Set Theory (F01): Assignment on well-orders

Starred exercises count towards ‘base-line credit’ only.

*1. Which of the following is true and which false? Give reasons.
(i) if \( A = (A, \leq) \) is a well-order, then so is \( A^* = (A, \geq) \);
(ii) if \( A = (A, \leq) \) is a well-order and \( B \) is a subset of \( A \), then \( A \uparrow B \) is a well-order;
(iii) if \( A = (A, \leq) \) is an order and \( A \uparrow B \) is a well-order for some subset \( B \) of \( A \), then \( A \) is a well-order;
(iv) if \( A = (A, \leq) \) and \( A^* = (A, \geq) \) are both well-orders, then \( A \) is finite.

2*. (i) Given a well-order \( A = (A, \leq) \), when is an element \( a \) of \( A \) a limit point of \( A \)?
(ii) Given an example of a well-order that contains no limit points. One limit point. Two limit points. \( n \) limit points for any natural number \( n \). \( \mathbb{N}_0 \) limit points.
(iii) EXCISED
(iv) Show that if \( A = (A, \leq) \) has a limit point then \( A^* = (A, \geq) \) is not a well-order. (It therefore follows, given a well-order \( A = (A, \leq) \), that \( A^* = (A, \leq) \) is also a well-order iff \( A \) contains no limit points.)

3. Let \( A = (A, \leq) \) be any order. Show that \( A \) is a well-order if \( A_a \) is a well-order for any element \( a \) of \( A \).

4. (i) Show, using the comparison theorem, that \((\mathbb{N}, \leq) \) (\( \mathbb{N} \) the set of natural numbers) is similar to an initial segment of any infinite well-order.
(ii) Show, again using the comparison theorem, that any other infinite well-order with this property (being similar to an initial segment of any infinite well-order) is similar to \((\mathbb{N}, \leq) \).

5. Show by transfinite induction on \( a \) that any well-order \( A_a \), for \( a \) in \( A \), is of the form \( B + C \) where \( B \) is a well-order without a last element and \( C \) is a finite well-order. (Since every well-order is of the form \( A_a \), it follows that every well-order can be put in the form \( B + C \).)