Set Theory (F01): Assignment on well-orders

Starred exercises count towards 'base-line credit' only.

*1. Which of the following is true and which false? Give reasons.

(i) if $\underline{A} = (A, \leq)$ is a well-order, then so is $\underline{A}^* = (A, \geq)$;

(ii) if $\underline{A} = (A, \leq)$ is a well-order and B is a subset of A, then $\underline{A} \restriction B$ is a well-order;

(iii) if $\underline{A} = (A, \leq)$ is an order and $\underline{A} \restriction B$ is a well-order for some subset B of A, then $\underline{A}$ is a well-order;

(iv) if $\underline{A} = (A, \leq)$ and $\underline{A}^* = (A, \geq)$ are both well-orders, then A is finite.

2*. (i) Given a well-order $\underline{A} = (A, \leq)$, when is an element *a* of A a limit point of $\underline{A}$?

(ii) Given an example of a well-order that contains no limit points. One limit point. Two limit points. n limit points for any natural number n. $\aleph_0$ limit points.

(iii) EXCISED

(iv) Show that if $\underline{A} = (A, \leq)$ has a limit point then $\underline{A}^* = (A, \geq)$ is not a well-order. (It therefore follows, given a well-order $\underline{A} = (A, \leq)$, that $\underline{A}^* = (A, \leq)$ is also a well-order iff $\underline{A}$ contains no limit points.)

3. Let $\underline{A} = (A, \leq)$ be any order. Show that $\underline{A}$ is a well-order if $\underline{A}_a$ is a well-order for any element a of A.

4. (i) Show, using the comparison theorem, that $(\mathbb{N}, \leq)$ ($\mathbb{N}$ the set of natural numbers) is similar to an initial segment of any infinite well-order.

(ii) Show, again using the comparison theorem, that any other infinite well-order with this property (being similar to an initial segment of any infinite well-order) is similar to $(\mathbb{N}, \leq)$.

5. Show by transfinite induction on a that any well-order $\underline{A}_a$, for a in A, is of the form $\underline{B} + \underline{C}$ where $\underline{B}$ is a well-order without a last element and $\underline{C}$ is a finite well-order. (Since every well-order is of the form $\underline{A}_a$, it follows that every well-order can be put in the form $\underline{B} + \underline{C}$).