The Existence of Nonstandard Models for Arithmetic

An *arithmetical* model N is one whose domain is the set of natural numbers and for which there is, for each natural number n, a formula $A_n(x)$ such that N |= $A_n(x)[m/x]$ iff m = n. Recall that Th_N = {A: N|= A}. A model M for Th_N is said to be *nonstandard* if there is an element *a* in the domain of M for which M=|$A_n(x)[a/x]$ for any natural number n; and an element *a* of this sort is itself said to be *nonstandard*.

Theorem There exists a nonstandard countably infinite model of Th_N (whatever the arithmetical model N or formulas $A_n(x)$).

Proof Let $\Gamma = Th_N \cup \{\sim A_0(x), \sim A_1(x), ...\}$. Then Γ is finitely satisfiable. For take any finite subset Γ' of Γ. Clearly, it is included in $Th_N \cup \{\sim A_0(x), \sim A_1(x), ..., \sim A_n(x)\}$ for some n. But then N|=Th_N and N|=$\sim A_k(x)[(n+1)/x]$ for k = 0, 1, ..., n and Γ' is satisfiable.

It follows by compactness that Γ is satisfiable. So by Skolem-Lowenheim, Γ is true in a countably infinite interpretation (M, δ). But M is then a nonstandard model of Th_N with nonstandard element δ(x).