Chapter 6. Completeness

In this chapter we show that **SL** does, indeed, axiomatize the relation of truth-functional consequence and the set of tautologies. Using our notation we can write these results as follows.

Theorem *a.* (i) $\Gamma \vdash \mathbf{A}$ iff $\Gamma \vDash \mathbf{A}$
(ii) $\vdash \mathbf{A}$ iff $\vDash \mathbf{A}$

Let us first examine part (i). It has two halves, an "if" half and an "only if" half. The "only if" half says that anything we can derive from Γ by means of our axioms and rule is a consequence of Γ. This is called soundness and if it is true for our system, the system is said to be sound. A system that lacked soundness would allow "too many" formulas to be derived.

The "if" half says that every truth-functional consequence of Γ can be derived from Γ. This is called completeness or sufficiency, and if it is true our system is said to be sufficient or complete. A system that lacked completeness would allow "too few" formulas to be derived. The word complete is often used loosely for **sound and complete**. Now let us examine part (ii). This also has two halves. The "only if" half says that every theorem is a tautology and the "if" half says that every valid formula is a theorem. Notice that (ii) is a special case of (i), namely the case in which Γ is the empty set. The "if" half of part (i) is sometimes called strong completeness, and the "if" half of part (ii), weak completeness. (This distinction is not significant in the "only if" direction. See exercise 2 below.) If we do not specify otherwise, **completeness** will mean weak completeness.

Beginners often do not appreciate that a completeness theorem is a substantial result. But this is clear once one attends to the definitions of the concepts involved. First, derivability and theoremhood are syntactic notions while consequence and validity are semantic notion. In defining derivability and theoremhood one appeals only to the formal properties of the symbols, whereas in defining validity and consequence one also appeals to something altogether outside the language, namely, an assignment of truth or falsehood to formulas. Second, derivability and theoremhood are existential notions whereas consequence and validity are universal. To say that a formula is derivable from a set is to say that there exists a certain list of formulas; to say that it is a consequence of the set is to say that every valuation either makes something in the set false or makes the formula true. A completeness result shows that two terms with very different definitions have identical extensions.

The idea of completeness can also be expressed in terms of consistency and satisfiability as the following theorem shows. Notice that the new formulation again asserts a connection between a syntactic notion and a semantic one, a universal notion (that <u>no</u> derivation leads from Γ to $\bot$) and an existential one (that <u>some</u> valuation verifies Γ).

Theorem *b.* (i) SL is strongly complete iff every consistent set of formulas is satisfiable.
(ii) SL is weakly complete iff every consistent formula is satisfiable.

<u>Proof of theorem 6.2</u> Suppose the system is strongly complete. Γ consistent implies $\Gamma \nvdash \bot$, which implies (by completeness) $\Gamma \nvDash \bot$, which means that there is an assignment of truth values that verifies everything in Γ, which means that Γ is satisfiable. Conversely, suppose that every consistent formula is satisfiable. $\Gamma \nvdash \mathbf{A}$ implies (by 5.23) $\Gamma \cup \{\neg \mathbf{A}\} \nvdash \bot$, which means that $\Gamma \cup \{\neg \mathbf{A}\}$ is consistent. By our supposition, then, $\Gamma \cup \{\neg \mathbf{A}\}$ is satisfiable, which implies $\Gamma \nvDash \mathbf{A}$. The proof of ii is similar and is left as an exercise.

Let us now prove that **SL** is sound and strongly complete. Soundness can be proved by a routine induction. Suppose $\Gamma \vdash \mathbf{A}$. The basis step includes the case in which $\mathbf{A} \in \Gamma$ as well as the cases in which A is an axiom. In the former case any valuation that makes Γ true will, <u>inter alia</u>, make **A** true, so $\Gamma \vDash \mathbf{A}$. To prove that the claim holds in the latter case it is sufficient to show that if **A** is an axiom then <u>every</u> valuation makes **A** true. We prove this for **A** an instance of **A3** and leave the remaining four subcases as exercises. If **A** is an instance of **A3** then **A** must be of the form $\neg \mathbf{B} \vee (\mathbf{B} \vee \mathbf{C})$. Consider an arbitrary valuation α. If $\alpha \vDash \neg \mathbf{B}$ then, by the clause for disjunction in the truth definition, $\alpha \vDash \mathbf{A}$. If not then $\alpha \vDash \mathbf{B}$, so $\alpha \vDash (\mathbf{B} \vee \mathbf{C})$, so again $\alpha \vDash \mathbf{A}$. So **A** is true in every valuation. To prove the inductive step we must show that modus ponens preserves the property of being a truth functional consequence of Γ. Suppose $\Gamma \vDash (\neg \mathbf{A} \vee \mathbf{B})$ and $\Gamma \vDash \mathbf{A}$. If $\alpha \vDash \Gamma$ then the second supposition implies that $\alpha \vDash \mathbf{A}$ and first supposition further implies that $\alpha \vDash \mathbf{B}$. So if $\alpha \vDash \Gamma$ then $\alpha \vDash \mathbf{B}$, i.e., $\Gamma \vDash \mathbf{B}$. Thus, modus ponens does preserve the property in question and our proof is finished.

