

Chapter 8. Arbitrary truth-functional languages

For the sake of concreteness the discussion in this chapter has so far been confined to a single formal language \mathcal{L} . It should be clear, however, that all of the concepts and results have a more general application; they apply to languages generated from different sets of constituents. We might take the constituents to be subscripted \mathbf{q} 's instead of subscripted \mathbf{p} 's, for example, or to be a mixture of the two. We call a language that--apart from its truth-functional constituents--is the same as \mathcal{L} , a truth-functional language, and let ' \mathcal{L}_C ' denote the truth functional language whose constituents are the members of C . It is convenient to regard the languages that will be introduced in subsequent chapters as just such languages. The truth-functional constituents of the traditional propositional modal logics, on this view, include expressions like $\Box\Box(\mathbf{p}\vee\neg\mathbf{q})$. Those of classical predicate logic include expressions like $\hat{x}\mathbf{Fxx}$; those of modal predicate logic, $\Box\hat{x}\Box\mathbf{Fxx}$. Care must be taken that, when the notion of truth-functional constituent is modified in this way, formulas still have a unique reading in the sense of theorem 2.3 and that the problem **is x a formula?** is still decidable. But as long as these conditions are met the notions of truth-value under a valuation, truth-functional validity and consequence, and provability and derivability in **SL** will still be sensible and the proofs of soundness, completeness and decidability will still be correct. For the languages introduced in subsequent chapters it will be clear that these conditions are met and so the concepts and results of this chapter may be cited without further ado. The question of what general properties of constituents might guarantee that the conditions are met is taken up in the problems.

The languages of Part II, in addition to being of the same kind as \mathcal{L} , will also contain \mathcal{L} as a part. In this case the truth-functionally valid formulas and argument-pairs are just the substitution instances of those of \mathcal{L} . More precisely, \mathcal{L}_{C_2} is said to be an extension of \mathcal{L}_{C_1} if $C_1 \subseteq C_2$. Then the following result obtains.

Theorem *a.* Suppose \mathcal{L}_{C_2} is an extension of \mathcal{L}_{C_1} . (i) If \mathbf{A} is a truth-functional consequence of Γ in \mathcal{L}_{C_1} and Γ', \mathbf{A}' is the result of a uniform substitution of constituents of \mathcal{L}_{C_2} for constituents of \mathcal{L}_{C_1} in Γ, \mathbf{A} then \mathbf{A}' is a truth-functional consequence of Γ' in \mathcal{L}_{C_2} . (ii) If there are as many constituents in C_1 as C_2 and \mathbf{A}' is a truth-functional consequence of Γ' in \mathcal{L}_{C_2} then there is some argument pair Γ, \mathbf{A} such that Γ', \mathbf{A}' is the result of a uniform substitution of constituents of \mathcal{L}_{C_2} for constituents of

\mathcal{L}_{C_1} in Γ, A and A is a truth-functional consequence of Γ in \mathcal{L}_{C_1} .

Proof. Note first that if \mathcal{L}_{C_2} is an extension of \mathcal{L}_{C_1} then truth under an interpretation is independent of language, i.e., if $\alpha \subseteq C_1$ and A is a formula of \mathcal{L}_{C_1} , then $\alpha \models A$ in \mathcal{L}_{C_1} iff $\alpha \models A$ in \mathcal{L}_{C_2} . (If \mathcal{L}_{C_2} is not an extension of \mathcal{L}_{C_1} , this may not be the case. See exercise 1 below.) To prove part (i) suppose that: the argument pair Γ, A is truth-functionally valid in \mathcal{L}_{C_1} ; Γ', A' is obtained from Γ, A by uniformly substituting constituents D_i' of \mathcal{L}_{C_2} for the constituents D_i of the latter argument-pair ($i=1,2,\dots$); but Γ', A' is not truth-functionally valid in \mathcal{L}_{C_2} . Then there is some valuation $\alpha \subseteq C_2$ such that $\alpha \models \Gamma'$ and $\alpha \not\models A'$. Let β be the set of all the D_i such that D_i' is a constituent of Γ', A' and $D_i' \in \alpha$. By theorem 2.3, β assigns the same truth values to the members of Γ, A as α assigns to the members of Γ', A' . This contradicts the supposition that Γ, A is truth-functionally valid in \mathcal{L}_{C_1} , thus completing the proof. To prove part (ii) suppose A' is truth functional consequence of Γ' in \mathcal{L}_{C_2} . Let C_1', C_2', \dots be an enumeration (without repetitions) of the constituents of \mathcal{L}_{C_2} and let C_1, C_2, \dots be a similar enumeration of the constituents of \mathcal{L}_{C_1} . For any formula B' of \mathcal{L}_{C_2} , let B be result of replacing each occurrence of a constituent C_i' in B' by the corresponding C_i . Let Γ be the set of all B such that B' is a member of Γ' . Clearly Γ', A' is obtainable from Γ, A by a uniform substitution of constituents and A and the members of Γ are formulas of \mathcal{L}_{C_1} . It remains only to verify that Γ, A is truth functionally valid in \mathcal{L}_{C_1} . Suppose $\alpha \models \Gamma$. Let $\beta = \{C_i' : C_i \in \alpha\}$. By theorem 2.3, $\beta \models \Gamma'$. Since A' is a truth functional consequence of Γ' in \mathcal{L}_{C_2} , $\beta \models A'$. By theorem 2.3 again $\alpha \models A$. So Γ, A is truth functionally valid in \mathcal{L}_{C_1} and the proof in this direction is complete.

