat{\Sigma}u \supset \mathbf{K}zu)$, gives $\mathbf{K}zv$, from which Σz by (df_2). Now, Υac and Υcz give Υaz by (ax_2). This, together with the major premiss, yields Πbz and, by (df_3), also $(\Sigma b \vee \Sigma z) \supset \exists u(\Upsilon bu \wedge \Upsilon zu)$, from which Σz gives an u such that $\Upsilon bu \wedge \Upsilon zu$. This, together with Σz, gives Σu and $\widehat{\Sigma}u$ by Theorem 1 and (df_6). Finally, $\Upsilon bu \wedge \widehat{\Sigma}u$ and $\forall u(\Upsilon bu \wedge \widehat{\Sigma}u \supset \mathbf{K}zu)$ give $\mathbf{K}zu$, which contradicts Υzu and Theorem 3. □

Theorem 38: The following syllogisms are valid:

(38.1) (eae, eee–2–NQM) (38.2) (eio, eoo–2–NQM)

Proof. Follows from Theorem 35 by Theorem 15. □

Theorem 39: The following syllogisms are valid:

(39.1) (aee, eee–2–QNM) (39.2) (aoo, eoo–2–QNM)

Proof. Follows from Theorem 36 and Theorem 37 by Theorem 15. □

Theorem 40: The following syllogisms are valid:

(40.1) (aai, eao, aei, eeo–3–QQQ) $\forall z(\Upsilon bz \supset \Pi az) \wedge \forall z(\Upsilon bz \supset \Pi cz) \vdash_{\mathcal{A}} \Pi ac$
(40.2) (aii, aoo, eio, eoo–3–QQQ) $\forall z(\Upsilon bz \supset \Pi az) \wedge \Pi cb \vdash_{\mathcal{A}} \Pi ac$
(40.3) (iai, oao, iei, oeo–3–QQQ) $\Pi ab \wedge \forall z(\Upsilon bz \supset \Pi cz) \vdash_{\mathcal{A}} \Pi ac$

Proof. 40.1 follows by Theorem 13 from Theorem 40.2. 40.2 and 40.3 follow from aii-1-QQQ (Theorem 27.2) by $\mathbb{Q}^{\mathbf{i}}$-conversion (Theorem 12.5). □

Theorem 41: The following syllogisms are valid:

(41.1) (aai, eao–3–QXQ) $\forall z(\Upsilon bz \supset \Pi az) \wedge \Upsilon cb \vdash_{\mathcal{A}} \Pi ac$
(41.2) (aii, eio–3–QXQ) $\forall z(\Upsilon bz \supset \Pi az) \wedge \exists z(\Upsilon bz \wedge \Upsilon cz) \vdash_{\mathcal{A}} \Pi ac$

Proof. Follows from aii-1-QXQ (Theorem 28.2) by $\mathbb{X}^{\mathbf{a}}$-conversion (Theorem 12.9) and $\mathbb{X}^{\mathbf{i}}$-conversion (Theorem 12.1) in the minor premiss. □

Theorem 42: The following syllogisms are valid:

(42.1) (aai–3–QXM) (42.3) (aii–3–QXM)
(42.2) (eao–3–QXM) (42.4) (eio–3–QXM)

Proof. Follows from Theorem 41 by Theorem 16. □

Theorem 43: The following syllogisms are valid:

(43.1) (iai–3–QXM) $\Pi ab \wedge \Upsilon cb \vdash_{\mathcal{A}} \exists z(\Upsilon cz \wedge \overline{\Pi} az)$
(43.2) (oao–3–QXM) $\Pi ab \wedge \Upsilon cb \vdash_{\mathcal{A}} \neg \overline{\mathbf{E}} ac$

Proof. 43.1: By $\mathbb{Q}^{\mathbf{i}}$-conversion (Theorem 12.5) in the major premiss the syllogism aii-1-XQM (Theorem 29.3) gives $\mathbb{M}^{\mathbf{i}}ca$, which yields $\mathbb{M}^{\mathbf{i}}ac$ by $\mathbb{M}^{\mathbf{i}}$-conversion (Theorem 12.8).

43.2: Negating the conclusion gives $\overline{\mathbf{E}}ac$, from which we have either $\mathbf{E}ac$ or $\Sigma a \wedge \Upsilon ac$ by (df_7). In the first case, Υcb gives $\mathbf{E}ab$ by (ax_4), contradicting Πab and (df_3). In the latter case, Υcb gives Υab by (ax_2). This together with Σa gives $\Sigma a \wedge \Sigma b$ by Theorem 1, contradicting Πab and (df_3). □

Theorem 44: The following syllogism is valid: (oao, oeo–3–XQM) $\neg \Upsilon ab \wedge \forall z(\Upsilon bz \supset \Pi cz) \vdash_{\mathcal{A}} \neg \overline{\mathbf{E}} ac$

Proof. Negating the conclusion gives $\overline{\mathbf{E}}ac$ and Σc by Theorem 2, which contradicts $\forall z(\Upsilon bz \supset \Pi cz)$ and Theorem 8. □

Theorem 45: The following syllogisms are valid:

(45.1) (eao, eeo–3–XQM) $\forall z(\Upsilon bz \supset \neg \Upsilon az) \wedge \forall z(\Upsilon bz \supset \Pi cz) \vdash_{\mathcal{A}} \neg \overline{\mathbf{E}} ac$
(45.2) (iai, iei–3–XQM) $\exists z(\Upsilon bz \wedge \Upsilon az) \wedge \forall z(\Upsilon bz \supset \Pi cz) \vdash_{\mathcal{A}} \exists z(\Upsilon cz \wedge \overline{\Pi} az)$
(45.3) (aai, aei–3–XQM) $\Upsilon ab \wedge \forall z(\Upsilon bz \supset \Pi cz) \vdash_{\mathcal{A}} \exists z(\Upsilon cz \wedge \overline{\Pi} az)$

(45.4) (aii–3–XQM) $\boldsymbol{\Upsilon}ab \wedge \Pi cb \vdash_{\mathcal{A}} \exists z(\boldsymbol{\Upsilon}cz \wedge \overline{\Pi}az)$
(45.5) (eio, eoo–3–XQM) $\forall z(\boldsymbol{\Upsilon}bz \supset \neg\boldsymbol{\Upsilon}az) \wedge \Pi cb \vdash_{\mathcal{A}} \neg\overline{\mathbf{E}}ac$

Proof. 45.1 follows from eio-1-XQM (Theorem 29.4) by $\mathbb{Q}^{\mathbf{a}}$-conversion (Theorem 12.12) in the minor premiss. 45.2: The major premiss gives $\boldsymbol{\Upsilon}bz \wedge \boldsymbol{\Upsilon}az$, which, together with the minor premiss, gives Πcz and, by (df_4), $\overline{\Pi}cz$. Thus, we have $\exists z(\boldsymbol{\Upsilon}az \wedge \overline{\Pi}cz)$, which yields $\exists z(\boldsymbol{\Upsilon}cz \wedge \overline{\Pi}az)$ by Theorem 11. 45.3 follows from 45.2 by Theorem 13. 45.4 and 45.5 follow from aii-1-XQM (Theorem 29.3) and eio-1-XQM (Theorem 29.4) by $\mathbb{Q}^{\mathbf{i}}$-conversion (Theorem 12.5) in the minor premiss. □

Theorem 46: The following syllogisms are valid:

(46.1) (eao, eeo–3–NQX) $\mathbf{K}ab \wedge \forall z(\boldsymbol{\Upsilon}bz \supset \Pi cz) \vdash_{\mathcal{A}} \neg\boldsymbol{\Upsilon}ac$
(46.2) (eio, eoo–3–NQX) $\mathbf{K}ab \wedge \Pi cb \vdash_{\mathcal{A}} \neg\boldsymbol{\Upsilon}ac$

