Truth-maker Semantics for Intuitionistic Logic

I wish to propose a new semantics for intuitionistic logic, which is in some ways a cross between the construction-oriented semantics of Brouwer-Heyting-Kolmogorov (as expounded in Troelstra & van Dalen [1998], for example) and the condition-oriented semantics of Kripke [1965]. The new semantics is of some philosophical interest, because it shows how there might be a common semantical underpinning for intuitionistic and classical logic and how intuitionistic logic might thereby be tied to a realist conception of the relationship between language and the world. The new semantics is also of some technical interest; it gives rise to a framework, intermediate between the frameworks of the two other approaches, within which several novel questions and approaches may be pursued.

I begin with a philosophical discussion and conclude with a long technical appendix. In principle, the two can be read independently of one another but it is preferable if the reader first gains a formal and informal understanding of the semantics and then goes back and forth between the philosophical and technical exposition.[1]

§1 The Standard Semantics

Let me remind the reader of the two standard forms of semantics for intuitionistic logic. Under the construction-oriented semantics of B-H-K, the basic notion is that of a construction (or proof) *establishing* a statement. We then have something like the following rules for the sentential connectives:

(i) a construction establishes $B \wedge C$ if it is the combination of a construction that establishes B and a construction that establishes C;

(ii) a construction establishes $B \vee C$ if it selects a construction that establishes B or a construction that establishes C;

(iii) a construction establishes $\neg B$ if in application to any construction that establishes B it yields a construction that establishes a contradiction;

(iv) a construction establishes $B \supset C$ if in application to any construction that establishes B it yields a construction that establishes C.

We might take $\perp$ to be a sentential constant that is established by all and only those constructions that establish a contradiction. Clause (iii) then follows from clause (iv) under the definition of $\neg B$ as $B \supset \perp$.

Under the condition-oriented semantics, the basic notion is that of a condition *verifying* or

[1]An earlier version of this paper was presented at a conference on truthmakers in Paris, 2011, and at a conference on the philosophy of mathematics in Bucharest, 2012. I should like to thank the participants of these two conferences for helpful comments and also an anonymous referee for the journal. After completing the paper, I learned that Ciardelli's thesis [2009] on inquisitive logic contains some related work. In particular, the system HH of the appendix is similar to the system for inquisitive logic while lemma 22 corresponds to the disjunctive-negative normal form theorem for inquisitive logic. It would be worthwhile to explore the connections between the two approaches in more detail. I should like to thank Ivano Ciardelli for bringing his thesis to my attention and for helpful correspondence.

forcing a statement. For reasons that will become apparent, I shall henceforth talk of *states* rather than conditions. The clauses for the various connectives are then as follows:

(i) a state verifies B ∧ C if it verifies B and verifies C;

(ii) a state verifies B ∨ C if it verifies B or verifiers C;

(iii) a state verifies ¬B if no extension of the state verifies B;

(iv) a state verifies B ⊃ C if any extension of the state that verifies B also verifies C.

We might now take ⊥ to be a sentential constant that is never verified. Clause (iii) then follows from clause (iv) under the definition of ¬B as B ⊃⊥; and in what follows we shall usually suppose that ¬B has been so defined.

There are a number of obvious differences between these two forms of semantics, to which I shall later advert. But there is one striking difference to which I should immediately draw the reader's attention. The construction-oriented semantics is an instance of what one might call an *exact* semantics. A construction, when it establishes a given statement, is wholly, or exactly, relevant to establishing the statement. This is evident when we consider the different clauses for the different kinds of statement. Thus given that the construction c_1 is wholly relevant to establishing the statement A_1 and the construction c_2 wholly relevant to establishing the statement A_2, the combination of the two constructions will be wholly relevant to establishing the conjunction of the two statements; and similarly for the other kinds of statement.[2]

Things are entirely different under the condition-oriented semantics; a state that verifies a given statement will not, as a rule, be wholly relevant to verifying the statement. Indeed, the relation of verifying, or forcing, is 'persistent' or 'monotonic':

if a state verifies a given statement then so does any extension of the state.

But this means that if even if some state were exactly to verify a given statement, we could add further content to it wily-nilly and still get a state that verified the statement, even though most or all of the additional content was irrelevant to its truth. For example, the state of the ball's being red will exactly verify the statement that it is red. But then the state of the ball's being red and of its raining in Timbuktu will also verify that the ball is red.

Now there is a natural thought, which we might call *Exactification*, according to which, for any inexact verifier, there should be an underlying exact verifier. In other words, if a given state inexactly verifies a statement then it should be an extension of a state that exactly verifies the statement. Since any extension of an exact verifier will be an inexact verifier, it follows from this requirement that a state will be an inexact verifier for a given statement just in case it is an extension of an exact verifier. There must, that is to say, be an account of inexact verification in terms of an underlying notion of exact verification.

We therefore face the problem, with which I shall be principally concerned in the rest of the paper, of saying what the underlying notion of exact verification might plausibly be taken to be in the case of the Kripke semantics. When a state makes a statement to be true, then what is the sub-state that is wholly relevant, and not simply partly relevant, to the statement being true?

In considering this question, it will be helpful to consider the corresponding question for

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[2]As David McCarty has pointed out to me, this is only true for statements of the form ¬B = (B ⊃⊥) when it is assumed that there are constructions that establish ⊥.

classical logic by way of a foil. There is a semantics for classical logic in terms of possible worlds. The basic semantical notion is that of a statement being *true at* (or *verified by*) a possible world and the clauses for the various kinds of truth-functional statement then go as follows:

(i) a world verifies B ∧ C if it verifies B and verifies C;
(ii) a world verifies B ∨ C if it verifies B or verifiers C;
(iii) a world verifies ¬B if it fails to verify B.
(iv) a world verifies B ⊃ C if it fails to verify B or verifies C;

The relevant notion of verification here is inexact. We do not, of course, have the previous kind of proof of this for, since the possible worlds are already complete, there is no possibility of adding any further irrelevant material to them. But it is evident in the case of an ordinary atomic statement, such as the statement that the ball is red, that the relevant notion of verification *will* be inexact. For any possible world that verifies the statement will contain a great deal of content (such as its raining in Timbuktu) that is irrelevant to its true.

We therefore face the problem, in the classical case too, of saying what the underlying notion of exact verification might plausibly be taken to be.

§2 Problems in Exactifying the Semantics for Intuitionistic Logic

It turns out that the problem, in the classical case, has already been solved by van Fraassen [69] (although he was working in a somewhat different context and had a somewhat different motivation in mind). In stating this semantics, we take as basic the twin notions of a state (*exactly*) *verifying* and of a state (*exactly*) *falsifying* a given statement. We also take the *fusion* or *sum* of two states to be the smallest state to contain them both. Thus the fusion of the states of the ball being red and of its being round will be the state of its being red and round. We then have the following clauses for the different kinds of truth-functional statement[3]:

(i)(a) a state verifies B ∧ C if it is the fusion of a state that verifies B and a state that verifies C,
(b) a state falsifies B ∧ C if it falsifies B or falsifies C;

(ii)(a) a state verifies B ∨ C if it verifies B or verifies C,
(b) a state falsifies B ∨ C if it is the fusion of a state that falsifies B and a state that falsifies C;

(iii)(a) a state verifies ¬B if it falsifies B,
(b) a state falsifies ¬B if it verifies B;

(iv)(a) a state verifies B ⊃ C if it falsifies B or verifies C,
(b) a state falsifies B ⊃ C if it is the fusion of a state that verifies B and a state that falsifies C.

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[3]There are a number of alternatives to the clauses below that might also be adopted, but the differences between them will not matter for our purposes.

It can then be shown that, under suitable assumptions, a statement will be true in a possible world iff there is some state within the world that exactly verifies it.

What we therefore require is a semantics that stands to Kripke's forcing semantics as van Fraassen's semantics stands to the possible worlds semantics. Now to some extent we can, to this end, take over the clauses from van Fraassen's semantics. For within an intuitionistic context, clause (i)(a) for the verification of a conjunction and clause (ii)(b) for the verification of a disjunction both look very reasonable. Indeed, (i)(a) corresponds pretty closely to the H-B-K clause for conjunction and (ii)(b) to the H-B-K clause for disjunction. It is true that the combination of two constructions in the constructive semantics for conjunction would not normally be regarded as a commutative operation (with the combination of two constructions being the same regardless of the order in which they are taken), while the operation of fusion under the forcing semantics will be commutative; and, likewise, the selection of a construction in the constructive semantics for disjunction would not normally be identified with the construction itself, while no corresponding distinction is made for the clause for disjunction under the forcing semantics. But these differences are relatively minor and would appear to have no essential bearing on the import of the semantics.

However, clause (iii) for the verification or falsification of a negation and clause (iv) for the verification or falsification of a conditional are seriously off target in application to the intuitionistic case. For the clause for negation will entail that B and $\neg\neg B$ are verified by the same states and the clause for the conditional will entail that $B \supset C$ and $\neg B \vee C$ are verified by the same states and so there will be no semantical difference between each of these pairs of formulas, as is required under an intuitionistic form of the semantics.

The problem then is to find alternative clauses for negation and for the conditional and, if we assume that the negation $\neg B$ of a formula can be defined as $B \supset \bot$, then the problem is simply to find a suitable clauses for the conditional operator $\supset$ and the falsum constant $\bot$.

In solving this problem, there are a number of other related problems that one might also wish to solve:

(1) The clauses for conjunction and disjunction in van Fraassen's semantics for classical logic correspond to the clauses for conjunction and disjunction in the constructive semantics; and one might hope that, in the same way, the new clauses for negation and for the conditional, whatever they might be, will correspond to the clauses for negation and the conditional in the constructive semantics. However, it is not so clear in this case in what the correspondence might consist. For the combination of constructions can be taken to correspond to the fusion of states and the selection of a construction to the construction itself. But what, within an unvariegated ontology of states, can plausibly be taken to correspond to a function from constructions to constructions?

(2) The fusion of two states can be defined in terms of the extension relation, the relation that holds between two states when one extends the other. For the fusion of two states will be their least upper bound; it will be the state that extends each of them and that is itself extended by any state that extends each of them. Thus the fusion of two states can be defined within the ontological and conceptual resources of the original Kripke semantics; and one would like the clauses for negation and for the conditional also to stay within these resources. Somehow, in stating these clauses, we need only appeal to the domain of states and to the extension relation on

those states.

(3) Within the Kripke semantics, there is a fundamental difference between the clauses for conjunction and disjunction, on the one hand, and the clauses for negation and the conditional, on the other. For the clauses for the former only appeal to the state that is itself in question. Thus a conjunction will be verified by a state if each conjunct is verified by that very state; and similarly for disjunction. But the clauses for the latter appeal to other states. Thus a negated statement will be verified by a state only if the statement fails to be verified by all states that extend the given state; and similarly for the conditional.

Consider, by way of an example, a state in which the statement B fails to obtain. We cannot conclude on this basis that ¬B is verified by the state, for that will depend upon the behavior of the states that extend the given state. If none of them verifies B, then ¬B will indeed be verified by the state; but if one of them does verify B, then ¬B will not be verified by the state. In this respect, negation and the conditional within the forcing semantics are more analogous to the modal operators within the possible worlds semantics rather than to the classical truth-functional connectives; just as the truth of □B or of ◊B at a given world will depend upon the behavior of B at other worlds, so the verification of ¬B or of B ⊃ C at a given state will depend upon the behavior of B or of B and C at other states.