A corollary to this theorem is that an extension of a truth functional language preserves the relation of truth-functional consequence (and therefore the property of truth-functional validity) of the original language. More precisely:

Corollary. If \mathcal{L}_{C_2} is an extension of \mathcal{L}_{C_1} and Γ, A is an argument pair of \mathcal{L}_{C_1} then A is a truth-functional consequence of Γ in \mathcal{L}_{C_1} iff A is a truth-functional consequence of Γ in \mathcal{L}_{C_2} .

Proof. Γ, A is a substitution instance of itself, so if Γ, A is truth-functionally valid in \mathcal{L}_{C_1} then the theorem implies it is valid in \mathcal{L}_{C_2} . To prove the other direction suppose Γ, A is an argument pair of \mathcal{L}_{C_1} that is truth-functionally valid in \mathcal{L}_{C_2} .

By the theorem, Γ, \mathbf{A} is a substitution instance of a (possibly different) argument-pair Δ, \mathbf{B} that is truth-functionally valid in \mathcal{L}_{C_1} . But, since \mathcal{L}_{C_1} is an extension of itself, another application of the theorem shows Γ, \mathbf{A} is truth-functionally valid.

This completes the proof.

As long as we deal with extensions of \mathcal{L} , therefore, truth-functional consequence between \mathcal{L} -formulas and truth functional validity of \mathcal{L} -formulas are language-independent notions.

Exercises and problems

1[e]. Describe two truth functional languages \mathcal{L}_1 and \mathcal{L}_2 , an expression \mathbf{A} and a set α such that: each language has the unique readability property, both languages contain \mathbf{A} as a formula, α is a valuation for both languages, and $\alpha \models \mathbf{A}$ in \mathcal{L}_1 but $\alpha \not\models \mathbf{A}$ in \mathcal{L}_2 . (Hint: Make \mathbf{A} a constituent of one language, but not the other.) Explain why this cannot happen if \mathcal{L}_2 is an extension of \mathcal{L}_1 .

2[e]. (i) State and prove a generalization of theorem 2.3 that permits constituents of one language to be replaced by constituents of another language. (ii) Using the result proved in (i), show that the assumption that \mathcal{L}_{C_2} is an extension of \mathcal{L}_{C_1} can be dropped in theorem 8.1.

3[p]. (General readability). Let C be an arbitrary set of constituents. (Note that any sequence of "symbols" can be a member of C .) Suppose C conforms to the following three conditions:

- (i) no constituent begins with $(^n \neg$, for $(^n$ an arbitrary sequence of n ('s, $n \geq 0$;
- (ii) no constituent properly begins with $(^n C$, for C a constituent;
- (iii) no constituent is empty or of the form $((^n D$ where D begins a constituent.

Let the grade of a formula \mathbf{A} of \mathcal{L}_C be the least number of steps required to generate \mathbf{A} . Show by induction on the sum of the grades of the formulas \mathbf{A} and \mathbf{B} that no formula \mathbf{A} properly begins a formula \mathbf{B} . Then deduce the unique readability theorem for \mathcal{L}_C . Although this result is not completely general, it is general enough to cover all of the different kinds of truth-functional language that are usually considered.