Proof. Follows from eio-1-NQX (Theorem 31.2) by $\mathbb{Q}^{\mathbf{a}}$-conversion (Theorem 12.12) and $\mathbb{Q}^{\mathbf{i}}$-conversion (Theorem 12.5) in the minor premiss. □

Theorem 47: The following syllogisms are valid:

(47.1) (aai–3–QNM) (47.4) (eio–3–QNM)
(47.2) (eao–3–QNM) (47.5) (iai–3–QNM)
(47.3) (aii–3–QNM) (47.6) (oao–3–QNM)

Proof. Follows from Theorem 42 and 43 by Theorem 14. □

Theorem 48: The following syllogisms are valid:

(48.1) (eao, eeo–3–NQM) (48.4) (aii–3–NQM)
(48.2) (iai, iei–3–NQM) (48.5) (eio, eoo–3–NQM)
(48.3) (aai, aei–3–NQM)

Proof. Follows from Theorem 45 by Theorem 14. □

Theorem 49: The following syllogism is valid: (oao, oeo–3–NQX)
$(\exists z(\boldsymbol{\Upsilon}bz \wedge \mathbf{K}az) \vee \exists zv(\widehat{\mathbf{E}}bz \wedge \widehat{\mathbf{E}}av \wedge \forall u(\boldsymbol{\Upsilon}au \wedge \widehat{\Sigma}u \supset \mathbf{K}zu))) \wedge$
$\forall z(\boldsymbol{\Upsilon}bz \supset \Pi cz) \vdash_{\mathcal{A}} \neg\boldsymbol{\Upsilon}ac$

Proof. First, assume $\boldsymbol{\Upsilon}bz \wedge \mathbf{K}az$. This, together with the minor premiss, yields Πcz. Negating the conclusion gives $\boldsymbol{\Upsilon}ac$, which yields $\mathbf{K}cz$ by Theorem 6, contradicting Πcz and Theorem 7.

Second, assume $\widehat{\mathbf{E}}bz \wedge \widehat{\mathbf{E}}av \wedge \forall u(\boldsymbol{\Upsilon}au \wedge \widehat{\Sigma}u \supset \mathbf{K}zu)$. $\widehat{\mathbf{E}}av$ yields $\boldsymbol{\Upsilon}av \wedge \widehat{\Sigma}v$ by (ax_3) and (df_6). This, together with $\forall u(\boldsymbol{\Upsilon}au \wedge \widehat{\Sigma}u \supset \mathbf{K}zu)$, gives $\mathbf{K}zv$ and, by (df_2), also Σz. Moreover, $\widehat{\mathbf{E}}bz$ yields $\boldsymbol{\Upsilon}bz$ by (ax_3), from which the minor premiss gives Πcz and, by (df_3), $(\Sigma c \vee \Sigma z) \supset \exists u(\boldsymbol{\Upsilon}cu \wedge \boldsymbol{\Upsilon}zu)$. Now, Σz gives $\boldsymbol{\Upsilon}cu \wedge \boldsymbol{\Upsilon}zu$ and, by Theorem 1 and (df_6), also $\widehat{\Sigma}u$. Negating the conclusion gives $\boldsymbol{\Upsilon}ac$, which, together with $\boldsymbol{\Upsilon}cu$, gives $\boldsymbol{\Upsilon}au$ by (ax_2). Thus, we have

$\mathbf{\Upsilon}au \wedge \widehat{\Sigma}u$, from which $\forall u(\mathbf{\Upsilon}au \wedge \widehat{\Sigma}u \supset \mathbf{K}zu)$ gives $\mathbf{K}zu$, contradicting $\mathbf{\Upsilon}zu$ and Theorem 3. □

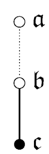
Theorem 50: The following syllogism is valid: (oao, oeo–3–NQM)

Proof. Follows from Theorem 49 by Theorem 15. □

4.3. Proofs of invalidity

Theorem 51: The syllogism (aaa–1–XNN) is invalid.

Proof. Let $\mathfrak{A}$ be a model for $\mathcal{A}$ with A = $\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, $\mathbf{E}^{\mathfrak{A}} = \{\mathfrak{bc}, \mathfrak{cc}\}$ and $\mathbf{\Upsilon}^{\mathfrak{A}} = \{\mathfrak{ab}, \mathfrak{ac}\} \cup \mathbf{E}^{\mathfrak{A}} \cup \mathrm{Id}(A)$. Then we have $\mathfrak{A} \models \mathbf{\Upsilon}ab \wedge \widehat{\mathbf{E}}bc$, that is $\mathfrak{A} \models \mathbb{X}^{\mathbf{a}}ab \wedge \mathbb{N}^{\mathbf{a}}bc$, but also $\mathfrak{A} \not\models \widehat{\mathbf{E}}ac$ and $\mathfrak{A} \not\models \mathbb{N}^{\mathbf{a}}ac$.

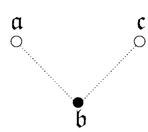


□

Theorem 52: The following syllogisms are invalid:

(52.1) (aii–1–XNN) (52.3) (iai–3–NXN)
(52.2) (aii–3–XNN)

Proof. Let $\mathfrak{A}$ be a model for $\mathcal{A}$ with A = $\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, $\mathbf{E}^{\mathfrak{A}} = \{\mathfrak{bb}\}$ and $\mathbf{\Upsilon}^{\mathfrak{A}} = \{\mathfrak{ab}, \mathfrak{cb}\} \cup \mathrm{Id}(A)$. Then we have $\mathfrak{A} \models \mathbf{\Upsilon}ab$, that is $\mathfrak{A} \models \mathbb{X}^{\mathbf{a}}ab$, and $\mathfrak{A} \models \mathbf{\Upsilon}cb \wedge \widehat{\mathbf{E}}bb$, from which $\mathfrak{A} \models \mathbb{N}^{\mathbf{i}}bc \wedge \mathbb{N}^{\mathbf{i}}cb$. But at the same time we have $\mathfrak{A} \not\models \exists z((\mathbf{\Upsilon}cz \wedge \widehat{\mathbf{E}}az) \vee (\mathbf{\Upsilon}az \wedge \widehat{\mathbf{E}}cz))$, that is $\mathfrak{A} \not\models \mathbb{N}^{\mathbf{i}}ac$, which invalidates 52.1 and 52.2. Moreover, we have $\mathfrak{A} \models \mathbf{\Upsilon}ab \wedge \widehat{\mathbf{E}}bb \wedge \mathbf{\Upsilon}cb$, from which $\mathfrak{A} \models \mathbb{N}^{\mathbf{i}}ab \wedge \mathbb{X}^{\mathbf{a}}cb$, which invalidates 52.3.