This appears to suggest that whether or not a negative or conditional statement is verified at a given state is not a matter which simply depends upon the intrinsic content of the state. I say 'suggests' rather than 'entails' because, although the semantics provides an extrinsic characterization of when such a statement is verified, it is not ruled out that there might be something intrinsic to the content of the given state which determines what its extensions should be. However, there is nothing in the semantics itself which provides any indication of what the relevant intrinsic content might be.

Again, this is a major difference from the B-H-K semantics. For in that case it may plausibly be regarded as intrinsic to a given construction that it establishes what it does. Other constructions may of course be involved in saying what a given construction is - a construction for a conditional, for example, must take any construction for the antecedent into a construction for the consequent. But once the construction has been specified, it should be evident from the construction itself what it is capable of establishing.

This feature of the Kripke semantics raises a problem for the project of Exactification. For exact verification is most naturally regarded as an internal matter; we must somehow be able to see it as intrinsic to the content of a given state that it exactly verifies what it does. But this means that the inexact relation of forcing from the Kripke semantics must also be an intrinsic matter. It must be intrinsic to any given forcing condition that it contains a given exact verifier and intrinsic to the content of the exact verifier that it verifies a given statement. So, despite appearances to the contrary, we must somehow be able to see the verification of a statement as arising from the intrinsic content of its verifying state.

(4) Finally, one might have the general aim of understanding the relationship between the two kinds of semantics and of understanding, in particular, how it is possible for the clauses for conjunction, negation and the conditional to take such different forms.

§3 An Exact Semantics for Intuitionistic Logic

Let me now propose a solution to these various problems. Essential to the solution is the idea of a *conditional connection*. Let s and t be any two states. Then we shall suppose that there is a further state of s's *leading to* t, which we denote by $s \rightarrow t$. Intuitively, the presence of $s \rightarrow t$ indicates that t will be present under the presence or addition of s.

If states corresponded exactly to statements, so that for each statement A there was a state of A's obtaining, then $s \rightarrow t$ could be taken to correspond to the state of (B ⊃ C)'s obtaining, where B corresponded to the state s and C to the state t. But it is far from clear that such a state will exist. For a state must be 'determinate' in the sense that it will verify a disjunction only if it verifies one of the disjuncts. But this means that we cannot in general take there to be the state of (B ∨ C)'s obtaining even when there is a state of B's obtaining and a state of C's obtaining. For the state of (B ∨ C)'s obtaining, were there to be such a state, would verify (B ∨ C) without necessarily verifying either B or C and thereby fail to be determinate.

If this difficulty can arise for the state corresponding to B ∨ C then what assurance can there be that it will not also arise for the state corresponding to B ⊃ C? Within the context of classical logic - at least as it is usually conceived - no such assurance can be given. For the statement B ⊃ C is logically equivalent to ¬B ∨ C and hence there will be no essential difference between the conditional state B ⊃ C and the disjunctive state ¬B ∨ C.

However, matters are fundamentally different within the context of intuitionistic logic. For the equivalence between B ⊃ C and ¬B ∨ C will not be valid and, indeed, it can be shown that nothing amiss will result from taking B ⊃ C to correspond to a determinate state, given that B and C also correspond to determinate states.[4] Thus the kind of circumstance that prevents the existence of a conditional state in the context of classical logical does not arise in the context of intuitionistic logic.

With conditional connections on the scene, let us return to the question of finding a suitable clause for the conditional. Suppose that t_1, t_2, ... are the (exact) verifiers of B. For each verifier t_i of B select a verifier u_i of C. Then the verifiers of the conditional B ⊃ C may be taken to be the fusions of the conditional connections $t_1 \rightarrow u_1$, $t_2 \rightarrow u_2$, ... for each such selection of verifiers. Thus a verifier for the conditional will tell us how to pass from any verifier of the antecedent to a verifier for the consequent. The verifier is plausibly taken to be exact since it is wholly composed of states that are relevant to verifying the conditional, one for each way of verifying its antecedent.[5]

There is a similar clause for negation, given the definition of ¬B as B ⊃⊥. For suppose that t_1, t_2, ... are the verifiers of B. Then a verifier for ¬B will be a fusion of conditional connections $t_1 \rightarrow u_1$, $t_2 \rightarrow u_2$, ... in which each of u_1, u_2, ... is a verifier for ⊥. We might call a state *contradictory* if it is a verifier for ⊥. Then an alternative way of stating the clause for negation is that a verifier for ¬B should be a fusion of conditional connections $t_1 \rightarrow u_1$, $t_2 \rightarrow u_2$, ... for which each of the states u_1, u_2, ... is contradictory.

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[4]Call a formula A *prime* if B or C is a logical consequence of A whenever B ∨ C is a logical consequence of A. It may then be shown that B ⊃ C is prime whenever C is prime; and more general results along these lines can also be established.

[5]A related proposal for the conditional is to be found in Ciardelli [2009].

Combining the previous clauses for conjunction and disjunction from the exact semantics for classical logic with the current clauses for the conditional and negation gives us the exact counterpart to the Kripke semantics that we have been looking for.

It should be noted that the present version of the exact semantics (like the previous classical version) is hyper-intensional in the sense that logically equivalent statements may not be verified by the same states. Consider, for example, the statements p and $p \vee (p \wedge q)$. Then a state may exactly verify $p \vee (p \wedge q)$ through exactly verifying $(p \wedge q)$ and therefore not be an exact verifier for p. This means that there is no obvious way to model these versions of the exact semantics within a familiar semantical scheme - by treating exact verifiers as minimal verifiers, for example - since the familiar semantical schemes are not hyper-intensional in this way. We appear to have something essentially new.

Once given these semantical clauses, we are in a position to provide a preliminary account of the various logical notions, such as validity, consequence and equivalence. Validity and equivalence may be defined in terms of consequence in the usual way, with a statement being valid (or logically true) if it is a consequence of zero statements and with two statements being equivalent if they are consequences of one another. We may therefore focus on the notion of consequence.

I here give a somewhat crude account of consequence in terms of verification; and later, in section 6, I shall give a more refined account in terms of truth. The obvious definition is that the statement C should be taken to be a consequence of $A_1, A_2, ...$ if, necessarily, C is verified by a given state whenever $A_1, A_2, ...$ are verified by that state. But given that there are now two notions of verification in play - the exact and the inexact - there will be two corresponding notions of consequence. Accordingly, we may say that statement C is an *exact* consequence of the statements $A_1, A_2, ...$ if C is exactly verified by any state that is the fusion of states that exactly verify $A_1, A_2, ...$ (strictly speaking, by any state *in any model*). Thus $A \wedge B$ will be an exact consequence of A and B, since the fusion of exact verifiers for A and B will be an exact verifier for $A \wedge B$.

The notion of exact consequence and the corresponding notion of exact equivalence are of considerable interest in their own right but they do not correspond to the normal notions of consequence or equivalence for intuitionistic logic. The statement A, for example, will not in general be an exact consequence of $A \wedge B$, since the fusion of exact verifiers for A and B will not in general be an exact verifier for A; and the statement $A \supset A$ will not be exactly valid, i.e. exactly verified by every state, since most states will be irrelevant to its truth.

A closer correspondence with the usual notions is obtained with the inexact notion. Statement C may be said to be an *inexact* consequence of $A_1, A_2, ...$ if C is inexactly verified by any state that inexactly verifiers each of $A_1, A_2, ...$ (in this case, nothing is gained or lost by considering the fusion of the verifiers for $A_1, A_2, ...$). The statement A will now be an inexact consequence of $A \wedge B$ since any inexact verifier for $A \wedge B$, i.e. any state that contains an exact verifier for $A \wedge B$, will be an inexact verifier for A, i.e. will contain an exact verifier for A; and, likewise, the statement $A \supset A$, will be inexactly valid, i.e. inexactly verified by every state, since every state will contain an exact verifier for $A \supset A$.

A philosophically more perspicuous formulation of the definition of inexact consequence may be obtained by requiring that any fusion of exact verifiers for $A_1, A_2, ...$ should contain an exact verifier for C since, given the underlying notion of exact verification, this makes clear that any verifier wholly relevant to the premises A_1, A_2, ... should contain a verifier wholly relevant to the conclusion C.

With these definitions in place, it can be shown that C will be derivable from A_1, A_2, ... in intuitionistic logic iff it is an inexact consequence of A_1, A_2, ... and it can be shown, in particular, that C is a theorem of intuitionistic logic iff it is inexactly valid (proofs of this and all other results are given in the appendix). The exact semantics constitutes, in this way, a semantics for intuitionistic logic.

§4 Contradictory States

I wish in the remaining informal part of the paper to discuss some of the philosophical implications of the present semantics. Let me begin by discussing the role of contradictory states, since this is somewhat different from the role they play in other, more familiar, semantical schemes.

We should, in the first place, allow there to be contradictory states, ones that verify the falsum constant $\perp$. For suppose that there are no contradictory states. Then $\neg B$ $(B \supset \perp)$ will have a verifier iff B does not have a verifier and, in that case, the verifier of $\neg B$ will be the null state, since it will be the fusion of the empty set of states. But we can then see that the formula $\neg A \vee \neg\neg A$ will be valid, thereby giving us the intermediate system KC of Jankov [1957] rather than the weaker system of intuitionistic logic. For either A will have no verifiers, in which case the null state will verify $\neg A$ and hence verify $\neg A \vee \neg\neg A$ or A will have a verifier, in which case $\neg A$ will have no verifier and the null state will verify $\neg\neg A$ and hence verify $\neg A \vee \neg\neg A$.

The use of contradictory states might also be allowed under the forcing semantics; for we could take there to be a state that (inexactly) verified every statement whatever as long as it was taken to be an extension of any other state. But nothing would thereby be gained; the admission of such a state would 'do no work'. Or again, contradictory states could be admitted into the previous exact semantics for classical logic (as was van Fraassen's original intention) . But for the purposes of getting classical, as opposed to relevance, logic, nothing is thereby gained.

In the present case, by contrast, it is essential to admit contradictory states if the correspondence with intuitionistic logic is to be preserved. Indeed, not only must we allow for the existence of a contradictory state, we must also allow for the existence of an arbitrary finite number of contradictory states. Suppose, for example, that there were just one contradictory state. Call it f. There could then only be a single verifier s for $\neg B$ $(B \supset \perp)$, one that was the fusion of $s_1 \rightarrow f$, $s_2 \rightarrow f$, ..., where s_1, s_2, ... were the verifiers of B. But any state that verified $\neg B \supset (C \vee D)$ would then verify either $\neg B \supset C$ or $\neg B \supset D$ (depending upon whether the single verifier s of $\neg B$ verified C or verified D); and so it would follow that $(\neg B \supset C) \vee (\neg B \supset D)$ was a consequence of $\neg B \supset (C \vee D)$, thereby giving us the system KP of Kreisel and Putnam [1957] rather than the weaker system of intuitionistic logic. And similarly when any other finite upper bound on the number of contradictory states is in place.[6]

When, intuitively, will a state s exactly verify the falsum constant $\perp$? We require, at the very least, that the state should be an inexact verifier for every statement, i.e. it should either be

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[6]My paper 'Constructing the Impossible' shows how to extend a space of possible states to a space of possible and impossible states.

or should contain an exact verifier for every statement. But we might impose an additional relevance requirement. Suppose that the statements of interest concern the color of some object - it could be red, blue, green etc. Then under the minimum requirement, the state of its being both red and blue might well count as a contradictory state. But by the same token, the result of adding any further state to *s* would also count as a contradictory state. Under the more substantive requirement, by contrast, there would be no guarantee that the enlarged state would remain contradictory. We will therefore need to distinguish, in this case, between contradictory states, which *exactly* verify $\perp$, and inconsistent states which need only *inexactly* verify $\perp$. Fortunately, it makes no difference to the resulting notions of consequence or validity whether we adopt the more of less substantive requirement; and so when it comes to contradictory states, we may ignore these issues of relevance and allow any extension of a contradictory state also to be a contradictory state.