The proof of strong completeness is more difficult. By theorem 6.2 it is sufficient to show that every consistent set is satisfiable. Given a consistent set Γ, we must try to find a valuation under which all the members of Γ are true. Call the set of all formulas that are true under a valuation α the theory of α and call a set of formulas a valuation theory if it is the theory of α for some α . Our problem, then, is to show that as long as $\bot$ is not derivable from Γ, Γ is contained in a valuation theory. It is convenient to note first that valuation theories can be characterized in purely syntactic terms.

Theorem *c.* Γ is a valuation theory iff it satisfies both of the following:

1. $\neg\mathbf{A} \in \Gamma$ iff $\mathbf{A} \notin \Gamma$
2. $(\mathbf{A}\vee\mathbf{B}) \in \Gamma$ iff $\mathbf{A}\in\Gamma$ or $\mathbf{B}\in\Gamma$

Proof. First suppose Γ is the set of all formulas true in valuation α. The truth definition guarantees that 1 and 2 hold. Now suppose Γ satisfies 1 and 2. Let α be the set of all sentence-letters in Γ. Then for all sentence-letters $\mathbf{A}$, $\mathbf{A}\in\Gamma$ iff $\mathbf{A}\in\alpha$. So membership in Γ meets the three conditions that define truth in α. It follows that Γ is the theory of α, and hence, a valuation theory. This proves theorem 6.3.

We also know that valuation theories have another property: they are consistent. For, by soundness, if $\bot$ were derivable from the theory of α, it would be true in α, which is clearly impossible. So we are trying to show that an arbitrary consistent set Γ is contained in some larger consistent set Γ^* that satisfies properties 1 and 2 above. Property 1 suggests that Γ^* will in general have to be quite large--for every formula of $\mathcal{L}$, it will have to either contain that formula itself or its negation. If it contained both, of course, it would be inconsistent; but property 1 seems to require that we construct a set as large as possible, short of inconsistency. It turns out that we can do just that, and that doing so will indeed produce a valuation theory. We give the details below.

definition *d.* A set is maximal consistent if it is consistent and it is not contained in any other consistent set.

Theorem *e.* Every maximal consistent set is a valuation theory.

Suppose that Γ is a maximal consistent set. By theorem 6.3 it is sufficient to show that it satisfies conditions 1 and 2. Condition 1 is equivalent to the condition that exactly one of $\mathbf{A}$ and $\neg\mathbf{A}$ are members of Γ. This condition can fail only if both $\mathbf{A}$ and $\neg\mathbf{A}$ are members of Γ or neither $\mathbf{A}$