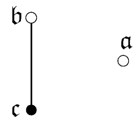


□

Theorem 53: The following syllogisms are invalid:

(53.1) (eae–1–XNN) (53.4) (eio–2–XNN)
(53.2) (eae–2–XNN) (53.5) (eio–3–XNN)
(53.3) (eio–1–XNN)

Proof. Let $\mathfrak{A}$ be a model for $\mathcal{A}$ with A = $\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, $\mathbf{E}^{\mathfrak{A}} = \{\mathfrak{bc}, \mathfrak{cc}\}$, $\mathbf{\Upsilon}^{\mathfrak{A}} = \mathbf{E}^{\mathfrak{A}} \cup \mathrm{Id}(A)$. Then we have $\mathfrak{A} \models \forall z(\mathbf{\Upsilon}bz \supset \neg\mathbf{\Upsilon}az)$, that is $\mathfrak{A} \models \mathbb{X}^{\mathbf{e}}ab \wedge \mathbb{X}^{\mathbf{e}}ba$, and $\mathfrak{A} \models \widehat{\mathbf{E}}bc$, from which $\mathfrak{A} \models \mathbb{N}^{\mathbf{a}}bc$ and $\mathfrak{A} \models \mathbb{N}^{\mathbf{i}}bc \wedge \mathbb{N}^{\mathbf{i}}cb$. But at the same time we have $\mathfrak{A} \not\models \mathbf{K}ac$, which invalidates 53.1 and 53.2. Moreover, we have $\mathfrak{A} \not\models \exists z(\mathbf{\Upsilon}cz \wedge \mathbf{K}az)$, $\mathfrak{A} \not\models \exists zv(\widehat{\mathbf{E}}cz \wedge \widehat{\mathbf{E}}av \wedge \forall u(\mathbf{\Upsilon}au \wedge \widehat{\Sigma}u \supset \mathbf{K}zu))$ and $\mathfrak{A} \not\models \mathbb{N}^{\mathbf{o}}ac$, which invalidates 53.3-53.5.

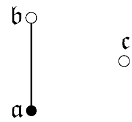


□

Theorem 54: The following syllogisms are invalid:

(54.1) (aee–2–NXN) (54.2) (aoo–2–NXN)

Proof. Let $\mathfrak{A}$ be a model for $\mathcal{A}$ with $\mathrm{A} = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, $\mathbf{E}^{\mathfrak{A}} = \{\mathfrak{ba}, \mathfrak{aa}\}$, $\mathbf{\Upsilon}^{\mathfrak{A}} = \mathbf{E}^{\mathfrak{A}} \cup \mathrm{Id(A)}$. Then we have $\mathfrak{A} \models \widehat{\mathbf{E}}ba$, that is $\mathfrak{A} \models \mathbb{N}^{\mathbf{a}}ba$, and $\mathfrak{A} \models \forall z(\mathbf{\Upsilon}cz \supset \neg\mathbf{\Upsilon}bz)$, that is $\mathfrak{A} \models \mathbb{X}^{\mathbf{e}}bc$ and $\mathfrak{A} \models \mathbb{X}^{\mathbf{o}}bc$. But at the same time we have $\mathfrak{A} \not\models \exists z(\mathbf{\Upsilon}cz \wedge \mathbf{K}az)$ and $\mathfrak{A} \not\models \exists zv(\widehat{\mathbf{E}}cz \wedge \widehat{\mathbf{E}}av \wedge \forall u(\mathbf{\Upsilon}au \wedge \widehat{\Sigma}u \supset \mathbf{K}zu))$, from which $\mathfrak{A} \not\models \mathbb{N}^{\mathbf{o}}ac$ and $\mathfrak{A} \not\models \mathbb{N}^{\mathbf{e}}ac$.

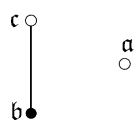


□

Theorem 55: The following syllogisms are invalid:

(55.1) (eao–3–XNN) (55.2) (oao–3–XNN)

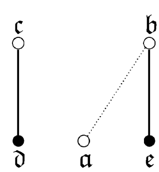
Proof. Let $\mathfrak{A}$ be a model for $\mathcal{A}$ with $\mathrm{A} = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, $\mathbf{E}^{\mathfrak{A}} = \{\mathfrak{cb}, \mathfrak{bb}\}$, $\mathbf{\Upsilon}^{\mathfrak{A}} = \mathbf{E}^{\mathfrak{A}} \cup \mathrm{Id(A)}$. Then we have $\mathfrak{A} \models \forall z(\mathbf{\Upsilon}bz \supset \neg\mathbf{\Upsilon}az)$, that is $\mathfrak{A} \models \mathbb{X}^{\mathbf{e}}ab$ and $\mathfrak{A} \models \mathbb{X}^{\mathbf{o}}ab$, and $\mathfrak{A} \models \widehat{\mathbf{E}}cb$, that is $\mathfrak{A} \models \mathbb{N}^{\mathbf{a}}cb$. But at the same time we have $\mathfrak{A} \not\models \exists z(\mathbf{\Upsilon}cz \wedge \mathbf{K}az)$ and $\mathfrak{A} \not\models \exists zv(\widehat{\mathbf{E}}cz \wedge \widehat{\mathbf{E}}av \wedge \forall u(\mathbf{\Upsilon}au \wedge \widehat{\Sigma}u \supset \mathbf{K}zu))$, from which $\mathfrak{A} \not\models \mathbb{N}^{\mathbf{o}}ac$.



□

Theorem 56: The syllogism (aoo–2–XNN) is invalid.

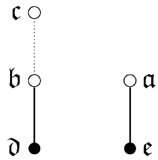
Proof. Let $\mathfrak{A}$ be a model for $\mathcal{A}$ with $\mathrm{A} = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}\}$, $\mathbf{E}^{\mathfrak{A}} = \{\mathfrak{cd}, \mathfrak{dd}, \mathfrak{be}, \mathfrak{ee}\}$ and $\mathbf{\Upsilon}^{\mathfrak{A}} = \{\mathfrak{ba}\} \cup \mathbf{E}^{\mathfrak{A}} \cup \mathrm{Id(A)}$. Then we have $\mathfrak{A} \models \mathbf{\Upsilon}ba$, that is $\mathfrak{A} \models \mathbb{X}^{\mathbf{a}}ba$, and $\mathfrak{A} \models \widehat{\mathbf{E}}cd \wedge \widehat{\mathbf{E}}be \wedge \forall u(\mathbf{\Upsilon}bu \wedge \widehat{\Sigma}u \supset \mathbf{K}du)$, that is $\mathfrak{A} \models \mathbb{N}^{\mathbf{o}}bc$. But at the same time we have $\mathfrak{A} \not\models \exists z(\mathbf{\Upsilon}cz \wedge \mathbf{K}az)$ and $\mathfrak{A} \not\models \exists zv(\widehat{\mathbf{E}}cz \wedge \widehat{\mathbf{E}}av \wedge \forall u(\mathbf{\Upsilon}au \wedge \widehat{\Sigma}u \supset \mathbf{K}zu))$, from which $\mathfrak{A} \not\models \mathbb{N}^{\mathbf{o}}ac$.



□

Theorem 57: The syllogism (oao–3–NXN) is invalid.

Proof. Let $\mathfrak{A}$ be a model for $\mathcal{A}$ with $A=\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}\}$, $\mathbf{E}^{\mathfrak{A}}=\{\mathfrak{bd}, \mathfrak{dd}, \mathfrak{ae}, \mathfrak{ee}\}$ and $\boldsymbol{\Upsilon}^{\mathfrak{A}}=\{\mathfrak{cb}, \mathfrak{cd}\}\cup \mathbf{E}^{\mathfrak{A}}\cup \mathrm{Id}(A)$. Then we have $\mathfrak{A}\models \widehat{\mathbf{E}}bd\wedge \widehat{\mathbf{E}}ae\wedge \forall u(\boldsymbol{\Upsilon}au\wedge \widehat{\Sigma}u\supset \mathbf{K}du)$, that is $\mathfrak{A}\models \mathbb{N}^{\mathbf{o}}ab$, and $\mathfrak{A}\models \boldsymbol{\Upsilon}cb$, that is $\mathfrak{A}\models \mathbb{X}^{\mathbf{a}}cb$. But at the same time we have $\mathfrak{A}\not\models \exists z(\boldsymbol{\Upsilon}cz\wedge \mathbf{K}az)$, $\mathfrak{A}\not\models \exists z\widehat{\mathbf{E}}cz$ and $\mathfrak{A}\not\models \exists zv(\widehat{\mathbf{E}}cz\wedge \widehat{\mathbf{E}}av\wedge \forall u(\boldsymbol{\Upsilon}au\wedge \widehat{\Sigma}u\supset \mathbf{K}zu))$, from which $\mathfrak{A}\not\models \mathbb{N}^{\mathbf{o}}ac$.