Given that different inconsistent states inexactly verify the very same statements, viz. all of them, it may be wondered how, intuitively, they are to be distinguished. But this question is readily answered once we appreciate that states have mereological structure; for different inconsistent states may differ in the states of which they are composed. Taking up our previous example of the colored object, let us suppose that there is an inconsistent state of its being red and green and let us further suppose that it verifies the statement that the object is blue not by containing the state of the object's being blue but by being itself an exact verifier for the statement that the object is blue. Likewise, let us suppose that there is a state of the object's being red and blue. Then these states may well exactly verify the very same statements (concerning the color of the object), but they will still differ in that one is composed of the states of the object's being red and its being green while the other is composed of the states of the object's being red and its being blue.

§5 The Ancillary Problems

Let us now turn to the four ancillary problems noted above and indicate how they might be solved within the present approach.

(1) We wanted the clause for the conditional under the exact semantics to correspond to its clause under the constructive semantics. A construction for the conditional under the constructive semantics is essentially constituted by a function from constructions to constructions; and it was hard to see how a state might be analogously constituted by a function from states to states. But this becomes clear once conditional connections are at our disposal. For suppose that we have a function from states to states, taking t_1 to u_1, t_2 to u_2, ... Let us say that the state *s* *encodes* this function if it is the fusion of the conditional connections $t_1 \to u_1$, $t_2 \to u_2$, ... The verifying state for the conditional under the exact semantics is then essentially given by the function which it encodes.

(2) In providing an underlying exact semantics, we did not want to go beyond the resources of the original Kripke semantics. The present version of the semantics does go beyond these resources since, in stating the clause for the conditional, we have appealed to the notion of a conditional connection $s \to t$. But this is not essential to its formulation. As I have already noted, a conditional connection $s \to t$ should satisfy the constraint that its fusion with *s* contain *t*. But we may then define the conditional connection $s \to t$ to be the least state to satisfy this constraint. It should, that is to say, satisfy the constraint and be contained in any other state that

satisfies the constraint. In this way, we need make no appeal to any notion beyond the extension relation.

(3) We wanted it to be intrinsic to the content of a state that it exactly verified a given statement. But this did not appear to be so under the Kripke semantics, since a state's verifying a conditional, for example, seemed to turn upon the absence of extensions which verified the antecedent while failing to verify the consequent. This problem disappears once conditional connections are brought into the picture. For a state's verifying a given conditional will now turn upon its being constituted by suitable conditional connections; and it will be the presence of these confirming conditional connections, which are internal to the state, rather than the absence of falsifying extensions, which are external to the state, that will account for the truth of a conditional.[7]

(4) We wanted to have a better understanding of the relation between the condition-oriented and the construction-oriented semantics; and we have what is, at least, a partial solution to this problem. When we examine the proof of 'soundness' for the exact semantics, we see that it works though constructing an inexact counterpart to the exact semantics. Suppose we are given an exact model, conforming to the clauses proposed above. Let us define a state to be an inexact verifier for a statement just in case it contains an exact verifier for the statement. We thereby obtain an inexact model; and it can be shown that the resulting notion of inexact verification will conform to the clauses of the Kripke semantics.[8] Thus these clauses are precisely the ones would expect to hold under the 'inexactification' of the exact semantics.

Not every Kripke model can be obtained, in this way, as an inexact counterpart of an exact model. But it can be shown that every Kripke model is essentially equivalent to one that *is* the inexact counterpart of an exact model. Thus the exact semantics will yield all of the inexact models required for the purposes of completeness.

A problem remains. For we still need to account for the relationship between the exact form of the condition-oriented semantics (with its domain of states ordered by extension) and the constructive semantics (with its typology of different constructions). This is by no means straightforward, not only because there is considerable controversy over how the constructive semantics is to be formulated but also because the clause for the conditional under the E-semantics involves a loss of information - different functions can lead to the same verifying state. But the two are now so similar in form that there would appear to a reasonable expectation that they can be related in a philosophically illuminating way.

§6 Validity and Consequence

I believe that the present semantics provides alternative philosophical foundations, for both classical and intuitionistic logic; and, in the present section, I wish to discuss how this is so.

There are two main ways of understanding the 'conditions' in Kripke's semantics - either

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[7]The last part of the appendix contains further discussion of this point.

[8]A qualification is in order. For the exact semantics allows states that verify $\perp$ while no such states are admitted under the forcing semantics. But as I have mentioned, no harm would come from allowing such states.

as states of the world or as states of information; and there are two corresponding ways to understand the forcing relation - either as *verifying*, with a state of the world making a statement true, or as *warranting*, with a state of information warranting the assertion of a statement. It has been natural to regard the conditions as states of information rather than as states of the world. Partly this is because it is in more in keeping with the spirit of intuitionism, but it is perhaps partly because there is a special difficulty in understanding conditions to be states of the world. For if a state of the world is to verify a statement then we want it to be intrinsic to the content of the state that it does verify the statement. But it is hard to see how this might be so under the forcing semantics. This is not such a difficulty when we construe conditions as states of information, since we might reasonably think of the increase of information as subject to constraints that do not flow from the intrinsic content of the initial state - either because they are intrinsic to the state though not to its content (as when we take a state to be given both by some content and by rules for how the content is to be extended) or because they are extrinsic to the state (as when there is an external source - either in our mind or in the world - for the states of information that might become available to us).

As I have mentioned, the exact semantics provides us with a way of seeing how verification might flow from the intrinsic content of the verifying state; and so there would appear to be no bar to construing the semantics in realist fashion, as relating the truth of a statement to states of the world.

Not only can we construe the present semantics in realist fashion, we can also see it as providing a common basis for intuitionistic and classical logic. To this end, we must refine our previous account of consequence. Our previous account was in terms of verification; C was to be a consequence of A if, necessarily, every verifier of A is a verifier of C. But a more natural account is in terms of truth: C is to be a consequence of A if, necessarily, C is true whenever A is true.

Within the present framework, the definition of truth requires that we single out some states as actual or real. These are the *facts*. A statement can then be taken to be true if it is verified by a fact; and since we now have a range of facts from which to choose a verifier, we can take the relevant notion of verification to be exact. Thus a statement will be true if it is *exactly* verified by some fact. In contrast to the previous verifier-oriented conception of consequence, we no longer require that whatever verifies the premises should verify the conclusion but merely that there should exist an actual verifier for the conclusion whenever there exists an actual verifier for the premises.

To which class valid of statements or arguments will the present truth-oriented conception lead? Let us designate the class of facts as 'reality'. Then the answer to our question will critically depend upon which assumptions are made about the constitution of reality.[9] There are four assumptions that are relatively unproblematic and which might plausibly be granted on any philosophical view. These are:

<u>Non-Vacuity</u> Reality is non-empty (i.e. some fact is real);

<u>Consistency</u> Reality is consistent (i.e. every real fact is consistent);

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[9]My paper 'Tense and Reality' (chapter 8 of Fine [2005]) also deals with the constitution of reality, but from a somewhat different point of view.

Part Every part of a real fact is also real;

Finite Fusion The fusion of any two real facts is also real.

But there are two other assumptions which are much more controversial and which would appear to depend upon one's general metaphysical outlook. These are:

Closure The fusion of all facts exists and is itself a fact

Completeness Any state is either a fact or incompatible with a fact.

We might take the fusion of all facts, given that it exists, to be the actual world. Thus Closure postulates the existence of the actual world. Completeness, on the other hand, tells us that reality is complete: for any state, reality settles the question of whether it obtains either because it is itself a fact or because it is incompatible with a fact.

Closure might be doubted by someone who thought that there was an infinitude of facts and who thought that this infinitude was potential in the sense that there was no getting 'passed it' and taking all of the facts themselves to constitute an über-fact. Thus just as it has been supposed that there is no über-set, constituted by all of the sets, so there is no über-fact, composed of all of the facts; they are always in the making, so to speak, and remain, in their entirety, forever beyond our grasp.[10]

The assumptions of Closure and Completeness both belong to a classical conception of reality under which reality is complete, both in the sense of constituting a complete totality and in the sense of settling the question of whether any given state obtains. We might call a conception of reality *anti-classical* if it rejects both Closure and Completeness (but retains the four uncontroversial assumptions) and we might call it *semi-classical* if it accepts one of Closure or Completeness while rejecting the other.

As might be expected, the classical conception of reality, combined with our truth-theoretic account of consequence, leads to the classical notion of consequence; the consequences of a given set of statements are the classical consequences. Likewise, the anti-classical conception of reality, combined with the truth-theoretic account of consequence, leads to the intuitionistic notion of consequence; the consequences of a given set of statements are the intuitionistic consequences.

But what of the semi-classical conceptions? Somewhat surprisingly, they lead, not to the classical notion of consequence nor to something in between the classical and the intuitionistic notions, but to the intuitionistic notion. Thus full rejection of the classical conception of reality is not required if we are to endorse the intuitionistic notion of consequence; it suffices to reject one strand - be it Closure or Completeness - in the classical conception.[11]

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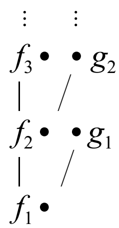
[10]The modeling given below is somewhat awkward in this regard since it takes there to be a set of facts and even takes there to be a fusion - even if an inconsistent fusion - of such facts. But the awkwardness simply arises from our desire to adopt an external point of view; it is in much the same way that we may wish to consider models for set theory in which the domain of quantification is taken to be a set.

[11]A qualification is in order. This result is correct for sentential logic but not for predicate logic. In case the Completeness assumption is made, the resulting logic will be, not intutionistic predicate logic, but intuitionistic predicate logic plus the double negation of all general statements of the law of the excluded middle (such as $\neg\neg\forall x(Fx \vee \neg Fx)$).

How, one might ask, can reality be complete and yet the Law of Exclude Middle not hold? Surely the one entails the other. To see why this is not so, consider a situation in which the consistent states f_1, f_2, ..., g_1, g_2, ... are depicted below (with f_1 included in f_2 and g_1 etc.):



For the sake of concreteness, we might think of f_k as the state of there being at least k objects and of g_k as the state of there being exactly k objects. Taking the *facts* to be f_1, f_2, ..., the Completeness assumption will be satisfied, since any consistent state is either an f_k and hence a fact or a g_k and hence incompatible with f_{k+1}. But letting p be the statement that there are finitely many objects, no fact will verify either p or ¬p and hence no fact will verify p ∨ ¬p.

We have seen how partial rejection of the classical conception of the reality may be compatible with the endorsement of intuitionistic logic. But there is a way in which *full* acceptance of the classical conception may also be compatible with the endorsement of intuitionistic logic. For we have so far taken a statement to be logically true if, necessarily, there is a fact that verifies it. But we may wish to insist upon a stricter criterion of logical truth and require not merely that some fact verify the statement but that the null fact verify it. In other words, nothing is required of the world for the statement to be true.