nor $\neg\mathbf{A}$ is a member of Γ. In the first case (by 5.17) Γ is inconsistent, contradicting our initial assumption. In the second case, both $\Gamma\cup\{\mathbf{A}\}$ and $\Gamma\cup\{\neg\mathbf{A}\}$ are proper extensions of Γ. But at least one of these is consistent. For otherwise (by the deduction theorem) $\mathbf{A}\supset\bot$ and $\neg\mathbf{A}\supset\bot$ are both derivable from Γ, so (by A1 and 5.2) $\bot$ is derivable from Γ, contradicting the consistency of Γ. Hence either $\Gamma\cup\{\mathbf{A}\}$ or $\Gamma\cup\{\neg\mathbf{A}\}$ is a consistent set properly containing Γ. So this case also contradicts our initial assumption. This proves that Γ satisfies condition 1. We can use this fact in proving that it also satisfies condition 2. First suppose $\mathbf{A}\in\Gamma$. If $\mathbf{A}\vee\mathbf{B} \notin \Gamma$ then by condition 1 $\neg(\mathbf{A}\vee\mathbf{B}) \in \Gamma$. By 5.18 $\Gamma\vdash \neg\mathbf{A}$. By 5.17 $\Gamma\vdash\bot$. This contradicts the assumption that Γ is consistent, so $\mathbf{A}\vee\mathbf{B} \in \Gamma$ after all. A similar argument establishes that if $\mathbf{B}\in\Gamma$ then $(\mathbf{A}\vee\mathbf{B})\in\Gamma$. Now suppose that $(\mathbf{A}\vee\mathbf{B})\in\Gamma$. If neither $\mathbf{A}$ nor $\mathbf{B}$ is a member of Γ then, by condition 1, $\neg\mathbf{A}$ and $\neg\mathbf{B}$ are both members of Γ. By 5.21 and modus ponens $\Gamma\vdash\mathbf{B}$. By 5.17 $\Gamma\vdash\bot$. Again, this contradicts the assumption that Γ is consistent. So if $\mathbf{A}\vee\mathbf{B}$ is a member of Γ then either $\mathbf{A}$ or $\mathbf{B}$ must also be, i.e., Γ does satisfy condition 2.

We are now ready to proceed with our plan for proving completeness.

Theorem f. (Lindenbaum's lemma) For every consistent set Γ there is a maximal consistent extension, i.e., a set Γ' such that $\Gamma\subseteq\Gamma'$ and Γ' is maximal consistent.

<u>Proof</u> Let $\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}_3,\ldots$ be an enumeration of all the formulas of $\mathscr{L}$. (We leave the proof of the existence of such an enumeration to the exercises.) We construct Γ' in stages. We start with Γ. At each subsequent stage i, we add $\mathbf{A}_i$ if it is consistent to do so. Finally we define Γ' to be the set of formulas that occur in any of these stages. More formally, the definition can be written as follows:

$$\begin{aligned}
&\Gamma_0 = \Gamma\\
&\Gamma_{i+1} = \begin{cases}\Gamma_i \cup \{\mathbf{A}_{i+1}\} & \text{if that is consistent and}\\ \Gamma_i & \text{otherwise.}\end{cases}\\
&\Gamma' = \{\mathbf{A}: \mathbf{A} \in \Gamma_i \text{ for some i}\}.
\end{aligned}$$

Note first that Γ_i is consistent for each i. For Γ_0 is just Γ, which is consistent by hypothesis, and consistency is clearly preserved by the rules for constructing each succeeding Γ_i. It follows that Γ' is consistent. For if it were not, there would be a derivation of $\bot$ from assumptions in Γ'. Since a derivation consists of only a finite sequence of formulas, this derivation would actually be a derivation of $\bot$ from one of the Γ_i's, which would mean that the Γ_i's were not consistent after all. It remains only to show that there is no larger consistent set. Suppose there was such a set, $\mathbf{\Gamma^+}$. Then there would be a formula **A** that was a member of $\mathbf{\Gamma^+}$ but not $\mathbf{\Gamma'}$. **A** must have occurred somewhere the enumeration of formulas of $\mathscr{L}$, say $\mathbf{A}=\mathbf{A_i}$. Since **A** is not in $\mathbf{\Gamma'}$, it was not added at stage i, i.e., $\mathbf{\Gamma_{i-1}}\cup\{\mathbf{A_i}\}$ is inconsistent. But we were supposing that $\mathbf{\Gamma^+}$ contains both $\mathbf{\Gamma'}$ and $\{\mathbf{A}\}$, so this would mean that $\mathbf{\Gamma^+}$ itself was inconsistent.

We have now assembled all the ingredients needed for the proof that **SL** is strongly complete. Let us recapitulate how the proof proceeds. We want to show that every consistent set is satisfied by some valuation. By Lindenbaum's lemma every consistent set is contained in a maximal consistent set. By Theorem 6.5 a maximal consistent set is a valuation theory, i.e., a set of formulas that are all true in some valuation α. So every consistent set **is** satisfiable.

There is a corollary of the strong completeness that is of interest in itself and that will be of use in the first section of chapter 6.

Say that a set of formulas $\mathbf{\Delta}$ is <u>finitely satisfiable</u> if each finite subset of $\mathbf{\Delta}$ is satisfiable. Then the result is:

Theorem g. (Compactness) If $\mathbf{\Delta}$ is finitely satisfiable it is satisfiable.

