Question 3. Let $P = \{p_1, p_2, ... \}$ be a (possibly infinite) group of people and suppose that certain finite non-empty subgroups $Q_1, Q_2, ...$ of P are 'incompatible'. Call a subgroup P' of P *harmonious* if it contains no incompatible subgroup.

(i) Show that there is a maximally harmonious subgroup of P. (Hint: follow the proof strategy of Lindenbaum's Lemma);

(ii) Show that there may be no *unique* maximally harmonious subgroup (Hint: suppose that the only incompatible subgroup is $\{p_1, p_2\}$);

(iii) Show that result under (i) still holds when the incompatible subgroups are allowed to be infinite but there are only finitely many of them. (Hint: look at a smallest subgroup of P that overlaps with each of the incompatible subgroups; and let the maximally harmonious subgroup be its complement.)

Answer to 3(iii).

Suppose that the incompatible subgroups are $Q_1, Q_2, ..., Q_n$. Since the subgroups are non-empty, we may choose an object $a_1, a_2, ..., a_n$ from each. Let $Q = \{a_1, a_2, ..., a_n\}$. Then Q overlaps with each of $Q_1, Q_2, ..., Q_n$. It follows that there is a (set-theoretically) smallest subset of P that overlaps with each of $Q_1, Q_2, ..., Q_n$. For if Q is not such a subset, there is a smaller subset Q' of Q that overlaps with each of $Q_1, Q_2, ..., Q_n$. If Q' is not a smallest subset of this sort, there is a smaller subset Q'' still; and so on. Since Q is finite, this process cannot continue indefinitely; and so eventually we arrive at a smallest subset Q* of the required sort.

Let P' = P - Q*. Then P' is harmonious for, since each Q_i, i = 1, 2, ..., n, overlaps Q*, it is not a subset of P - Q*. P' is also maximally harmonious. For suppose $P' \subsetneq P'' \subseteq P$. Then P'' contains at least one object a from P - P'. But one of $Q^* \cap Q_1, Q^* \cap Q_2, ..., Q^* \cap Q_n$ is {a} since otherwise, each Q_i, for i = 1, 2, ..., n, contains an object b_i distinct from a_i and so $\{b_1, b_2, ..., b_n\}$ is a smaller set than Q* that overlaps with each of $Q_1, Q_2, ..., Q_n$.