This distinction was ignored by the logical positivists when they claimed that logical truths were lacking in factual content - all logical truths were regarded by them as alike in this respect. But the distinction becomes very clear once we consider what might verify a statement of the form A ∨ ¬A as opposed to a statement of the form A ⊃ A. The former will be verified by whatever verifies A or verifies ¬A and this, in general, will be a substantive, or 'non-null', fact. A verifier is guaranteed to exist but what verifies the statement will, as a rule, be a substantive fact, varying with how things turn out. On the other hand, a statement of the form A ⊃ A will be verified by the null fact, regardless of how things turn out. For consider the verifiers s_1, s_2, ... of A. For each s_k, the connection $s_k \to s_k$ will be the null fact, since the fusion of the null fact with s_k will always contain s_k. Thus the fusion of all $s_k \to s_k$ will be the null fact and hence be a verifier for A ⊃ A. Of course, a classical logician may wish to define A ⊃ B as ¬A ∨ B and hence will not be willing to accept a difference in logical status between A ⊃ A and ¬A ∨ A (or A ∨ ¬A). But from the present perspective, the proposed definition cannot be accepted since the verifiers for A ⊃ B and ¬A ∨ B will not in general be the same.

However, it should be pointed out that the present validation of intuitionistic logic is somewhat at odds with the standard intuitionist position. For every instance of ¬A ∨ A - or any other classical logical truth - will be true and, indeed, necessarily true. It is just that it may not be *logically* true, since something substantive of the world may be required for it to be true. Still, the present approach does provide a clear and reasonably well-motivated way in which someone might combine a classical stance on reality with an intuitionistic stance towards its logic.

The present distinction between the two kinds of logical truth is also in sharp contrast to how the related distinction plays out within the earlier truthmaker semantics for classical logic (with separate clauses for the verification and falsification of a statement). Within the classical semantics, every statement, if true, will be made true by a substantive non-null fact (unless it contains an atomic sentence that already has a trivial verifier). Thus it is only be adopting the intuitionist style of semantics, with its characteristic clause for the conditional, that the present distinction can get any grip.

The picture of intuitionistic logic that emerges from the preceding considerations is very different from the picture familiar from the writings of Dummett and commonly accepted in the philosophical literature. On the Dummettian view, the intuitionist will adopt a semantics for his language in terms of assertion conditions (i.e. conditions of warranted assertability) while the classical philosopher will adopt a semantics in terms of truth-conditions (relating the truth of the statement to how things are in the world); and it is because of this difference in their semantics that there will be a difference in their logic.

On the present view, there is no difference in the semantics embraced by the intuitionist or classical philosopher.[12] The semantical clauses telling us when a logically complex statement is verified will be exactly the same. Moreover, although these clauses will have an intuitionist 'flavor' to them, they can be understood in straightforward realist fashion, relating the verification of a statement, not to our knowledge of the world, but to the world itself.

The difference in logic will arise, not from a semantical difference, but either from a metaphysical difference, concerning the constitution of reality, or from a meta-logical difference, concerning the nature of logical truth. Indeed, in the first case, it will be possible for the intuitionist to regard reality as complete as long as he also regards it as suitably open-ended. Of course, the semantically oriented philosopher may always attempt to attribute the metaphysical difference to a semantical difference. Just as the disagreement over the Law of the Excluded Middle is to be attributed, on his view, to a semantic difference, so it is, he might argue, for the disagreement over whether reality is complete or is closed. But given the present diagnosis of the difference between the intuitionist and classical philosopher, the disagreement over Completeness or Closure is much more plausibly regarded as a metaphysical disagreement that arises once a common semantics is in place.

Technical Appendix

A (*sentential*) *atom* is either one of the sentence letters p_1, p_2, or the falsum constant $\bot$. We use α, β, γ, ... for atoms and p, q, r, ... for sentence letters. Formulas are constructed from the atoms in the usual way by means of the connectives $\vee$, $\wedge$ and $\supset$.

Recall that a relational structure (S, $\sqsubseteq$), for S a set and $\sqsubseteq$ a binary relation on S, is a *partial order* if $\sqsubseteq$ is a reflexive, transitive and antisymmetric relation on S. A *K-frame* is a partially

[12]Rumfitt (in Rumfitt [2012] and other work) also provides a common semantical basis for intuitionistic and classical logic. But his approach is somewhat different from my own. He works with an 'inexact' rather than an 'exact' notion of verification and he adopts a broader conception of the possible states, under which a possible state may verify a disjunction without verifying either disjunct. Our approaches have the common virtue of applying with equal ease to the mathematical and empirical domains.

order set $(S, \sqsubseteq)$ and a *K-model* is a triple $(S, \sqsubseteq, \varphi)$, where $(S, \sqsubseteq)$ is a K-frame and φ (valuation) is a relation between the states of S and the sentence letters, subject to:

Hereditary Condition $\varphi s\mathrm{p}$ & $s \sqsubseteq t \Rightarrow \varphi t\mathrm{p}$.

Relative to a K-model $(S, \sqsubseteq, \varphi)$, we define what it is for a formula A to be *verified by* a state s of S ($s \models$ A):

K(i)(a) $s \models$ p if $\varphi s\mathrm{p}$

(b) $s \models \bot$ never

K(ii) $s \models$ B $\vee$ C if $s \models$ B or $s \models$ C;

K(iii) $s \models$ B $\wedge$ C if $s \models$ B and $s \models$ C;

K(iv) $s \models$ B $\supset$ C if $t \models$ C whenever $t \models$ B and $s \sqsubseteq t$.

The following standard result is proved by induction on the formula A:

Lemma 1 (Hereditary) For any K-model $(S, \sqsubseteq, \varphi)$ and states s and t of S,

$s \models$ A and $s \sqsubseteq t$ implies $t \models$ A

Where Δ is a set of formulas, we say $s \models \Delta$ (relative to a model) if $s \models$ A for every A in Δ, and we say that the formula C is an (*intuitionistic*) *consequence of* the set of formulas Δ - in symbols, $\Delta \models_{\mathrm{I}}$ C - if, for any model $\mathcal{M} = (S, \sqsubseteq, \varphi)$ and state s in S, $s \models$ C whenever $s \models \Delta$. Let us use $\Delta \vdash_{\mathrm{I}}$ C to denote that C is derivable from Δ under some standard system for intuitionist logic. We state without proof:

Theorem 2 (Soundness and Completeness) $\Delta \models_{\mathrm{I}}$ C iff $\Delta \vdash_{\mathrm{I}}$ C

We turn to the corresponding exact semantics. Recall that a partial order $(S, \sqsubseteq)$ is said to be *complete* - or to be a *complete semi-lattice* - if each subset T of S has a least upper bound. We denote the least upper bound of each subset T of a complete semi-lattice $(S, \sqsubseteq)$ by $\bigsqcup T$ (and also use obvious variants of this notation). It is readily shown that each subset T in a complete semi-lattice $(S, \sqsubseteq)$ will also have a greatest lower bound (glb), which we denote by $\sqcap T$; for $\sqcap T$ may be defined as $\bigsqcup\{u: u \sqsubseteq t$ for each $t \in T\}$.

Given two states s and t in a complete partial order, their *residuation* $s \rightarrow t$ is defined to be $\sqcap\{u: s \sqcup u \sqsupseteq t\}$ (I called these states *conditional connections* in the informal exposition above) and the order itself is said to be *residuated* if it satisfies:

Residuation Condition $s \sqcup (s \rightarrow t) \sqsupseteq t$.

Let us note the following facts about residuation (use of which will often be implicit):

Lemma 3 For any elements s, t, u of a complete residuated semi-lattice $(S, \sqsubseteq)$:

(i) $(s \rightarrow t) \sqsubseteq t$

(ii) $s \sqcup (s \rightarrow t) \sqsubseteq s \sqcup t$

(iii) $t \sqsubseteq u \Rightarrow (s \rightarrow t) \sqsubseteq (s \rightarrow u)$

(iv) $s \rightarrow (s \sqcup t) \sqsubseteq (s \rightarrow t)$.

Proof (i) $(s \rightarrow t)$ is the glb of $\{u: s \sqcup u \sqsupseteq t\}$ and so, since $s \sqcup t \sqsupseteq t$, $(s \rightarrow t) \sqsubseteq t$.

(ii) Since $s \sqsubseteq s$ and since, by (i) above, $(s \rightarrow t) \sqsubseteq t$, $s \sqcup (s \rightarrow t) \sqsubseteq s \sqcup t$.

(iii) $s \rightarrow t$ is the glb of $\{v: s \sqcup v \sqsupseteq t\}$ and $s \rightarrow u$ the glb of $\{v: s \sqcup v \sqsupseteq u\}$. But given $t \sqsubseteq u$, $\{v: s \sqcup v \sqsupseteq u\} \supseteq \{v: s \sqcup v \sqsupseteq t\}$; and so $(s \rightarrow t) \sqsubseteq (s \rightarrow u)$.

(iv) From (ii), $s \sqcup (s \rightarrow t) \sqsubseteq s \sqcup t$; and so, given that $s \rightarrow (s \sqcup t)$ is the glb of $\{u: s \sqcup u \sqsubseteq s \sqcup t\}$, $s \rightarrow (s \sqcup t) \sqsubseteq (s \rightarrow t)$.

A E-*frame* ('E' for exact) is a partial order (S, ⊑) which is residuated and complete. An E-*model* is an ordered triple (S, ⊑, φ), where (S, ⊑) is an E-frame and φ (the valuation) is a relation holding between the members of S and the sentential atoms, subject to:

Falsum Condition $\varphi s\bot$ implies $\varphi s'$p for some state $s' \sqsubseteq s$.

There would be no difference in the notions of consequence and validity if we strengthened the condition to:

Strict Falsum Condition $\varphi s\bot$ implies φsp.

A state s in a E-model (S, ⊑, φ) is said to be *contradictory* if $\varphi s\bot$, to be *inconsistent* if $s \sqsupseteq t$ for some contradictory state t, and to be *consistent* otherwise. Two states s and t are said to be *compatible* if their fusion $s \sqcup t$ is consistent and *incompatible* otherwise. Note that we do not impose the Hereditary Condition on E-models. However, there would be no difference in the notions of consequence and validity if we insisted upon this condition for the falsum constant ⊥:

Hereditary Condition for ⊥ $\varphi s\bot \;\&\; s \sqsubseteq t \Rightarrow \varphi t\bot$.

We have the following clauses for when a formula A is exactly verified by a state in a E-model (S, ⊑, φ):

E(i) s ||- α if $\varphi s\alpha$;

E(ii) s ||- B ∨ C if s ||- B or s ||- C;

E(iii) s ||- B ∧ C if for some s_1 and s_2, s_1 ||- B, s_2 ||- C and $s = s_1 \sqcup s_2$;

E(iv) s ||- B ⊃ C if there is a function taking each t ||- B into a u_t ||- C for which

$$s = \bigsqcup\{t \rightarrow u_t : t \Vdash \mathrm{B}\}.$$

It is illuminating to state the clauses in 'algebraic' form. Relative to a E-model (S, ⊑, φ), let us use [A] for $\{s: s \Vdash \mathrm{A}\}$. For subsets T and U of S, let $T \sqcup U = \{t \sqcup u: t \in T \text{ and } u \in U\}$ and let $T \rightarrow U = \{s: s = \bigsqcup\{t \rightarrow u_t: t \in T\}$ for some function $u: T \rightarrow U\}$. We then have the following identities:

[B ∨ C] = [B] ∪ [C]

[B ∧ C] = [B] ⊔ [C]

[B ⊃ C] = [B] → [C].