<u>Proof</u> Suppose $\mathbf{\Delta}$ is not satisfiable. **SL** is complete, so by theorem 6.2 $\mathbf{\Delta}$ must be inconsistent. This means that there is a derivation of $\bot$ from $\mathbf{\Delta}$. But derivations are finite lists so this must be a derivation of $\bot$ from some finite subset $\mathbf{\Delta'}$ of $\mathbf{\Delta}$. By soundness, $\mathbf{\Delta'}\vDash\bot$. By theorem 4.5 (ii) $\mathbf{\Delta'}$ is not satisfiable and hence $\mathbf{\Delta}$ is not finitely satisfiable.

It is worth noting that strong completeness follows from compactness and weak completeness. For by compactness if $\mathbf{\Delta}$ is not satisfiable then some finite subset $\mathbf{\Delta'}$ of $\mathbf{\Delta}$ is not satisfiable. By theorem 4.5 (ii), $\mathbf{\Delta'}\vDash\bot$. By the corollary to theorem 4.3, $\vDash(\mathbf{D_1}\supset\ldots\supset(\mathbf{D_n}\supset\bot))$, where $\mathbf{D_1},\ldots,\mathbf{D_n}$ are the formulas of $\mathbf{\Delta'}$. By weak completeness $\vdash(\mathbf{D_1}\supset\ldots\supset(\mathbf{D_n}\supset\bot))$, and finally, by the corollary to theorem 5.7, $\mathbf{\Delta'}\vdash\bot$. But if $\bot$ is derivable from assumptions $\mathbf{\Delta'}$ then it is derivable (by the very same derivation) from the larger set $\mathbf{\Delta}$. So we have shown that if $\mathbf{\Delta}$ is not satisfiable then it is not consistent.

<u>Exercises and problems</u>

1**[e]**. Show that there exists an enumeration of the formulas of $\mathscr{L}$. [Hint: Associate a positive number with every symbol of the language. Associate with each formula the sum of the numbers associated with the occurrences of its symbols and show that the set of formulas can be enumerated by listing any formulas associated with the number 1, then any associated with the number 2, and so forth. Note that this method can be used generally to show that there is an enumeration for any set of expressions of any language with enumerable vocabulary.]

2**[e]**. Consider the systems that would result by emending the **SL** as indicated below. In each case state whether the resulting system would be sound and complete.

a. Add as an axiom schema: $(A\supset B)\vee(B\supset A)$
b. Add as a rule: $\mathbf{A\supset B},\mathbf{B}\ /\ \mathbf{A}$
c. Replace **A** and **B** in A1-A5 by $\mathbf{p}_1$ and $\mathbf{p}_2$ respectively and add a rule which ensures that if $\Gamma\vdash A$ and Γ',A' is a substitution instance of Γ,A then $\Gamma'\vdash A'$.

3**[e]**. A system of axioms and rules is <u>strongly sound</u> if everything derivable from a set of formulas is a consequence of that set. It is <u>weakly</u> sound if every theorem is valid. It has the <u>consistency property</u> if every satisfiable set is consistent. Prove that if the deduction theorem holds then these three conditions are all equivalent.

4**[e]**. Prove part ii of theorem 6.2.

5**[e]**. Prove that every maximal consistent set Γ satisfies the following properties.

a. $\mathbf{(A\wedge B)}\in\Gamma$ iff $\mathbf{A}\in\Gamma$ and $\mathbf{B}\in\Gamma$.
b. $\mathbf{(A\supset B)}\in\Gamma$ iff $\mathbf{A}\notin\Gamma$ or $\mathbf{B}\in\Gamma$.
c. $\mathbf{(A\equiv B)}\in\Gamma$ iff either $\mathbf{A}\in\Gamma$ and $\mathbf{B}e\Gamma$ or $\mathbf{A}\notin\Gamma$ and $\mathbf{B}\notin\Gamma$.
d. $\bot\notin\Gamma$.
e. $\top\in\Gamma$.
f. If $\Sigma\subseteq\Gamma$ and $\Sigma\vdash\mathbf{A}$ then $\mathbf{A}\in\Gamma$.

6**[e]**. Prove the converse of theorem 6.5.

