Chapter 7. Decidability

A general problem is one of the form **is** x **a** P**?**, where x is a variable that ranges over a certain class of entities and P is a predicate defined on that class. The following are examples of general problems: is the sequence w of Roman letters a word of English?; is the natural number n a prime number?; is the sequence of symbols E a formula of $\mathfrak{L}$?; is the formula **A** a theorem of **SL**?. An instance of a general problem is obtained by specifying a value for the variable 'x'. The following are instances of the previous general problems: is **banana** a word of English?; is 57 a prime?; is $\neg(\mathbf{p}_1 \vee \mathbf{p}_2)$ a formula of $\mathfrak{L}$**?**; is $\neg(\mathbf{p}_1 \vee \mathbf{p}_2) \vee \mathbf{p}_1$ a theorem of **SL**?.

A general problem is decidable if there is an effective procedure for solving each instance of the problem. We shall not attempt an exact definition of effective procedure, but let us note the following features: an effective procedure consists of a finite set of instructions; the instructions can be applied in a mechanical manner; and they always terminate in a yes or no answer in a finite number of steps. Returning to our examples, we see that the second problem, **is** n **a prime?** is decidable. For an answer can be determined by considering each of the numbers between one and n in turn and checking whether it divides n. The decidability of the third problem, **is** E **a formula?** is left as a problem.

Our concern in this chapter is to outline an argument showing that the last of the problems, **is A a theorem?**, is decidable. First, note that by the completeness theorem it suffices to establish that the problem **is A valid?** is decidable. Second, note that the truth-value of a formula **A** in a valuation depends only upon the truth functional constituents of **A**. (See corollary to theorem 2.3.) It is now easy to describe the procedure for determining whether or not a formula **A** is valid. List all of the valuations that only contain sentence letters occurring in **A**. Clearly there are only a finite number of such valuations (2^n if **A** contains n sentence letters). Now consider each valuation in turn and determine whether or not **A** is true in that valuation. If **A** is true in each of the valuations, it is valid, otherwise it is not valid.

This is just the method of truth tables, which is presented in many introductory logic texts. These matters are taken up in more detail in the exercises and problems.

Exercises and problems

1**[e]**. a. The argument above for the decidability of the question **is A a valid**

formula? presupposes that there is an effective procedure for listing all the valuations containing only sentence letters that occur in **A**. Describe one such procedure.
b. It also presupposes that the question **is formula A true in valuation α?** is decidable. Give an argument to show that it is.

2**[e]**. a. Given two English words $\mathbf{W_1}$ and $\mathbf{W_2}$, show that the question 'Does $\mathbf{W_1}$ alphabetically precede $\mathbf{W_2}$?' is decidable. b. Use the result of a to show that given a word-list **L** of arbitrary finite length, the question 'Is **L** in alphabetical order?' is decidable.

3**[p]**. Show that the general problem 'is the expression **E** a formula?' is decidable. Hint: perform the following procedure. At stage 1, write down the sentence letters in **E**. Assume that certain expressions are written down at stage n, n>0. Then at stage (n+1), write down any expression that appears in **E** and is the disjunction or negation of expressions written down at stage n. Can you think of more efficient procedures?

4**[p]**. (decidability of sentential logic) a. Show that the problem 'is formula **A** true in the finite valuation α?' is decidable. Hint: Determine the truth-values of successively larger subformulas of A. Note that this procedure could be combined with the procedure for determining formulahood. For given an expression **E** and a finite valuation α, one could determine the truth-values relative to α of successively larger formulas that appear in **E**. Eventually, one would conclude that **E** is either a formula verified by α, a formula falsified by α, or a non-formula.

b. By putting together problem 3, part a of this problem, and the first corollary to 2.3, give a more rigorous proof that the problem 'is expression **E** a valid formula?' is decidable.