Note that the clause for [A] in each case provides us with a description of how the members of [A] are constituted from the verifiers for the component formulas and in such a way as to make clear that they are indeed verifiers for A.

We say s ||> A (s *inexactly* verifies A) if s' ||- A for some $s' \sqsubseteq s$. It can then be shown that inconsistent states inexactly verify every formula:

Lemma 4 (Quodlibet) For any formula A and inconsistent state s in a E-model, s ||> A.

Proof By induction on the complexity of the formula A.

A = α Given that s is inconsistent, there is a contradictory state $s' \sqsubseteq s$. But s' ||- ⊥ by definition; so s'' ||- α for some $s'' \sqsubseteq s'$ by the Falsum Condition; and so s ||> α.

A = B ∨ C By IH, s ||> B. But then s' ||- B for some $s' \sqsubseteq s$; so s' ||- B ∨ C; and so s ||> B ∨ C.

A =B ∧ C By IH, s ||> B and s ||> C. So s' ||- B and s'' ||- C for some $s', s'' \sqsubseteq s$. But then $s' \sqcup s''$ ||- B ∧ C; and, since $s' \sqcup s'' \sqsubseteq s$, s ||> B ∧ C.

A = B ⊃ C By IH, s ||> C. So s' ||- C for some $s' \sqsubseteq s$. For each t ||- B, we set $u_t = s'$. So $t \rightarrow u_t = t \rightarrow s' \sqsubseteq s' \sqsubseteq s$. But $\bigsqcup(t \rightarrow u_t)$ ||- B ⊃ C; and since $\bigsqcup(t \rightarrow u_t) \sqsubseteq s$, s ||> B ⊃ C.

Let us now show how to go from an E-model to a K-model and from a K-model to an E-model. This will enable us to transfer completeness for the one kind of modeling to the other.

Given an E-model $\mathcal{M} = (S, \sqsubseteq, \varphi)$, we define a corresponding K-model $\mathcal{M}_K = (S_K, \sqsubseteq_K, \varphi_K)$ by:

(i) $S_K = \{s \in S: s \text{ is consistent}\}$;

(ii) $\sqsubseteq_K = \sqsubseteq \restriction S_K$;

(iii) $\varphi_K = \{(s, p): s \in S_K, p \text{ is a sentence letter, and, for some } s' \sqsubseteq s, \varphi s'p\}$.

It should be evident that $\mathcal{M}_K$, as so defined, is indeed a K-model (it follows, in particular, from clause (iii) for φ_K that the Hereditary Condition will be satisfied). If we did not cut away the inconsistent points, then $\mathcal{M}_K$ would be a modified Kripke model in the sense of Veldman [1976]. Thus the E-models provide a natural underpinning for the models that he uses in his completeness proof.

We now show that inexact verification in an E-model behaves in the same way as forcing in the corresponding K-model:

Theorem 5 (E/K) Let $\mathcal{M} = (S, \sqsubseteq, \varphi)$ be an E-model and $\mathcal{M}_K = (S_K, \sqsubseteq_K, \varphi_K)$ the corresponding K-model. Then for any state $s \in S_K$ and any formula A,

$s \Vdash\!> A$ in $\mathcal{M}$ iff $s \models A$ in $\mathcal{M}_K$.

Proof By induction on A.

A = ⊥ Never $s \Vdash\!> \bot$ in $\mathcal{M}$ since s is consistent; and never $s \models \bot$ in $\mathcal{M}_K$ by clause K(i)(b).

A = p, p $s \Vdash\!> p$ in $\mathcal{M}$ iff $s' \Vdash p$ in $\mathcal{M}$ for some $s' \sqsubseteq s$

iff $\varphi s'p$ for some $s' \sqsubseteq s$

iff $\varphi_K sp$

iff $s \models p$ in $\mathcal{M}_K$.

A = B ∨ C Suppose $s \Vdash\!> B \vee C$ (in $\mathcal{M}$). Then $s' \Vdash B \vee C$ for some $s' \sqsubseteq s$. So $s' \Vdash B$ or $s' \Vdash C$. By IH, $s' \models B$ or $s' \models C$ (in $\mathcal{M}_K$). But then $s' \models B \vee C$; and so, by the Hereditary Lemma, $s \models B \vee C$.

Now suppose $s \models B \vee C$ (in $\mathcal{M}_K$). Then $s \models B$ or $s \models C$. By IH, $s \Vdash\!> B$ or $s \Vdash\!> C$; and so for some $s' \sqsubseteq s$, $s' \Vdash B$ or $s' \Vdash C$. In either case, $s' \Vdash B \vee C$; and so $s \Vdash\!> B \vee C$.

A =B ∧ C Suppose $s \Vdash\!> B \wedge C$. Then $s' \Vdash B \wedge C$ for some $s' \sqsubseteq s$. So for some s_1' and s_2', $s_1' \Vdash B$, $s_2' \Vdash C$ and $s' = s_1' \sqcup s_2'$. By IH, $s_1' \models B$ and $s_2' \models C$; by the Hereditary Lemma, $s \models B$ and $s \models C$; and so $s \models B \wedge C$.

Now suppose $s \models B \wedge C$. Then $s \models B$ and $s \models C$. By IH, $s \Vdash\!> B$ and $s \Vdash\!> C$; and so for some $s_1, s_2 \sqsubseteq s$, $s_1 \Vdash B$ and $s_2 \Vdash C$. But then $s' = s_1 \sqcup s_2 \Vdash B \wedge C$; and, since $s' \sqsubseteq s$, $s \Vdash\!> B \wedge C$.

A = B ⊃ C Suppose $s \Vdash\!> B \supset C$. Then $s' \Vdash B \supset C$ for some $s' \sqsubseteq s$ of the form $\bigsqcup(t \rightarrow u_t)$ (where the $t \Vdash B$ and the $u_t \Vdash C$). Consider now any consistent $t \sqsupseteq s$ for which $t \models B$ (with a view to establishing $t \models C$). By IH, $t \Vdash\!> B$; and so, for some $t' \sqsubseteq t$, $t' \Vdash B$. Now $t' \rightarrow u_{t'} \sqsubseteq s' \sqsubseteq s \sqsubseteq t$ and also $t' \sqsubseteq t$. So $u_{t'} \sqsubseteq (t' \rightarrow u_{t'}) \sqcup t' \sqsubseteq s' \sqcup t' \sqsubseteq t$. Since $u_{t'} \Vdash C$, $t \Vdash\!> C$ and so, by IH, $t \models C$.

Now suppose $s \models B \supset C$ and consider a $t \Vdash B$ (with a view to establishing $s \sqcup t \Vdash\!> C$). We distinguish two cases:

(a) t is compatible with s. Then $t, s \sqcup t \in S_K$. Given $t \in S_K$, $t \models B$ by IH; given $s \sqcup t \in S_K$, $s \sqcup t \models B$ by the Hereditary Condition; given $s \models B \supset C$, $s \sqcup t \models C$; and so $s \sqcup t \Vdash\!> C$ by IH.

(b) t is incompatible with s. Then $s \sqcup t$ is inconsistent; and so, by Quodlibet, $s \sqcup t \Vdash\!> C$. Thus, in either case, $s \sqcup t \Vdash\!> C$ and so $u \Vdash C$ for some $u \sqsubseteq s \sqcup t$. For each t for which $t \Vdash B$, we set $u_t = u$. Then $t \rightarrow u_t = t \rightarrow u \sqsubseteq t \rightarrow (s \sqcup t) \sqsubseteq t \rightarrow s \sqsubseteq s$. So $s' = \bigsqcup(t \rightarrow u_t) \sqsubseteq s$ and, given that $s' \Vdash B \supset C$, $s \Vdash\!> B \supset C$.

We turn to the transformation in the opposite direction. In this case, not every K-model can be straightforwardly transformed into an E-model and we need to impose some further conditions on the K-model.

Let $(S, \sqsubseteq)$ be a partially ordered set. A subset T of S is then said to be *downward-closed* if $t \in T$ and $s \sqsubseteq t$ implies $s \in T$. Let $[t] = \{u \sqsubseteq t: u \in S\}$. A subset T of S is said to be *principal* if it is of the form $[t]$ for some element t of S and is said to be *non-principal* otherwise. Two elements s and t of S are said to be *comparable* if either $s \sqsubseteq t$ or $t \sqsubseteq s$ and to be *incomparable* otherwise. An element s of $(S, \sqsubseteq)$ is said to be *root* if $s \sqsubseteq t$ for each element t of S and $(S, \sqsubseteq)$ itself is said to be *rooted* if it has a root. By the antisymmetry of $\sqsubseteq$, a root, if it exists, is unique and, clearly, in the case of a complete partial order, the root is identical to the null element. Finally, the partial order $(S, \sqsubseteq)$ is said to be *tree-like* if:

(a) no infinite ascending chain of elements $s_1 \sqsubset s_2 \sqsubset \dots$ of S has an upper bound;

(b) no two incomparable elements of S have an upper bound; and

(c) there is a root.

The following result on tree-like structures will later be useful:

Lemma 6 Given that the partial order $(S, \sqsubseteq)$ is tree-like, no principal downward-closed subset of S contains a non-empty non-principal downward-closed subset of S.

Proof Suppose that T is a non-empty non-principal downward-closed subset of S. Let t_1 be a member of T. Then $T \neq [t_1]$ since otherwise T would be principal; and so, since T is downward-closed, $[t_1] \subset T$. Let t_2 be a member of $T - [t_1]$. Then not $t_2 \sqsubseteq t_1$ since otherwise $t_2 \in [t_1]$. If t_1 and t_2 were incomparable then we would be done since no principal downward-closed subset $[t]$ could contain T without t being an upper bound for t_1 and t_2, contrary to (b) above.

So t_1 and t_2 are comparable and hence $t_1 \sqsubset t_2$. Continuing in this way, we may construct a chain $t_1 \sqsubset t_2 \sqsubset t_3 \sqsubset \dots$ of members of T. If the chain is finite, then it is of the form $t_1 \sqsubset t_2 \sqsubset t_3 \sqsubset \dots t_n$ with T the downward closure of $\{t_1, t_2, t_3, \dots, t_n\}$ and $T = [t_n]$ principal after all. If the chain $t_1 \sqsubset t_2 \sqsubset t_3 \sqsubset \dots$ is infinite, then no principal downward closed subset of T can contain $\{t_1, t_2, t_3, \dots\}$ on pain of violating condition (a) above.

Given a *K-model* $\mathcal{M} = (S, \sqsubseteq, \varphi)$, we define a corresponding structure $\mathcal{M}_E = (S_E, \sqsubseteq_E, \varphi_E)$ by:

(i) $S_E = \{T \subseteq S: T$ is non-empty and downward closed$\}$;

(ii) $\sqsubseteq_E = \subseteq \upharpoonright S_E$

(iii) $\varphi_E = \{([s], p): s \in S$ and $\varphi s p\} \cup \{(T, \alpha): T$ is a non-principal downward-closed subset of $S_E\}$.

The transformation of $\mathcal{M}$ into $\mathcal{M}_E$ does indeed provide us with an E-model:

Lemma 7 When $\mathcal{M} = (S, \sqsubseteq, \varphi)$ is a tree-like K-model, $\mathcal{M}_E = (S_E, \sqsubseteq_E, \varphi_E)$ is an E-model.

Proof (i) Clearly, $(S_E, \sqsubseteq_E)$ is a partial order.