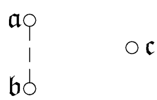


□

Theorem 58: The following syllogisms are invalid:

(58.1) (aaa, aea–1–NQX) (58.2) (aii, aoi–1–NQX)

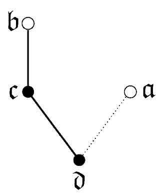
Proof. Let $\mathfrak{A}$ be a model for $\mathcal{A}$ with $A=\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, $\widetilde{\mathbf{E}}^{\mathfrak{A}}=\{\mathfrak{ab}\}$, $\boldsymbol{\Upsilon}^{\mathfrak{A}}=\widetilde{\mathbf{E}}^{\mathfrak{A}}\cup \mathrm{Id}(A)$. Then we have $\mathfrak{A}\models \widetilde{\mathbf{E}}ab$, from which $\mathfrak{A}\models \mathbb{N}^{\mathbf{a}}ab$. Moreover, we have $\mathfrak{A}\models \forall z(\boldsymbol{\Upsilon}cz\supset \Pi bz)$, from which for all $\circledast\in\{\mathbf{a}, \mathbf{e}, \mathbf{i}, \mathbf{o}\}$: $\mathfrak{A}\models \mathbb{Q}^{*}bc$. But at the same time we have $\mathfrak{A}\models \forall z(\boldsymbol{\Upsilon}cz\supset \neg\boldsymbol{\Upsilon}az)$, from which $\mathfrak{A}\not\models \mathbb{X}^{\mathbf{a}}ac$ and $\mathfrak{A}\not\models \mathbb{X}^{\mathbf{i}}ac$.



□

Theorem 59: The syllogism (aaa–1–QNX) is invalid.

Proof. Let $\mathfrak{A}$ be a model for $\mathcal{A}$ with $A=\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}$, $\mathbf{E}^{\mathfrak{A}}=\{\mathfrak{bc}, \mathfrak{cd}, \mathfrak{bd}, \mathfrak{cc}, \mathfrak{dd}\}$, $\boldsymbol{\Upsilon}^{\mathfrak{A}}=\{\mathfrak{ad}\}\cup \mathbf{E}^{\mathfrak{A}}\cup \mathrm{Id}(A)$. Then we have $\mathfrak{A}\models \forall z(\boldsymbol{\Upsilon}bz\supset \Pi az)$, that is $\mathfrak{A}\models \mathbb{Q}^{\mathbf{a}}ab$. Moreover, we have $\mathfrak{A}\models \mathbf{E}bc$, from which $\mathfrak{A}\models \mathbb{N}^{\mathbf{a}}bc$. But at the same time we have $\mathfrak{A}\not\models \boldsymbol{\Upsilon}ac$, that is $\mathfrak{A}\not\models \mathbb{X}^{\mathbf{a}}ac$.



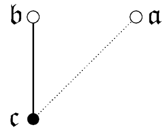
□

Theorem 60: The following syllogisms are invalid:

(60.1) (eae–1–QNX) (60.2) (eio–1–QNX)

Proof. Let $\mathfrak{A}$ be a model for $\mathcal{A}$ with $A=\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, $\mathbf{E}^{\mathfrak{A}}=\{\mathfrak{bc}, \mathfrak{cc}\}$, $\boldsymbol{\Upsilon}^{\mathfrak{A}}=\{\mathfrak{ac}\}\cup \mathbf{E}^{\mathfrak{A}}\cup \mathrm{Id}(A)$. Then we have $\mathfrak{A}\models \forall z(\boldsymbol{\Upsilon}bz\supset \Pi az)$, that is $\mathfrak{A}\models \mathbb{Q}^{\mathbf{e}}ab$.

Moreover, we have $\mathfrak{A} \models \mathbf{E}bc$, from which $\mathfrak{A} \models \mathbb{N}^{\mathbf{a}}bc$ and $\mathfrak{A} \models \mathbb{N}^{\mathbf{i}}bc$. But at the same time we have $\mathfrak{A} \models \boldsymbol{\Upsilon}ac$, from which $\mathfrak{A} \not\models \mathbb{X}^{\mathbf{e}}ac$ and $\mathfrak{A} \not\models \mathbb{X}^{\mathbf{o}}ac$.

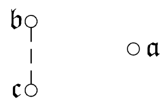


□

Theorem 61: The following syllogisms are invalid:

(61.1) (aii–1–QNX) (61.2) (aii–3–QNX)

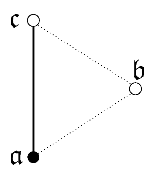
Proof. Let $\mathfrak{A}$ be a model for $\mathcal{A}$ with $A = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, $\widetilde{\mathbf{E}}^{\mathfrak{A}} = \{\mathfrak{bc}\}$, $\boldsymbol{\Upsilon}^{\mathfrak{A}} = \widetilde{\mathbf{E}}^{\mathfrak{A}} \cup \mathrm{Id}(A)$. Then we have $\mathfrak{A} \models \forall z(\boldsymbol{\Upsilon}bz \supset \Pi az)$, that is $\mathfrak{A} \models \mathbb{Q}^{\mathbf{a}}ab$. Moreover, we have $\mathfrak{A} \models \widetilde{\mathbf{E}}bc$, from which $\mathfrak{A} \models \mathbb{N}^{\mathbf{i}}bc$ and $\mathfrak{A} \models \mathbb{N}^{\mathbf{i}}cb$. But at the same time we have $\mathfrak{A} \models \forall z(\boldsymbol{\Upsilon}cz \supset \neg\boldsymbol{\Upsilon}az)$, from which $\mathfrak{A} \not\models \mathbb{X}^{\mathbf{i}}ac$.



□

Theorem 62: The syllogism (iai, oao–3–QXQ) is invalid.

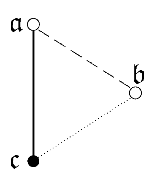
Proof. Let $\mathfrak{A}$ be a model for $\mathcal{A}$ with $A = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, $\mathbf{E}^{\mathfrak{A}} = \{\mathfrak{ca}, \mathfrak{aa}\}$, $\boldsymbol{\Upsilon}^{\mathfrak{A}} = \{\mathfrak{cb}, \mathfrak{ba}\} \cup \mathbf{E}^{\mathfrak{A}} \cup \mathrm{Id}(A)$. Then we have $\mathfrak{A} \models \Pi ab$, that is $\mathfrak{A} \models \mathbb{Q}^{\mathbf{i}}ab$ and $\mathfrak{A} \models \mathbb{Q}^{\mathbf{o}}ab$. Moreover, we have $\mathfrak{A} \models \boldsymbol{\Upsilon}cb$ and $\mathfrak{A} \models \mathbb{X}^{\mathbf{a}}cb$. But at the same time we have $\mathfrak{A} \models \mathbf{E}ca$, from which $\mathfrak{A} \not\models \Pi ac$, that is $\mathfrak{A} \not\models \mathbb{Q}^{\mathbf{i}}ac$ and $\mathfrak{A} \not\models \mathbb{Q}^{\mathbf{o}}ac$.



□

Theorem 63: The syllogism (aoo–3–NQM) is invalid.