7**[e]**. State whether the following are true or false. Give reasons.

a. If **A** truth-functionally implies **B** then every maximal consistent set that contains **A** also contains **B**.

b. If **A** does not truth-functionally imply **B** then there is a maximal consistent set that contains **A** but not **B**.

c. Every consistent formula is a member of at least two distinct maximal consistent sets.

d. Every consistent set is a subset of at least two distinct maximal consistent sets.

e. If Γ is a maximal consistent set then the complement of Γ is not.

f. If Γ is a maximal consistent set and Γ' is the result of substituting **q** for **p** in every formula of Γ then Γ' is also maximal consistent.

g. If Γ is a maximal consistent set and Γ' is the result of interchanging **p** and **q** in every formula of Γ then Γ' is also maximal consistent.

8**[e]**. Show that for every number n there is a set Γ with n formulas such that every subset of Γ with fewer than n formulas is satisfiable, while Γ itself is not satisfiable.

9**[e]**. Show that compactness is equivalent to the following condition: $\{\mathbf{A}_1,\mathbf{A}_2,\ldots\}$ is satisfiable if, for every m, there is an $n > m$ such that $\{\mathbf{A}_1,\ldots,\mathbf{A}_n\}$ is satisfiable.

10**[e]**. Compactness, like strong completeness, can be expressed in terms of consequence as well as satisfiability:

(*) $\Delta \vDash \mathbf{A}$ implies $\Delta' \vDash \mathbf{A}$ for some finite subset Δ' of Δ.

a. Prove the equivalence of the two versions of compactness by showing that (*) holds iff theorem 6.7 does.

b. Without appealing to this equivalence, show that (*) follows from strong completeness.

11**[e]**. For sets of formulas Γ and Δ, say that $\Gamma \vDash \Delta$ iff any valuation making all the members of Γ true makes some members of Δ true (or, equivalently, iff any valuation making all the members of Δ false makes some member Γ false). Prove that if $\Gamma \vDash \Delta$ then $\Gamma' \vDash \Delta'$ for some finite subsets $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$. [Hint: Express $\Gamma \vDash \Delta$ in terms of satisfiability and apply compactness.]

12**[p]**. (Completeness via disjunctive normal form) Use the result of problem 15 or 16 *** of the previous chapter to provide an alternate proof of weak completeness. Hint: Suppose **A** is truth functionally valid. Then no valuation satisfies $\neg\mathbf{A}$. Let **B** be a disjunctive normal form of $\neg\mathbf{A}$. By soundness $\mathbf{B}$ is truth-functionally equivalent to $\neg\mathbf{A}$, so no valuation satisfies $\mathbf{B}$. Show first that this implies that $\mathbf{B}=\bot$ and that $\neg\mathbf{B}$ is therefore provable. Finally, use the fact that $\neg\mathbf{A}$ is provably equivalent to $\mathbf{B}$ to show that $\mathbf{A}$ is provable.

13**[p]**. (Model sets.) Let $\mathcal{L}^*$ be the language built from the sentence letters and the truth-functional connectives $\neg$, $\vee$, and $\wedge$. A set Γ of formulas of $\mathcal{L}^*$ is a <u>model set</u> if it satisfies the following conditions:

i) If $\neg\mathbf{A} \in \Gamma$ then $\mathbf{A} \notin \Gamma$.
ii) If $(\mathbf{A}\vee\mathbf{B}) \in \Gamma$ then $\mathbf{A}\in\Gamma$ or $\mathbf{B}\in\Gamma$.
iii) If $(\mathbf{A}\wedge\mathbf{B}) \in \Gamma$ then $\mathbf{A}\in\Gamma$ and $\mathbf{B}\in\Gamma$.

a. Without using Lindenbaum's lemma, prove that if Γ is a consistent set of formulas of $\mathbf{L}^*$ then there is a model set Δ such that $\Gamma \subseteq \Delta \subseteq \mathbf{SUB}(\Gamma)$. Hint: Construct Δ in stages. Start with Γ. At each subsequent stage add both conjuncts of every conjunction that was added at the previous stage and at least one disjunct of every disjunction that was added. Show that this can always be done consistently.

b. Use a to provide an alternative proof of completeness. Hint: Let Γ be a consistent set, and let Γ' be a set containing a negation normal equivalent of each formula in Γ (as defined in problem 14 *** of the previous chapter). Now expand Γ' to a model set Γ^+. Let α be the set of all sentence letters in Γ^+. Prove that Γ^+ is contained in the theory of α and that therefore Γ is as well. Note that this method of proving completeness establishes the additional fact that every finite consistent set is satisfiable by a finite valuation.