(ii) $(S_E, \sqsubseteq_E)$ is complete.

Pf. Suppose $S_1, S_2, \dots$ are any number of non-empty downward closed subsets of S. Let S' be s_0, where s_0 is the root element, if there are no $S_1, S_2, \dots$; and let $S' = S_1 \cup S_2 \cup \dots$ otherwise. Then $s_0 \in S'$ and so S' is also non-empty; clearly, S' is downward-closed; and so $S' \in S_E$. Moreover, it is evident that S' is the lub of $S_1, S_2, \dots$ under the relation $\sqsubseteq_E$ of set-theoretic inclusion on S_E.

(iii) $(S_E, \sqsubseteq_E)$ is residuated.

Pf. Take two non-empty downward closed subsets T and U of S. Let $V = \bigcup\{[u]: u \in U - T\} \cup \{s_0\}$. Clearly, $V \in S_E$. Also, $(T \cup V) \supseteq U$ and hence $(T \sqcup_E V) \sqsupseteq_E U$. For if $s \in U$ then either $s \in T$

$\subseteq (T \cup V)$ or $s \in U - T$, in which case $s \in [s] \subseteq V \subseteq (T \cup V)$. Moreover, for any $V' \in S_E$, $T \cup V' \supseteq U$ implies $V \subseteq V'$ and hence $(T \sqcup_E V') \sqsupseteq_E U$ implies $V \sqsubseteq_E V'$. For take any $u \in U - T$. Then clearly $u \in V'$, given $T \cup V' \supseteq U$. But then $[u] \subseteq V'$ given that V' is downward-closed; and so $V \subseteq V'$. Thus $V = (T \rightarrow_E U)$ and satisfies the Residuation Condition.

(iv) $\mathcal{M}_E$ conforms to the Falsum Condition.

Pf. From the definition of φ_E.

Note that $\mathcal{M}_E$ is far from being a typical E-model. As is easily shown, it satisfies the Strict Falsum Condition and the Hereditary Condition; and this means that imposing either or both of these conditions on E-models will not result in the validity of any further formulas. But here, as is often the case, the intended models for our logic will far outstrip those required to establish completeness.

Rather than directly establishing the equivalence between the K-model $\mathcal{M}$ and the E-model $\mathcal{M}_E$, we derive it from the corresponding equivalence between the E-model $\mathcal{M}$ and the K-model $\mathcal{M}_K$. But first we show that the one transformation is, in a way, the inverse of the other.

Theorem 8 Suppose $\mathcal{M} = (S, \sqsubseteq, \varphi)$ is a tree-like K-model. Then the map taking each member s of S into $[s]$ is an isomorphism from $\mathcal{M}$ onto $(\mathcal{M}_E)_K$.

Proof Let $(\mathcal{M}_E)_K = (S_{E/K}, \sqsubseteq_{E/K}, \varphi_{E/K})$. Then $S_{E/K}$ consists of the consistent elements of S_E, i.e. of those elements of S_E that do not contain a contradictory element, i.e of those elements of S_E that do not contain a non-empty non-principal subset of S. Each such element must itself be a principal subset of S and, by lemma 6, each principal subset of S_E will be such an element. Thus $S_{E/K} = \{[s]: s \in S\}$. Also for $s, t \in S$, $[s] \sqsubseteq_{E/K} [t]$ iff $[s] \subseteq [t]$, which holds iff $s \sqsubseteq t$, given that $[s]$ and $[t]$ are downward-closed. Finally, for $s \in S$, never $\varphi_{E/K}[s]\bot$ and never $\varphi s\bot$ and, for any sentence letter p, $\varphi_{E/K}[s]$p iff $\varphi_E[s']$p for some $[s'] \sqsubseteq [s]$. But given that $\mathcal{M}_E$ satisfies the Hereditary Condition, $\varphi_E[s']$p for some $[s'] \sqsubseteq [s]$ iff $\varphi_E[s]$p; and $\varphi_{E/K}[s]$p iff φsp, as required.

Corollary 9 (K/E) Let $\mathcal{M} = (S, \sqsubseteq, \varphi)$ be a tree-like K-model and $\mathcal{M}_E = (S_E, \sqsubseteq_E, \varphi_E)$ the corresponding E-model. Then for any $s \in S$:

$s \models$ A in $\mathcal{M}$ iff $[s] \Vert\!>$ A in $\mathcal{M}_E$.

Proof By the E/K theorem, $[s] \Vert\!>$ A in $\mathcal{M}_E$ iff $s \models$ A in $(\mathcal{M}_E)_K$ for each $s \in S$. By the isomorphism theorem above, $s \models$ A in $(\mathcal{M}_E)_K$ iff $s \models$ A in $\mathcal{M}$. But then $s \models$ A in $\mathcal{M}$ iff $[s] \Vert\!>$ A in $\mathcal{M}_E$, as required.

We turn to consequence. There are two somewhat different ways of defining the notion (and the cognate notion of validity) - one in terms of the preservation of verifiers and the other in terms of the preservation of truth - and there are variants on each approach, depending upon the form of verification or upon how the concept of truth is related to verification. Let us begin with the definitions in terms of verification.

We may say that the formula A is an *exact consequence of* the formulas $A_1, A_2, \ldots$ - in symbols, $A_1, A_2, \ldots \models_e C$ - if in any E-model $\mathcal{M}$ and any states $s_1, s_2, \ldots$ of $\mathcal{M}$, $s_1 \sqcup s_2 \sqcup \ldots \Vdash C$ whenever $s_1 \Vdash A_1$, $s_2 \Vdash A_2$, ... The notion of exact consequence is of great interest in its own right. However, our interest is in modeling the more usual notions of consequence and we shall say no more about it.

Where Δ is a set of formulas, we say *s* ||> Δ (relative to an E-model) if *s* ||- A for each A ∈ Δ. Substituting inexact verification for exact verification, we obtain the following three notions of consequence:

$\Delta \models_{i1}$ C if in any E-model $\mathcal{M}$ and any state *s* of $\mathcal{M}$, *s* ||> C whenever *s* ||- Δ;

$\Delta \models_{i2}$ C if in any E-model $\mathcal{M}$ and any consistent state *s* of $\mathcal{M}$, *s* ||> C whenever *s* ||> Δ;

$\Delta \models_{i3}$ C if in any E-model $\mathcal{M}$ and for the null-state *s* of $\mathcal{M}$, *s* ||> C whenever *s* ||- Δ.

These definitions differ in which states are taken to be relevant to establishing a countermodel to the argument from Δ to C, with the first allowing any state whatever, consistent or inconsistent, the second allowing only consistent states, and the third allowing only the null state. (I have given these definitions for the case in which the conclusion is required to be a single formula C, but they can be extended in the usual way to the case in which the conclusion is allowed to be an arbitrary set of formulas Γ, disjunctively interpreted.)

Although these various notions of inexact consequence are apparently of different strength, we may establish the equivalence of each of them to intuitionistic consequence. But first we need a standard result on 'tree' models. Suppose $\mathcal{M} = (S, \sqsubseteq, \varphi)$ is a K-model with root s_0. We define the corresponding *tree* model $\mathcal{M}_t = (S_t, \sqsubseteq_t, \varphi_t)$ by:

(i) S_t is the set of sequences $s_0 s_1 s_2 \ldots s_n$, n ≥0, for which $s_i \sqsubseteq s_{i+1}$ for all i = 0, 1, ..., n - 1;

(ii) $\sqsubseteq_t = \{(\boldsymbol{s}, \boldsymbol{t}) \in S_t \times S_t$: $\boldsymbol{s}$ is identical to or an initial segment of $\boldsymbol{t}\}$;

(iii) $\varphi_t = \{((\boldsymbol{s}, p)$: $\boldsymbol{s} \in S_t$, p is a sentence letter and φsp, for $\boldsymbol{s}$ terminating in $s\}$.

It should be clear that $\mathcal{M}_t$ is a tree model with root the unit sequence s_0.

We now have a standard inductive proof of:

<u>Theorem 10</u> Suppose that $\mathcal{M} \models (S, \sqsubseteq, \varphi)$ is a rooted K-model and let $\mathcal{M}_t \models (S_t, \sqsubseteq_t, \varphi_t)$ be the corresponding tree model. Then for any $s \in S$, any $\boldsymbol{s} \in S_t$ terminating in *s*, and any formula A:

$s \models$ A in $\mathcal{M}$ iff $\boldsymbol{s} \models$ A in $\mathcal{M}_t$.

From this, we obtain:

<u>Corollary 11</u> If not $\Delta \models_I$ C then, in some tree-like K-model with root s_0, $s_0 \models \Delta$ while not $s_0 \models$ C.

<u>Proof</u> Given not $\Delta \models_I$ C, it is readily shown that in some K-model $\mathcal{M}$ with root s_0, $s_0 \models \Delta$ while not $s_0 \models$ C. From the previous theorem, it follows that in the corresponding tree model $\mathcal{M}_t$, $s_0 \models \Delta$ while not $s_0 \models$ C; and it is readily verified that $\mathcal{M}_t$ is tree-like with root s_0.

We now have:

<u>Theorem 12</u> For any set of formulas Δ and formula C, the following are equivalent:

(i) $\Delta \models_{i1}$ C

(ii) $\Delta \models_{i2}$ C

(iii) $\Delta \models_{i3}$ C

(iv) $\Delta \models_I$ C.

<u>Proof</u> It is evident from the definitions of $\models_{i1}$, $\models_{i2}$ and $\models_{i3}$ that (i) implies (ii) and (ii) implies (iii); and so it suffices to establish that (iii) implies (iv) and (iv) implies (i).

(iii) implies (iv). Suppose not $\Delta \models_I$ C. By the previous theorem, in some tree-like K-model $\mathcal{M} = (S, \sqsubseteq, \varphi)$ with root s_0, $s_0 \models \Delta$ while not $s_0 \models$ C. We consider the corresponding E-model $\mathcal{M}_E$. By corollary 9, $[s_0]$ ||> Δ in $\mathcal{M}_E$ while not $[s_0]$ ||> C. But $[s_0]$ is the null state of $\mathcal{M}_E$; and so not $\Delta \models_{i3}$ C.

(iv) implies (i). Suppose not $\Delta \models_{i1}$ C. Then for some E-model $\mathcal{M}$ and state s of $\mathcal{M}$, $s \Vdash \Delta$ but not $s \Vdash$ C. Consider the corresponding K-model $\mathcal{M}_K$. Since not $s \Vdash$ C, s is consistent and hence a state of $\mathcal{M}_K$. By the E/K theorem, $s \models \Delta$ in $\mathcal{M}_K$ but not $s \models$ C and hence not $\Delta \models_I$ C.

We turn to the truth-theoretic notion of consequence. A statement may be taken to be true if it has an *actual* verifier. We make this idea precise within the E-semantics (and we might also do so within the K-semantics) by singling out a subset R of S to represent the states that actually obtain. Thus we may say that $\mathcal{M} = (S, R, \sqsubseteq, \varphi)$ is a *distinguished* E-model - or, more simply, a D-model - if $(S, \sqsubseteq, \varphi)$ is an E-model and R (for 'reality') is subject to the following four conditions:

Non-Vacuity R is non-empty

Consistency Each $s \in R$ is consistent

Part $s' \in R$ if $s \in R$ and $s' \sqsubseteq s$

Finite Fusion $s \sqcup t \in R$ if $s, t \in R$.