Proof. Let $\mathfrak{A}$ be a model for $\mathcal{A}$ with $A = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, $\mathbf{E}^{\mathfrak{A}} = \{\mathfrak{ac}, \mathfrak{cc}\}$, $\widetilde{\mathbf{E}}^{\mathfrak{A}} = \{\mathfrak{ab}, \mathfrak{ac}\}$ and $\boldsymbol{\Upsilon}^{\mathfrak{A}} = \{\mathfrak{bc}\} \cup \mathbf{E}^{\mathfrak{A}} \cup \widetilde{\mathbf{E}}^{\mathfrak{A}} \cup \mathrm{Id}(A)$. Then we have $\mathfrak{A} \models \widetilde{\mathbf{E}}ab$ and also $\mathfrak{A} \models \mathbb{N}^{\mathbf{a}}ab$. Moreover, we have $\mathfrak{A} \models \Pi bc$, that is $\mathfrak{A} \models \mathbb{Q}^{\mathbf{o}}cb$. But at the same time we have $\mathfrak{A} \models \mathbf{E}ac$, from which $\mathfrak{A} \not\models \mathbb{M}^{\mathrm{o}}ac$.



□

Theorem 64: The following syllogisms are invalid:

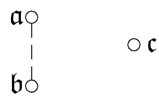
(64.1) (aoo–3–XQM) (64.2) (aoo–3–NQX)

Proof. Follows from Theorem 63 by Theorem 14 and 15. □

Theorem 65: The following syllogisms are invalid:

(65.1) (aai, aei–3–NQX) (65.3) (aii–3–NQX)
(65.2) (iai, iei–3–NQX)

Proof. Let $\mathfrak{A}$ be a model for $\mathcal{A}$ with $A = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, $\widetilde{\mathbf{E}}^{\mathfrak{A}} = \{\mathfrak{ab}\}$, $\boldsymbol{\Upsilon}^{\mathfrak{A}} = \widetilde{\mathbf{E}}^{\mathfrak{A}} \cup \mathrm{Id}(A)$. Then we have $\mathfrak{A} \models \widetilde{\mathbf{E}}ab$, from which $\mathfrak{A} \models \mathbb{N}^{\mathbf{a}}ab$ and $\mathfrak{A} \models \mathbb{N}^{\mathbf{i}}ab$, and we have $\mathfrak{A} \models \forall z(\boldsymbol{\Upsilon}bz \supset \Pi cz)$, that is $\mathfrak{A} \models \mathbb{Q}^{\mathbf{a}}cb$, $\mathfrak{A} \models \mathbb{Q}^{\mathbf{e}}cb$ and $\mathfrak{A} \models \mathbb{Q}^{\mathbf{i}}cb$. But at the same time we have $\mathfrak{A} \models \forall z(\boldsymbol{\Upsilon}cz \supset \neg\boldsymbol{\Upsilon}az)$, from which $\mathfrak{A} \not\models \mathbb{X}^{\mathbf{i}}ac$.

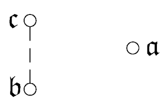


□

Theorem 66: The following syllogisms are invalid:

(66.1) (aai–3–QNX) (66.2) (iai–3–QNX)

Proof. Let $\mathfrak{A}$ be a model for $\mathcal{A}$ with $A = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, $\widetilde{\mathbf{E}}^{\mathfrak{A}} = \{\mathfrak{cb}\}$, $\boldsymbol{\Upsilon}^{\mathfrak{A}} = \widetilde{\mathbf{E}}^{\mathfrak{A}} \cup \mathrm{Id}(A)$. Then we have $\mathfrak{A} \models \forall z(\boldsymbol{\Upsilon}bz \supset \Pi az)$, that is $\mathfrak{A} \models \mathbb{Q}^{\mathbf{a}}ab$ and $\mathfrak{A} \models \mathbb{Q}^{\mathbf{i}}ab$. Moreover, we have $\mathfrak{A} \models \widetilde{\mathbf{E}}cb$, from which $\mathfrak{A} \models \mathbb{N}^{\mathbf{a}}cb$. But at the same time we have $\mathfrak{A} \models \forall z(\boldsymbol{\Upsilon}cz \supset \neg\boldsymbol{\Upsilon}az)$, from which $\mathfrak{A} \not\models \mathbb{X}^{\mathbf{i}}ac$.

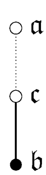


□

Theorem 67: The following syllogisms are invalid:

(67.1) (eao–3–QNX) (67.3) (eio–3–QNX)
(67.2) (oao–3–QNX)

Proof. Let $\mathfrak{A}$ be a model for $\mathcal{A}$ with $A = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, $\mathbf{E}^{\mathfrak{A}} = \{\mathfrak{cb}, \mathfrak{bb}\}$, $\boldsymbol{\Upsilon}^{\mathfrak{A}} = \{\mathfrak{ac}, \mathfrak{ab}\} \cup \mathbf{E}^{\mathfrak{A}} \cup \mathrm{Id}(A)$. Then we have $\mathfrak{A} \models \forall z(\boldsymbol{\Upsilon}bz \supset \Pi az)$, that is $\mathfrak{A} \models \mathbb{Q}^{\mathbf{e}}ab$ and $\mathfrak{A} \models \mathbb{Q}^{\mathbf{o}}ab$. Moreover, we have $\mathfrak{A} \models \mathbf{E}cb$, from which $\mathfrak{A} \models \mathbb{N}^{\mathbf{a}}cb$ and $\mathfrak{A} \models \mathbb{N}^{\mathbf{i}}cb$. But at the same time we have $\mathfrak{A} \models \boldsymbol{\Upsilon}ac$, from which $\mathfrak{A} \not\models \mathbb{X}^{\mathbf{o}}ac$.



□

4.4. Proofs of inconclusiveness

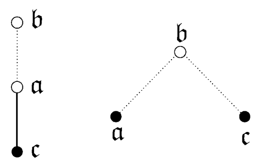
A premiss-pair is inconclusive if it does not yield any one of the syllogistic propositions listed on p. 116 with the major term *a* as predicate term and the minor term *c* as subject term. Since every syllogistic proposition implies either the affirmative or negative particular one-way possibility (Theorem 17), it suffices to give two counter-models in order to prove this kind of inconclusiveness. The first counter-model shows that the premisses do not yield an $\mathbb{M}^{o}ac$-conclusion, the second one that they do not yield an $\mathbb{M}^{\mathbf{i}}ac$-conclusion.

Theorem 68: The following premiss-pairs are inconclusive:

(68.1) (ia, oa, ie, oe–1–QQ) (68.3) (ei, ao, eo, ai–2–QQ)
(68.2) (ea, ae, aa, ee–2–QQ) (68.4) (oa, ie–2–QQ)

Proof. Let $\mathfrak{A}$ be a model for $\mathcal{A}$ with $A = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, $\mathbf{E}^{\mathfrak{A}} = \{\mathfrak{ac}, \mathfrak{cc}\}$, $\mathbf{\Upsilon}^{\mathfrak{A}} = \{\mathfrak{ba}, \mathfrak{bc}\} \cup \mathbf{E}^{\mathfrak{A}} \cup \mathrm{Id}(A)$. Then we have $\mathfrak{A} \models \forall z \Pi bz$, from which for all $* \in \{\mathbf{a}, \mathbf{e}, \mathbf{i}, \mathbf{o}\}$: $\mathfrak{A} \models \mathbb{Q}^{*}ba$ and $\mathfrak{A} \models \mathbb{Q}^{*}bc$ as well as $\mathfrak{A} \models \mathbb{Q}^{\mathbf{i}}ab$ and $\mathfrak{A} \models \mathbb{Q}^{\mathbf{o}}ab$. But at the same time we have $\mathfrak{A} \models \mathbf{E}ac$, from which $\mathfrak{A} \not\models \mathbb{M}^{o}ac$.