14**[p]**. (A language without letters) Let $\mathcal{L}^-$ be the language with all the connectives of $\mathcal{L}^+$ but no sentence letters. (Formulas are built up from $\top$ and $\bot$.)

a. Show that every formula of $\mathcal{L}^-$ is provably equivalent in our axiom system to either $\top$ or $\bot$.

b. Use a to prove that the axiom system is complete for $\mathcal{L}^-$ i.e., that every consistent set of formulas of $\mathcal{L}^-$ is satisfiable.

15**[p]**. Let $\mathbf{A}(\top)$ and $\mathbf{A}(\bot)$ be the results of substituting $\top$ and $\bot$ for all occurrences of some sentence letter $\mathbf{p}$ in $\mathbf{A}$.

a. Prove that $\mathbf{A(\top),A(\bot)}\vdash\mathbf{A}$. Hint: Use axiom A1 and examples ? i and j to show that $\vdash\mathbf{(p\equiv\top)}\vee\mathbf{(p\equiv\bot)}$. Use this fact and theorem 5.15 to prove the desired fact.

b. Use a repeatedly to prove $\Gamma\vdash\mathbf{A}$ where Γ is the set of all substitution instances of **A** that are the language $\mathcal{L}^-$ defined in the previous problem.

16**[p]**. (Another completeness proof) Suppose **A** is not provable. Then by problem 14b above there is a substitution instance **A'** of **A** such that **A'** is a formula of $\mathcal{L}^-$ and **A'** is not provable. By problem 15b **A'** is not valid. To finish this completeness proof it is sufficient to show that this implies that **A** is not valid. Do this by describing a valuation for **A** that agrees with the one that falsifies **A'**.

17**[p]**. (direct proof of compactness) Given a sequence $s=s_1,\ldots,s_n$ of truth values, $n>0$, let α_s be the valuation $\{\mathbf{p_i}{:}i<n \text{ and } s_i=T\}$. Thus $s=s_1,\ldots,s_n$ represents the assignment of truth value s_1 to $\mathbf{p_1}$, s_2 to $\mathbf{p_2}$, and so on. Say that r is an <u>extension</u> of s if r is a sequence of truth values of the form $s_1,\ldots,s_n,t_1,\ldots,t_m$. Suppose that for every $i>0$, s^i is a sequence of truth values such that $s^0=s$ and s^{i+1} is an extension of s^i for $i>0$. Now suppose Δ is a finitely satisfiable set of formulas. Call a sequence s <u>suitable for Δ</u> if, for each finite subset Δ' of Δ, there is an extension r of s such that α_r verifies Δ'. Consider the following truth-value sequences.

s^0 is the empty sequence. (I.e., it is a sequence representing the valuation that makes every sentence letter false.)

For all $i>0$, s^{i+1} is the result of appending T to s^i if that is suitable for Δ and the result of appending F otherwise.

a. Prove by induction that for each $i>0$, s^i is suitable for Δ.

b. Prove that the limit s (in the natural sense) of the s^i's represents a valuation that verifies Δ.

c. Use compactness to deduce strong completeness from weak completeness (thereby allowing a proof of weak completeness alone, as in problems 12 and 17, to be strengthened.)

18**[p]**. (completeness via sequents) An expression of the form $\Gamma\Rightarrow\Delta$ is called a <u>sequent</u>. The sequent $\Gamma\Rightarrow\Delta$ is <u>provable</u> if it can be shown that $\Gamma\Rightarrow\Delta$ on the basis of finitely many applications of the conditions listed in problem 19 of the previous chapter. (So a sequent is provable iff the set named on the left bears the relation **SC** to the set named on the right.) If the proof makes no appeal to the third condition, the sequent is provable without cut. The notion of truth-functional consequence can be extended as follows: $\Gamma \vDash \Delta$ iff, for every valuation α, either α makes something in Γ false or α makes something in Δ true.

a. (soundness) Show by induction on provable sequents that if $\Gamma \Rightarrow \Delta$ is a provable sequent then $\Gamma \vDash \Delta$.

b. (completeness) Show by induction on finite sets of formulas (cf section * of the appendix) that if $\Gamma \vDash \Delta$ for finite sets Γ and Δ then the sequent $\Gamma \Rightarrow \Delta$ is provable without cut. It follows from a and b that cut is redundant in the sense that any sequent provable with cut can also be proved without it.

c. Use a and b above and the completeness of **SL** to give an alternative proof that **SC1**=**SLD**.