We might think of the elements of R as the *facts*. Non-vacuity then says that there is a fact, Consistency that each fact is consistent, Part that parts of facts are facts, and Fusion that the fusion of any two facts is a fact. Non-vacuity, Part and Finite Fusion correspond, of course, to the defining conditions for an ideal.

These conditions are all very reasonable. But there are two other, somewhat more controversial conditions, which we may wish to adopt:

Closure $\bigsqcup R' \in R$ for each $R' \subseteq R$

Completeness For any state $s \in S$ either s is a member of R or is incompatible with a member of R.

Closure tells us that the facts are closed under arbitrary fusions (not just finite fusions) and Completeness tells us that the facts are, in a certain sense, complete.

Clearly, Closure implies Fusion; and it also implies Non-Vacuity (upon letting $R' = \varnothing$). Given Part, Closure is equivalent to:

Totality $\bigsqcup R \in R$.

We might call $r_\infty = \bigsqcup R$ (*total*) *reality*. Thus Closure tells us that total reality is itself a fact.

Given Totality and Part, R will be identical to $\{r: r \sqsubseteq r_\infty\}$. This means that, in the definition of a D-model, we may replace the set R with a single element r_∞ and talk of the parts of r_∞ instead of the members of R. Under this alternative definition, Non-Vacuity, Part, Fusion and Closure become redundant. Consistency becomes:

(*) r_∞ is consistent

and Completeness becomes:

(**) any $s \in S$ is either a part of r_∞ or incompatible with r_∞.

We might call an element w of S a (*classical*) *world* if it conforms to (*) and (**), i.e. if it is consistent and if any state s is a part of w or incompatible with w. Thus (*) and (**) amount to the assumption that reality is a classical world.

We may say that a D-model $\mathcal{M} = (S, R, \sqsubseteq, \varphi)$ is *closed* if R satisfies Closure, *complete* if R satisfies Completeness, and *classical* if R satisfies both Closure and Completeness.

Even if a given D-model is complete, let us say, or classical, it is not clear that there is any guarantee that it will be complete or classical, no matter how things might have turned out. So let us say that an E-model $\mathcal{M} = (S, \sqsubseteq, \varphi)$ is *thoroughly complete* (*closed*, *classical*) if for any consistent state s of S there is a complete (resp. closed, classical) D-model (S, R, $\sqsubseteq$, φ) in which $s \in R$. It is trivial that any E-model is thoroughly closed since we may let $R = \{r \in S: r \sqsubseteq s\}$.

Perhaps somewhat surprisingly:

Theorem 13 Any E-model is thoroughly complete.

Proof Say that a subset R_0 of S satisfies the *finite consistency condition* (*fcc*) if any finite fusion of members of R_0 is consistent. We may then successively extend R_0 to a complete subset of S (satisfying the other conditions) by taking each element of S in turn and adding it when, and only when, fcc is preserved. (The construction is exactly analogous to the construction of an ultrafilter from a set of elements in a Boolean algebra with the finite intersection property).

By contrast, in order for an E-model $\mathcal{M} = (S, \sqsubseteq, \varphi)$ to be thoroughly classical, each of its consistent states must be part of a classical world; and there is no general guarantee that the required classical worlds will exist.

We turn to the truth-theoretic definition of consequence. Given an E-model $\mathcal{M} = (S, R, \sqsubseteq, \varphi)$, say that A is *true in* $\mathcal{M}$ - $\models_{\mathcal{M}}$A - if r ||- A for some $r \sqsubseteq R$ (A is exactly verified by some fact). Note that as long as R satisfies Part, this condition will be equivalent to saying that A is *inexactly* verified by some fact. Thus in the present context, there is no need to distinguish between exact and inexact verification.

Given a class X of D-models, say that C is a *consequence of* Δ *relative to* X - $\Delta \models_X$ C - if C is true in any model of X in which Δ is true. We may now extend our previous result on the 'robustness' of the notion of intuitionistic consequence:

Theorem 14 $\Delta \models_X$ C is intuitionistic consequence for X the class of D-models, for X the class of closed D-models, and for X the class of complete D-models.

Proof Clearly, if $\Delta \models_I$ C then $\Delta \models_X$ C for any X. Also $\Delta \models_X$ C implies $\Delta \models_{X'}$ C for $X' \subseteq X$; and so it suffices to show that $\Delta \models_X$ C implies $\Delta \models_I$ C for X the class of closed D-models and X the class of complete D-models.

So suppose that not $\Delta \models_I$ C. Then from theorem 12, it follows that in some E-model $\mathcal{M} = (S, \sqsubseteq, \varphi)$ and for some element s of $\mathcal{M}$, s ||> Δ while not s ||> C. But we may now set $R = \{r \in S: r \sqsubseteq s\}$ to obtain a closed model (S, R, $\sqsubseteq$, φ) in which Δ is true (since s ||> Δ) but C is not true (since not r ||- C for each $r \sqsubseteq s$).

It is a little more work to show that not $\Delta \models_X$ C for X the class of complete models. We appeal to the particular tree model $\mathcal{M}_t \models (S_t, \sqsubseteq_t, \varphi_t)$ defined above. We know that for some such model with root s_0, $s_0 \models \Delta$ while not $s_0 \models$ C and so, in the corresponding E-model $(\mathcal{M}_t)_E$, s_0 ||> Δ while not s_0 ||> C. We now set $R = \{\boldsymbol{s}: \boldsymbol{s}$ is a sequence of s_0's$\}$. Adding R to $(\mathcal{M}_t)_E$, it is then readily verified that the resulting model is a complete D-model.

With X the class of classical models, things are quite different. First, we have that instances of Excluded Middle are verified at a classical world:

Lemma 15 Let w be a classical world of the E-model $\mathcal{M} = (S, \sqsubseteq, \varphi)$. Then w ||> (p ∨ ¬p) for each sentence letter p.

Proof Suppose not w ||> (p ∨ ¬p). Then not w ||> p and not w ||> ¬p and so, for some consistent $v \sqsupseteq w$, v ||> p. But then v is compatible with w and yet not a part of w.

We now have:

Theorem 16 $\Delta \models_X C$ is classical consequence for X the class of classical D-models.

Proof From the previous lemma and the fact that C will be a classical consequence of Δ if it is an intuitionistic consequence of Δ plus all instances $(p \vee \neg p)$ of Excluded Middle.

Still, there is a way in which a classical conception of reality is still compatible with the endorsement of intuitionistic logic. Say that a formula A is *degenerately valid* relative to the class X of E-models if A is inexactly verified (and hence exactly verified) by the null state of each model of X. Then:

Theorem 17 A is intuitionistically valid iff it is degenerately valid relative to the class of all thoroughly classical E-models.

Proof The direction from left to right is straightforward. Now suppose A is not intuitionistically valid. By the finite model property for intuitionistic sentential logic, it follows that in some finite K-model $\mathcal{M} = (S, \sqsubseteq, \varphi)$ with root s_0, not $s_0 \models A$; and it may be shown that $\mathcal{M}$ can also be assumed to be tree-like (modify the construction of $\mathcal{M}_t$ by requiring the elements in the sequences $s_0 s_1 s_2 \ldots s_n$ to be distinct). In the corresponding E-model $(\mathcal{M}_t)_E$, $[s_0]$ is the null state and not $[s_0] \Vdash\!> A$; and given the finitude of S, it is readily shown that $(\mathcal{M}_t)_E$ is thoroughly classical.

I wish now to establish some results which bear on the question of how the exact verification of a statement might be seen to arise from the intrinsic content of the verifying state. To this end, we might identify the content of a state with the set of formulas either exactly verified or inexactly verified by the state. We want it to be apparent from the logical form of these formulas that they are prime, i.e. that they entail a disjunction only if they entail a disjunct, and are thereby suitable as the content of a state. We also want it to be apparent from the logical form of these formulas, along with initial information about the verifying states of the atoms, that the state will verify the formulas that it does.

Let us deal first with primal issues. A formula is said to be *non-disjunctive* if it is formed without the help of ∨, i.e from ∧, ⊃ and ⊥, and a set of formulas Δ is said to be *non-disjunctive* if its members are non-disjunctive.

Lemma 18 Any non-disjunctive formula A is provably equivalent (within intuitionistic logic) to a conjunction of formulas of the form $D \supset \alpha$ for some non-disjunctive formula D and atom α that appears in A.

Proof By induction on A.

A = α A is then equivalent to $(\alpha \supset \alpha) \supset \alpha$.

A = (B ∧ C) By IH on B and C.

A = (B ⊃ C) By IH, C is equivalent to a conjunction of formulas of the form $D \supset \alpha$ for D non-disjunctive and α an atom appearing in C; and so $B \supset C$ is equivalent to a conjunction of formulas of the form $B \supset (D \supset \alpha)$, with α appearing in A and B non-disjunctive. But $B \supset (D \supset \alpha)$ is equivalent to $((B \wedge D) \supset \alpha)$.

The set of formulas Δ is said to be *prime* if, for any formulas A and B, $\Delta \vdash B$ or $\Delta \vdash C$ (within intuitionistic logic) whenever $\Delta \vdash B \vee C$.

Lemma 19 Any non-disjunctive set of formulas Δ is prime.

Proof By the previous lemma, we can assume that each formula of Δ is of the form $D \supset \alpha$ for α an atom. Suppose that not Δ |- B and not Δ |- C (to show not Δ |- B $\vee$ C). By the completeness theorem for intuitionistic logic, there are K-models $\mathcal{M}_1 = (S_1, \sqsubseteq_1, \varphi_1)$ and $\mathcal{M}_2 = (S_2, \sqsubseteq_2, \varphi_2)$ with respective root s_1 and s_2, with Δ true at s_1 in $\mathcal{M}_1$ and at s_2 in $\mathcal{M}_2$ but with B not true at s_1 in $\mathcal{M}_1$ and C not true at s_2 in $\mathcal{M}_2$. Clearly, we may suppose that S_1 and S_2 are disjoint. Choose a element s_o in neither S_1 nor S_2 and define the model $\mathcal{M} = (S, \sqsubseteq, \varphi)$ by:

(i) $S = \{s_o\} \cup S_1 \cup S_2$
(ii) $\sqsubseteq_1 = \{(s_o, s): s \in S\} \cup \sqsubseteq_1 \cup \sqsubseteq_2$
(iii) $\varphi = \{(s_o, \text{p}): \varphi_1 s_1\text{p} \text{ and } \varphi_2 s_2\text{p}\} \cup \varphi_1 \cup \varphi_2$.

It is easy to show with the help of the Hereditary Condition that B $\vee$ C is not true at s_o in $\mathcal{M}$. Suppose for reductio that Δ is not true at s_o in $\mathcal{M}$. So some formula D $\supset \alpha$ of Δ is not true at s_o in $\mathcal{M}$. It is readily shown that D is true at s_o in $\mathcal{M}$ while α is not. By the Hereditary Condition, D is true at s_1 in $\mathcal{M}_1$ and at s_2 in $\mathcal{M}_2$; and since D $\supset \alpha$ is true at s_1 in $\mathcal{M}_1$ and at s_2 in $\mathcal{M}_2$, it follows that α is true at s_1 in $\mathcal{M}_1$ and at s_2 in $\mathcal{M}_2$. So, clearly, α is not the falsum constant $\perp$ but a sentence letter p and so, by the definition of $\mathcal{M}$, φs_op and p is true at s_o in $\mathcal{M}$ after all.

As is well known, this result can also be established on the basis of the normalization theorem.