Let $\mathfrak{B}$ be a model for $\mathcal{A}$ with $B = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, $\mathbf{E}^{\mathfrak{B}} = \{\mathfrak{aa}, \mathfrak{cc}\}$, $\mathbf{\Upsilon}^{\mathfrak{B}} = \{\mathfrak{ba}, \mathfrak{bc}\} \cup \mathrm{Id}(A)$. Then we have $\mathfrak{B} \models \forall z \Pi bz$, from which for all $* \in \{\mathbf{a}, \mathbf{e}, \mathbf{i}, \mathbf{o}\}$: $\mathfrak{B} \models \mathbb{Q}^{*}ba$ and $\mathfrak{B} \models \mathbb{Q}^{*}bc$ as well as $\mathfrak{B} \models \mathbb{Q}^{\mathbf{i}}ab$ and $\mathfrak{B} \models \mathbb{Q}^{\mathbf{o}}ab$. But at the same time we have $\mathfrak{B} \models \mathbf{K}ac$, from which $\mathfrak{B} \not\models \mathbb{M}^{\mathbf{i}}ac$ by Theorem 10.



□

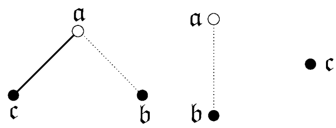
Theorem 69: The following premiss-pairs are inconclusive:

(69.1) (ae, ee–1–QN) (69.5) (ae, ee–3–QN)
(69.2) (ao, eo–1–QN) (69.6) (ie, oe–3–QN)
(69.3) (ie, oe–1–QN) (69.7) (ao, eo–3–QN)
(69.4) (ie, oe–2–QN)

Proof. Let $\mathfrak{A}$ be a model for $\mathcal{A}$ with $A = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, $\mathbf{E}^{\mathfrak{A}} = \{\mathfrak{ac}, \mathfrak{cc}, \mathfrak{bb}\}$, $\mathbf{\Upsilon}^{\mathfrak{A}} = \{\mathfrak{ab}\} \cup \mathbf{E}^{\mathfrak{A}} \cup \mathrm{Id}(A)$. Then we have $\mathfrak{A} \models \Pi ab$ and $\mathfrak{A} \models \forall z(\mathbf{\Upsilon}bz \supset \Pi az)$, from which for all $* \in \{\mathbf{a}, \mathbf{e}, \mathbf{i}, \mathbf{o}\}$: $\mathfrak{A} \models \mathbb{Q}^{*}ab$ as well as $\mathfrak{A} \models \mathbb{Q}^{\mathbf{i}}ba$ and $\mathfrak{A} \models \mathbb{Q}^{\mathbf{o}}ba$. Moreover, we have $\mathfrak{A} \models \mathbf{K}bc$, that is $\mathfrak{A} \models \mathbb{N}^{\mathbf{e}}bc$, $\mathfrak{A} \models \mathbb{N}^{\mathbf{e}}cb$ and $\mathfrak{A} \models \mathbb{N}^{\mathbf{o}}bc$ and $\mathfrak{A} \models \mathbb{N}^{\mathbf{o}}cb$. But at the same time we have $\mathfrak{A} \models \mathbf{E}ac$, from which $\mathfrak{A} \not\models \mathbb{M}^{\mathbf{o}}ac$.

Let $\mathfrak{B}$ be a model for $\mathcal{A}$ with $B = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, $\mathbf{E}^{\mathfrak{B}} = \{\mathfrak{bb}, \mathfrak{cc}\}$, $\mathbf{\Upsilon}^{\mathfrak{B}} = \{\mathfrak{ab}\} \cup \mathrm{Id}(A)$. Then we have $\mathfrak{B} \models \Pi ab$ and $\mathfrak{B} \models \forall z(\mathbf{\Upsilon}bz \supset \Pi az)$, that is for all $* \in \{\mathbf{a}, \mathbf{e}, \mathbf{i}, \mathbf{o}\}$: $\mathfrak{B} \models \mathbb{Q}^{*}ab$ as well as $\mathfrak{B} \models \mathbb{Q}^{\mathbf{i}}ba$ and $\mathfrak{B} \models \mathbb{Q}^{\mathbf{o}}ba$. Moreover, we have $\mathfrak{B} \models \mathbf{K}bc$, that is $\mathfrak{B} \models \mathbb{N}^{\mathbf{e}}bc$, $\mathfrak{B} \models \mathbb{N}^{\mathbf{e}}cb$ and $\mathfrak{B} \models \mathbb{N}^{\mathbf{o}}bc$, $\mathfrak{B} \models \mathbb{N}^{\mathbf{o}}cb$. But at the same time we have $\mathfrak{B} \models \Sigma c$ and $\mathfrak{B} \not\models \exists z(\mathbf{\Upsilon}az \wedge \mathbf{\Upsilon}cz)$, from which

$\mathfrak{B} \not\models (\Sigma a \vee \Sigma c) \supset \exists z(\Upsilon az \wedge \Upsilon cz)$ and by (df$_3$) also $\mathfrak{B} \not\models \Pi ac$. Thus, we have $\mathfrak{B} \not\models \exists z(\Upsilon cz \wedge \overline{\Pi} az)$, that is $\mathfrak{B} \not\models \mathbb{M}^{\mathbf{i}}ac$.



□

Theorem 70: The following premiss-pairs are inconclusive:

(70.1) (ae, ee–1–QX) (70.5) (ae, ee–3–QX)
(70.2) (ao, eo–1–QX) (70.6) (ie, oe–3–QX)
(70.3) (ie, oe–1–QX) (70.7) (ao, eo–3–QX)
(70.4) (ie, oe–2–QX)

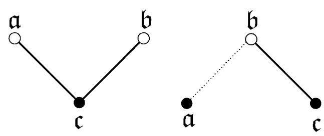
Proof. Follows from Theorem 69 by Theorem 14. □

Theorem 71: The following premiss-pairs are inconclusive:

(71.1) (ia, oa–1–QN) (71.2) (ia, oa–1–QX)

Proof. Let $\mathfrak{A}$ be a model for $\mathcal{A}$ with $A = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, $\mathbf{E}^{\mathfrak{A}} = \{\mathfrak{ac}, \mathfrak{bc}, \mathfrak{cc}\}$, $\Upsilon^{\mathfrak{A}} = \mathbf{E}^{\mathfrak{A}} \cup \mathrm{Id}(A)$. Then we have $\mathfrak{A} \models \Pi ab$, that is $\mathfrak{A} \models \mathbb{Q}^{\mathbf{i}}ab$ and $\mathfrak{A} \models \mathbb{Q}^{\mathbf{o}}ab$. Moreover, we have $\mathfrak{A} \models \mathbf{E}bc$, from which $\mathfrak{A} \models \mathbb{N}^{\mathbf{a}}bc$ and $\mathfrak{A} \models \mathbb{X}^{\mathbf{a}}bc$. But at the same time we have $\mathfrak{A} \models \mathbf{E}ac$, from which $\mathfrak{A} \not\models \mathbb{M}^{\mathrm{o}}ac$.

Let $\mathfrak{B}$ be a model for $\mathcal{A}$ with $B = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, $\mathbf{E}^{\mathfrak{B}} = \{\mathfrak{bc}, \mathfrak{cc}, \mathfrak{aa}\}$, $\Upsilon^{\mathfrak{B}} = \{\mathfrak{ba}\} \cup \mathbf{E}^{\mathfrak{B}} \cup \mathrm{Id}(A)$. Then we have $\mathfrak{B} \models \Pi ab$, that is $\mathfrak{B} \models \mathbb{Q}^{\mathbf{i}}ab$ and $\mathfrak{B} \models \mathbb{Q}^{\mathbf{o}}ab$. Moreover, we have $\mathfrak{B} \models \mathbf{E}bc$, from which $\mathfrak{B} \models \mathbb{N}^{\mathbf{a}}bc$ and $\mathfrak{B} \models \mathbb{X}^{\mathbf{a}}bc$. But at the same time we have $\mathfrak{B} \models \mathbf{K}ac$, from which $\mathfrak{B} \models \mathbb{M}^{\mathbf{i}}ac$ by Theorem 10.



□

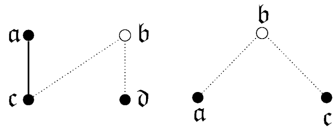
Theorem 72: The following premiss-pairs are inconclusive:

(72.1) (ia, ie–1–NQ) (72.2) (oa, oe–1–NQ)

Proof. Let $\mathfrak{A}$ be a model for $\mathcal{A}$ with $A = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}$, $\mathbf{E}^{\mathfrak{A}} = \{\mathfrak{aa}, \mathfrak{ac}, \mathfrak{cc}, \mathfrak{dd}\}$, $\Upsilon^{\mathfrak{A}} = \{\mathfrak{bc}, \mathfrak{bd}\} \cup \mathbf{E}^{\mathfrak{A}} \cup \mathrm{Id}(A)$. Then we have $\mathfrak{A} \models \Upsilon bc \wedge \mathbf{E}ac$, from which $\mathfrak{A} \models \mathbb{N}^{\mathbf{i}}ab$, and $\mathfrak{A} \models \Upsilon bd \wedge \mathbf{K}ad$, from which $\mathfrak{A} \models \mathbb{N}^{\mathbf{o}}ab$. Moreover, we have $\mathfrak{A} \models \forall z(\Upsilon cz \supset \Pi bz)$, that is $\mathfrak{A} \models \mathbb{Q}^{\mathbf{a}}bc$ and $\mathfrak{A} \models \mathbb{Q}^{\mathbf{e}}bc$. But at the same time we have $\mathfrak{A} \models \mathbf{E}ac$, from which $\mathfrak{A} \not\models \mathbb{M}^{\mathrm{o}}ac$.

Let $\mathfrak{B}$ be a model for $\mathcal{A}$ with $B = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, $\mathbf{E}^{\mathfrak{B}} = \{\mathfrak{aa}, \mathfrak{cc}\}$, $\Upsilon^{\mathfrak{B}} = \{\mathfrak{ba}, \mathfrak{bc}\} \cup \mathrm{Id}(A)$. Then we have $\mathfrak{B} \models \Upsilon ba \wedge \mathbf{E}aa$, from which $\mathfrak{B} \models \mathbb{N}^{\mathbf{i}}ab$, and $\mathfrak{B} \models \Upsilon bc \wedge \mathbf{K}ac$, from which $\mathfrak{B} \models \mathbb{N}^{\mathbf{o}}ab$. Moreover, we have $\mathfrak{B} \models \forall z(\Upsilon cz \supset \Pi bz)$, that is $\mathfrak{B} \models \mathbb{Q}^{\mathbf{a}}bc$

and $\mathfrak{B} \models \mathbb{Q}^{\mathbf{e}}bc$. But at the same time we have $\mathfrak{B} \models \mathbf{K}ac$, from which $\mathfrak{B} \not\models \mathbb{M}^{\mathbf{i}}ac$ by Theorem 10.



□

Theorem 73: The following premiss-pairs are inconclusive:

(73.1) (ia, ie–1–XQ) (73.2) (oa, oe–1–XQ)

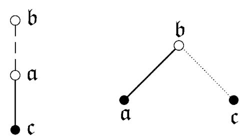
Proof. Follows from Theorem 72 by Theorem 14. □

Theorem 74: The following premiss-pairs are inconclusive:

(74.1) (aa, ae–2–NQ) (74.3) (ie–2–NQ)
(74.2) (ai, ao–2–NQ)

Proof. Let $\mathfrak{A}$ be a model for $\mathcal{A}$ with $A = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, $\mathbf{E}^{\mathfrak{A}} = \{\mathfrak{ac}, \mathfrak{cc}\}$, $\widetilde{\mathbf{E}}^{\mathfrak{A}} = \{\mathfrak{ba}, \mathfrak{bc}\}$, $\boldsymbol{\Upsilon}^{\mathfrak{A}} = \mathbf{E}^{\mathfrak{A}} \cup \widetilde{\mathbf{E}}^{\mathfrak{A}} \cup \mathrm{Id}(A)$. Then we have $\mathfrak{A} \models \widetilde{\mathbf{E}}ba$, from which $\mathfrak{A} \models \mathbb{N}^{\mathbf{a}}ba$ and $\mathfrak{A} \models \mathbb{N}^{\mathbf{i}}ba$. Moreover, we have $\mathfrak{A} \models \Pi bc$ and $\mathfrak{A} \models \forall z(\boldsymbol{\Upsilon} cz \supset \Pi bz)$, from which for all $\circledast \in \{\mathbf{a}, \mathbf{e}, \mathbf{i}, \mathbf{o}\}$: $\mathfrak{A} \models \mathbb{Q}^{*}bc$. But at the same time we have $\mathfrak{A} \models \mathbf{E}ac$, from which $\mathfrak{A} \not\models \mathbb{M}^{\mathrm{o}}ac$.

Let $\mathfrak{B}$ be a model for $\mathcal{A}$ with $B = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, $\mathbf{E}^{\mathfrak{B}} = \{\mathfrak{ba}, \mathfrak{aa}, \mathfrak{cc}\}$, $\boldsymbol{\Upsilon}^{\mathfrak{B}} = \{\mathfrak{bc}\} \cup \mathbf{E}^{\mathfrak{B}} \cup \mathrm{Id}(A)$. Then we have $\mathfrak{B} \models \mathbf{E}ba$, from which $\mathfrak{B} \models \mathbb{N}^{\mathbf{a}}ba$ and $\mathfrak{B} \models \mathbb{N}^{\mathbf{i}}ba$. Moreover, we have $\mathfrak{B} \models \Pi bc$ and $\mathfrak{B} \models \forall z(\boldsymbol{\Upsilon} cz \supset \Pi bz)$, from which for all $\circledast \in \{\mathbf{a}, \mathbf{e}, \mathbf{i}, \mathbf{o}\}$: $\mathfrak{B} \models \mathbb{Q}^{*}bc$. But at the same time we have $\mathfrak{B} \models \mathbf{K}ac$, from which $\mathfrak{B} \not\models \mathbb{M}^{\mathbf{i}}ac$ by Theorem 10.



□

Theorem 75: The following premiss-pairs are inconclusive:

(75.1) (aa, ae–2–XQ) (75.3) (ie–2–XQ)
(75.2) (ai, ao–2–XQ)

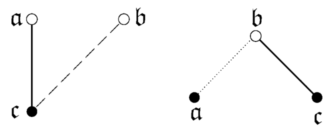
Proof. Follows from Theorem 74 by Theorem 14. □

Theorem 76: The following premiss-pairs are inconclusive:

(76.1) (aa, ea–2–QN) (76.3) (oa–2–QN)
(76.2) (ai, ei–2–QN)

Proof. Let $\mathfrak{A}$ be a model for $\mathcal{A}$ with $A=\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, $\mathbf{E}^{\mathfrak{A}}=\{\mathfrak{ac}, \mathfrak{cc}\}$, $\widetilde{\mathbf{E}}^{\mathfrak{A}}=\{\mathfrak{bc}\}$, $\boldsymbol{\Upsilon}^{\mathfrak{A}}=\mathbf{E}^{\mathfrak{A}}\cup\widetilde{\mathbf{E}}^{\mathfrak{A}}\cup \mathrm{Id}(A)$. Then we have $\mathfrak{A}\models \Pi ba$, $\mathfrak{A}\models \Pi bc$ and $\mathfrak{A}\models \forall z(\boldsymbol{\Upsilon} az\supset \Pi bz)$, that is $\mathfrak{A}\models \mathbb{Q}^{\mathbf{a}}ba$ and $\mathfrak{A}\models \mathbb{Q}^{\mathbf{o}}ba$. Moreover, we have $\mathfrak{A}\models \widetilde{\mathbf{E}}bc$, from which $\mathfrak{A}\models \mathbb{N}^{\mathbf{a}}bc$ and $\mathfrak{A}\models \mathbb{N}^{\mathbf{i}}bc$. But at the same time we have $\mathfrak{A}\models \mathbf{E}ac$, from which $\mathfrak{A}\not\models \mathbb{M}^{\mathrm{o}}ac$.