Let us now deal with the issue of content. To this end, we shall appeal to an extension of intuitionistic logic, which we might call *hybrid* intuitionistic logic (HH) after the corresponding nomenclature for modal logics with world-constants. The language of HH is obtained from that for intuitionistic logic by adding an arbitrary (finite or infinite) number of *state constants* $\underline{s}_1$, $\underline{s}_2$, ... We shall think of each state constant $\underline{s}_k$ as designating a particular state s_k in the sense that s_k is the sole verifier of $\underline{s}_k$. Call a formula of HH *definite* if it is constructed from the state constants $\underline{s}_1$, $\underline{s}_2$, ... by means of the connectives $\wedge$ and $\supset$. It is by means of the definite formulas that we shall make explicit the content of each state.

The logic of HH is obtained from minimal logic by adding the axiom scheme:

Definiteness $[(D \supset (B \vee C)] \supset [(D \supset B) \vee (D \supset C)]$, for any formulas B and C of the extended language and any definite formula D.

and by adding $\perp \supset$ p for each sentence letter p (however, we do not add $\perp \supset \underline{s}$ for any of the state constants $\underline{s}$). Using Lemma 18 above, we can show that it suffices to restrict D to definite formulas of the form $(D' \supset \underline{s})$.

We may show that definite formulas are like the state constants in designating a single state. Let us say that $\mathcal{M} = (S, \sqsubseteq, \varphi)$ is an E-model for the language of HH if $(S, \sqsubseteq)$ is an E-frame and if, in addition, we have:

Falsum Condition $\varphi s\perp$ implies $\varphi s'$p for some $s' \sqsubseteq s$; and

Statehood for each state constant $\underline{s}_k$ there is a state s_k such that $\varphi s\underline{s}_k$ iff $s = s_k$.

Note that the state constants are *not* subject to the Falsum Condition.

Given an E-model $\mathcal{M} = (S, \sqsubseteq, \varphi)$ for HH and a definite formula D, we use (D)*, or *d*, for the *corresponding* state of $\mathcal{M}$. Thus:

(i) $(\underline{s}_k)^* = s_k$;
(ii) $(D_1 \wedge D_2)^* = (D_1)^* \sqcup (D_2)^*$;
(iii) $(D_1 \supset D_2)^* = (D_1)^* \rightarrow (D_2)^*$.

Thus $\underline{s}_k$ in the formula D is, in effect, replaced with s_k, $\wedge$ with $\sqcup$, and $\supset$ with $\rightarrow$.

<u>Lemma 20 (Uniqueness)</u> Given any E-model $\mathcal{M} = (S, \sqsubseteq, \varphi)$ for HH and definite formula D, d = (D)* is the sole exact verifier of D.
<u>Proof</u> By a straightforward induction on D.

E-models for HH verify Definiteness and hence HH is sound with respect to the class of such models:
<u>Lemma 21</u> $[(D \supset (B \vee C)] \supset [(D \supset B) \vee (D \supset C)]$, for D a definite formula, is true at the root of any E-model $\mathcal{M} = (S, \sqsubseteq, \varphi)$.
<u>Proof</u> By the Uniqueness Lemma, d is the sole verifier of D in $\mathcal{M}$. Thus any verifier of $(D \supset (B \vee C)$ is of the form $d \to s$, where s is a verifier of B or of C. In either case, it is clear that $d \to s$ is also a verifier of $(D \supset B) \vee (D \supset C)$. So we may associate each $d \to s$ with itself and the sum of all the $(d \to s) \to (d \to s)$ will be the null state.

We use the state constants to indicate which states verify a given atom (p_k or $\perp$). Accordingly, we take a *state assignment* to be a set of formulas of the form $\alpha \equiv (A_1 \vee A_2 \vee ... \vee A_n)$, $n \geq 1$, one for each atom α, where $A_1, A_2, ..., A_n$ are definite formulas. Our main interest will be in when the formulas $A_1, A_2, ..., A_n$ are themselves state constants. A state assignment can be extended from atoms to all formulas.
<u>Lemma 22 (Assignment)</u> Let Δ be a state assignment. Then for each formula A of HH, there are definite formulas $A_1, A_2, ..., A_n$, $n \geq 1$, such that $A \equiv (A_1 \vee A_2 \vee ... \vee A_n)$ is derivable from Δ within HH.
<u>Proof</u> By induction on A. The result is obvious when A is an atom. There are three other cases:
<u>$A = (B \wedge C)$</u> By IH, $B \equiv (B_1 \vee B_2 \vee ... \vee B_k)$ and $C \equiv (C_1 \vee C_2 \vee ... \vee C_l)$ are derivable (with $k, l \geq 1$ and definite formulas on the right). But then $(B \wedge C) \equiv (B_1 \vee B_2 \vee ... \vee B_k) \wedge (C_1 \vee C_2 \vee ... \vee C_l)$ is derivable and we may use the Distributive Law to put the right hand side in the required form.
<u>$A = (B \vee C)$</u> Straightforward.
<u>$A = (B \supset C)$</u> By IH, $B \equiv (B_1 \vee B_2 \vee ... \vee B_k)$ is derivable (with $k \geq 1$ and definite formulas on the right). But $(B_1 \vee B_2 \vee ... \vee B_k) \supset C$ is provably equivalent within minimal logic to $(B_1 \supset C) \wedge (B_2 \supset C) \wedge ... \wedge (B_k \supset C)$ and so, by applying Distributivity, it suffices to consider the case in which $k = 1$. By IH again, $C \equiv (C_1 \vee C_2 \vee ... \vee C_l)$ is derivable (with $l \geq 1$ and definite formulas on the right). Using the Definiteness axiom, it follows that $[B_1 \supset (C_1 \vee C_2 \vee ... \vee C_l)] \equiv [(B_1 \supset C_1) \vee (B_2 \supset C_2) \vee ... \vee (B_l \supset C_l)]$ is derivable, where the right hand side is of the required form.

Suppose that $\mathcal{M} = (S, \sqsubseteq, \varphi)$ is a regular E-model in which S is a finite set $\{s_1, s_2, ..., s_n\}$ containing at least one contradictory state. We extend $\mathcal{M}$ to an E-model $\mathcal{M}^+ = (S, \sqsubseteq, \varphi^+)$ for HH by adding n new state constants $\underline{s}_1, \underline{s}_2, ..., \underline{s}_n$ to the original language and letting $\varphi^+ = \varphi \cup \{(s_k, \underline{s}_k): k = 1, 2, ..., n\}$. Let $\Delta_{\mathcal{M}}$ be the state assignment that, for each atom α of the original language, contains the formula $\alpha \equiv (\underline{s}_{k1} \vee \underline{s}_{k2} \vee ... \vee \underline{s}_{km})$, $m \geq 1$, where $s_{k1}, s_{k2}, ..., s_{km}$ are the states that exactly verify α in $\mathcal{M}$.

Putting together the previous results, we are in a position to show how each formula verified by a state in $\mathcal{M}^+$ can be seen to be verified by the state on the basis of its content:
<u>Theorem 23 (Constitution)</u> Let $\mathcal{M}^+$ be an E-model for HH as previously defined. Suppose that the formula A from the unextended language is inexactly verified by a state s of $\mathcal{M}$. Then A is

derivable within HH from $\Delta_{\mathcal{M}} \cup$ {D: D is a definite formula inexactly verified by *s* in $\mathcal{M}^+$}.
Proof Suppose A is inexactly verified by *s* in $\mathcal{M}$. By the Assignment Lemma, there are definite formulas $D_1, D_2, \ldots, D_m$, $m \geq 1$, such that $A \equiv (D_1 \vee D_2 \vee \ldots \vee D_m)$ is derivable from $\Delta_{\mathcal{M}}$ within HH. It should be evident that the formulas of $\Delta_{\mathcal{M}}$ are true at the root of $\mathcal{M}^+$. So by lemma 22, $A \equiv (D_1 \vee D_2 \vee \ldots \vee D_m)$ is also true at the root. Given that A is inexactly verified by *s*, some D_j is inexactly verified by *s*. But now from $A \equiv (D_1 \vee D_2 \vee \ldots \vee D_m)$ and D_j we can derive A.

Some comments on this result are in order:

(1) It is important to the significance (and non-triviality) of the result that the underlying formulas D should be definite, since otherwise there is no guarantee that they are prime and hence can legitimately be taken to correspond to a verifying state.

(2) The result has only been established for finite E-models. In order to extend it to infinite E-models (which would, in any case, be required in application to intuitionistic predicate logic), we would need to employ infinitary means to construct the underlying definite formulas.

(3) It is unfortunate that we have had to resort to minimal logic in order to be able to differentiate the different states at which a contradiction may be true. I do not know if there is any reasonable way in which this partial appeal to minimal logic might be avoided.

(4) It would be nice to be able to establish a version of Constitution Theorem with exact verification in place of in exact verification. Thus when A is *exactly* verified by a state *s*, it will be required that A is derivable from $\Delta_{\mathcal{M}} \cup$ {D: D is a definite formula *exactly* verified by *s*}. There is a difficulty in establishing this result under the present semantics. For we will wish to strengthen the Assignment Lemma and show that A will be exactly equivalent to $(A_1 \vee A_2 \vee \ldots \vee A_n)$ (and thereby have the same exact verifiers) whenever the same is true of the pairs of equivalent formulas under the assignment Δ. But when we examine the proof of the lemma, we see that this requires that $(A \vee A) \supset C$, for example, should be exactly equivalent to $(A \supset C) \wedge (A \supset C)$. But suppose that A has the one verifier *s* and C the two verifiers *t* and *u*. Then $(s \to t) \sqcup (s \to u)$ will be a verifier of $(A \supset C) \wedge (A \supset C)$ but not (as a rule) of $(A \supset C)$. There is a similar difficulty with the Definiteness Axioms.

We may overcome these difficulties by a modification to the semantics which is independently well-motivated. For we may take the verifiers of a formula to constitute a multi-set. Thus if [*s*] is the multi-set of verifiers for A, [*s*, *s*] will be the multiset of verifiers for (A ∨ A) (*s* will verify A 'twice', from the left and from the right). Similarly, the verifiers of (A ⊃ C) should be taken to encode a function from the multiset of verifiers for A into the multiset of verifiers for C. Thus if *s* verifies A twice we can associate *s* twice over with a verifier of C and if *t* verifies C twice, we can employ it twice in forming verifiers of (A ⊃ C) given a verifier of A.

The above version of the condition-oriented semantics constitutes a closer approximation to the construction-oriented semantics, since it permits us to 'select' a left and right verifier for (A ∨ A), though without distinguishing one as left and the other as right; and perhaps we can think of there being a gradual transition in this way from the one form of the semantics to the other.

We shall not go into details but the above considerations and results can be extended to quantificational intuitionistic logic. To this end, the language should be enriched with an existence predicate E in addition to the quantifiers and an E-model should be equipped with a domain *I* of possible individuals. The clause for the universal quantifier then takes the form:

$s \Vdash \forall x A(x)$ if there is a function taking each individual $i \in I$ and each state $t \Vdash E[i]$ into a $u_{i,t} \Vdash A[i]$ for which $s = \bigsqcup(t \to u_{i,t})$.

The universal quantification $\forall x A(x)$ is treated as equivalent, in effect, to the conjunction of the $Ei \supset A(i)$ for each possible individual *i*. However, this clause is not altogether satisfactory from a philosophical point of view; and it would be preferable to have a clause that could be stated without appeal to the full range of possible individuals.

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