Let $\mathfrak{B}$ be a model for $\mathcal{A}$ with $B=\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, $\mathbf{E}^{\mathfrak{B}}=\{\mathfrak{bc}, \mathfrak{cc}, \mathfrak{aa}\}$, $\boldsymbol{\Upsilon}^{\mathfrak{B}}=\{\mathfrak{ba}\}\cup\mathbf{E}^{\mathfrak{B}}\cup \mathrm{Id}(A)$. Then we have $\mathfrak{B}\models \Pi ba$ and $\mathfrak{B}\models \forall z(\boldsymbol{\Upsilon} az\supset \Pi bz)$, that is $\mathfrak{B}\models \mathbb{Q}^{\mathbf{a}}ba$ and $\mathfrak{B}\models \mathbb{Q}^{\mathbf{o}}ba$. Moreover, we have $\mathfrak{B}\models \mathbf{E}bc$, from which $\mathfrak{B}\models \mathbb{N}^{\mathbf{a}}bc$ and $\mathfrak{B}\models \mathbb{N}^{\mathbf{i}}bc$. But at the same time we have $\mathfrak{B}\models \mathbf{K}ac$, from which $\mathfrak{B}\not\models \mathbb{M}^{\mathbf{i}}ac$ by Theorem 10.



□

Theorem 77: The following premiss-pairs are inconclusive:

(77.1) (aa, ea–2–QX) (77.3) (oa–2–QX)

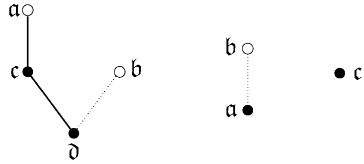
(77.2) (ai, ei–2–QX)

Proof. Follows from Theorem 76 by Theorem 14. □

Theorem 78: The premiss-pair (ao, eo–2–QX) is inconclusive.

Proof. Let $\mathfrak{A}$ be a model for $\mathcal{A}$ with $A=\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}$, $\mathbf{E}^{\mathfrak{A}}=\{\mathfrak{ac}, \mathfrak{cd}, \mathfrak{ad}, \mathfrak{cc}, \mathfrak{dd}\}$, $\boldsymbol{\Upsilon}^{\mathfrak{A}}=\{\mathfrak{bd}\}\cup\mathbf{E}^{\mathfrak{A}}\cup \mathrm{Id}(A)$. Then we have $\mathfrak{A}\models\forall z(\boldsymbol{\Upsilon} az\supset \Pi bz)$, that is $\mathfrak{A}\models \mathbb{Q}^{\mathbf{a}}ba$ and $\mathfrak{A}\models \mathbb{Q}^{\mathbf{e}}ba$. Moreover, we have $\mathfrak{A}\models\neg\boldsymbol{\Upsilon} bc$, that is $\mathfrak{A}\models \mathbb{X}^{\mathbf{o}}bc$. But at the same time we have $\mathfrak{A}\models \mathbf{E}ac$, from which $\mathfrak{A}\not\models \mathbb{M}^{\mathrm{o}}ac$.

Let $\mathfrak{B}$ be a model for $\mathcal{A}$ with $B=\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$, $\mathbf{E}^{\mathfrak{B}}=\{\mathfrak{aa}, \mathfrak{cc}\}$, $\boldsymbol{\Upsilon}^{\mathfrak{B}}=\{\mathfrak{ba}\}\cup \mathrm{Id}(A)$. Then we have $\mathfrak{B}\models \Pi ba$ and $\mathfrak{B}\models \forall z(\boldsymbol{\Upsilon} az\supset \Pi bz)$, that is $\mathfrak{B}\models \mathbb{Q}^{\mathbf{a}}ba$ and $\mathfrak{B}\models \mathbb{Q}^{\mathbf{e}}ba$. Moreover, we have $\mathfrak{B}\models\neg\boldsymbol{\Upsilon} bc$, that is $\mathfrak{B}\models \mathbb{X}^{\mathbf{o}}bc$. But at the same time we have $\mathfrak{B}\models \mathbf{K}ac$, from which $\mathfrak{B}\not\models \mathbb{M}^{\mathbf{i}}ac$ by Theorem 10.

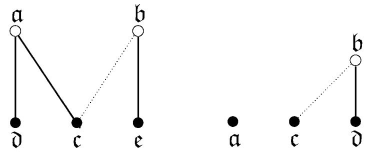


□

Theorem 79: The premiss-pair (oa–2–NQ) is inconclusive.

Proof. Let $\mathfrak{A}$ be a model for $\mathcal{A}$ with $A=\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}\}$, $\mathbf{E}^{\mathfrak{A}}=\{\mathfrak{ad}, \mathfrak{dd}, \mathfrak{ac}, \mathfrak{cc}, \mathfrak{be}, \mathfrak{ee}\}$ and $\boldsymbol{\Upsilon}^{\mathfrak{A}}=\{\mathfrak{bc}\}\cup\mathbf{E}^{\mathfrak{A}}\cup \mathrm{Id}(A)$. Then we have $\mathfrak{A}\models \widehat{\mathbf{E}}ad\wedge\widehat{\mathbf{E}}be\wedge\forall u(\boldsymbol{\Upsilon} bu\wedge \widehat{\Sigma}u\supset \mathbf{K}du)$, from which $\mathfrak{A}\models \mathbb{N}^{\mathbf{o}}ba$. Moreover, we have $\mathfrak{A}\models\forall z(\boldsymbol{\Upsilon} cz\supset \Pi bz)$, that is $\mathfrak{A}\models \mathbb{Q}^{\mathbf{a}}bc$. But at the same time we have $\mathfrak{A}\models \mathbf{E}ac$, from which $\mathfrak{A}\not\models \mathbb{M}^{\mathrm{o}}ac$.

Let $\mathfrak{B}$ be a model for $\mathcal{A}$ with $B=\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}\}$, $\mathbf{E}^{\mathfrak{B}}=\{\mathfrak{bd}, \mathfrak{aa}, \mathfrak{cc}, \mathfrak{dd}\}$ and $\Upsilon^{\mathfrak{B}}=\{\mathfrak{bc}\}\cup \mathbf{E}^{\mathfrak{B}}\cup \mathrm{Id}(A)$. Then we have $\mathfrak{B}\models \widehat{\mathbf{E}}aa\wedge\widehat{\mathbf{E}}bd\wedge\forall u(\Upsilon bu\wedge\widehat{\Sigma}u\supset \mathbf{K}au)$, from which $\mathfrak{B}\models\mathbb{N}^{\mathbf{o}}ba$. Moreover, we have $\mathfrak{B}\models\forall z(\Upsilon cz\supset\Pi bz)$, that is $\mathfrak{B}\models\mathbb{Q}^{\mathbf{a}}bc$. But at the same time we have $\mathfrak{B}\models\mathbf{K}ac$, from which $\mathfrak{B}\not\models\mathbb{M}^{\mathbf{i}}ac$ by Theorem 10.



□

Theorem 80: The premiss-pair (oa–2–XQ) is inconclusive.

Proof. Follows from Theorem 79 by Theorem 14. □

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