Chapter 5. Axiomatics

In the last chapter we introduced notions of sentential logic that correspond to logical truth and logically valid argument pair. In this chapter we introduce the notions of a **proof** of **A** and a derivation of **A** from Γ, which correspond to a demonstration of a truth and a deduction of a consequence from assumptions. A demonstration is meant to make evident that its conclusion is true. It starts with evident truths and proceeds by the application of rules of inference that evidently transmit truth. A deduction is meant to make evident that its conclusion follows from its assumptions. It starts either with evident truths or with assumed truths and proceeds by the application of rules of inference that transmits the property of following from.

We take an axiom system to be a generation scheme for formulas. The basis elements of the scheme are axioms and the rules are rules of inference. (This notion of axiom system will be extended subsequently.) A proof of **A** in an axiom system is simply a record of generation of **A** in that generational scheme. A derivation of **A** from Γ is a record of generation of **A** from Γ, i.e., a record of generation of **A** in the scheme obtained by adding the members of Γ to the axiom system as basis elements.

A is a theorem in an axiom system **S** if there is a proof of **A** in **S**. **A** is derivable from Γ in **S** if there is a derivation of **A** from Γ. (Here and elsewhere, reference to the axiom system is omitted when it is understood from context.) We sometimes say that the argument pair Γ,**A** is provable to indicate that **A** is derivable from Γ.

A relation between sets of formulas and formulas is axiomatized by an axiom system if that relation is the derivability relation for the axiom system, i.e., if the relation holds between Γ and **A** iff **A** is derivable from Γ. Γ is said to be consistent in an axiom system if $\perp$ is not derivable from Γ in that system, and inconsistent if $\perp$ is derivable. **A** is said to be consistent or inconsistent if {**A**} is. We write $\Gamma\vdash_S\mathbf{A}$ to indicate that **A** is derivable from Γ in axiom system **S**, $\vdash_S\mathbf{A}$ to indicate that **A** is provable in **S**, and $\text{Consis}_S(\Gamma)$ or $\text{Consis}_S(\mathbf{A})$ to indicate that Γ or **A** is consistent in S. If the axiom system is clear from context the subscripts will be dropped. We sometimes write $\mathbf{A}\vdash\mathbf{B}$ for $\{\mathbf{A}\}\vdash\mathbf{B}$ and $\Gamma,\mathbf{A_1},\ldots,\mathbf{A_n}\vdash\mathbf{B}$ for $\Gamma\cup\{\mathbf{A_1},\ldots,\mathbf{A_n}\}\vdash\mathbf{B}$.

In this chapter we introduce a system that axiomatizes the set of tautologies and the relation of truth-functional consequence. It is called **SL** (for sentential logic) and contains the following axioms and rule of inference.

Axioms

A1. $\neg\mathbf{A}\vee\mathbf{A}$

A2. $\neg(\neg\mathbf{A}\vee\mathbf{B})\vee(\neg(\neg\mathbf{B}\vee\mathbf{C})\vee(\neg\mathbf{A}\vee\mathbf{C}))$

A3. $\neg\mathbf{A}\vee(\mathbf{A}\vee\mathbf{B})$

A4. $\neg\mathbf{A}\vee(\mathbf{B}\vee\mathbf{A})$

A5. $\neg(\neg\mathbf{B}\vee\mathbf{C})\vee(\neg(\neg\mathbf{A}\vee\mathbf{C})\vee(\neg(\mathbf{A}\vee\mathbf{B})\vee\mathbf{C}))$

Rule of inference

R From **A** and $\neg\mathbf{A}\vee\mathbf{B}$ infer **B**

The use of metalinguistic variables in A1-A5 indicates that an axiom is obtained for each choice of **A**,**B** and **C**. For example, $\neg\mathbf{p}\vee(\mathbf{q}\vee\mathbf{p})$, $\neg\neg\mathbf{q}\vee(\mathbf{q}\vee\neg\mathbf{q})$, and $\neg(\mathbf{q}\vee\mathbf{r})\vee(\neg\neg\mathbf{q}\vee(\mathbf{q}\vee\mathbf{r}))$ are all axioms by A4. Thus each of A1-A5 corresponds to an infinite set whose members are all substitution instances of a single formula. We call such sets formula schemas (or, more simply, schemas) because they comprise all the formulas of a particular pattern. The members of a schema are its <u>instances</u>. Schemas may be denoted, as they are above, by "formulas" containing metalinguistic variables in place of sentence letters.

The rule of inference R is called <u>modus ponens</u> (MP). Clearly MP is also a kind of schema. A more precise account of the nature of axiomatic systems like SL is outlined in problem ** of chapter 2.

The axioms and rules of **SL** are perhaps more perspicuous when they are presented in abbreviated form.

<u>Axioms</u>
AA1. $\mathbf{A \supset A}$
AA2. $\mathbf{(A \supset B) \supset (B \supset C) \supset (A \supset C)}$
AA3. $\mathbf{A \supset (A \vee B)}$
AA4. $\mathbf{A \supset (B \vee A)}$
AA5. $\mathbf{(B \supset C) \supset (A \supset C) \supset (A \vee B) \supset C}$
<u>Rule of inference</u>
AR From **A** and $\mathbf{A \supset B}$ infer **B**

The members of the apparent basis of a derivation of **A** from Γ that are justified by their membership in Γ (rather than by appeal to an axiom) are the <u>assumptions</u> of the derivation. The assumptions of the derivation that are in its genuine basis are the <u>assumptions on which **A** rests</u>. Thus every assumption on which the conclusion of a derivation rests is an assumption of the derivation, but not every assumption of the derivation need be one on which the conclusion rests. Furthermore, every line of a derivation is derivable from the assumptions on which it rests. (See ***)

In the same way that a sentence letter can be taken to represent a sentence from an already interpreted language, proofs and derivations can be taken to represent demonstrations and deductions within such a language. When the letters are taken to represent propositions, the corresponding demonstrations and deductions will, of course, involve propositions rather than sentences.

Just as we find it convenient to abbreviate the formulas of $\mathfrak{L}$, we find it convenient to abbreviate derivations of **SL**. We shall eventually adopt a number of abbreviatory devices. But for now, let us note that the basic rules of abbreviation for formulas can be extended to derivations. We may say, quite generally, that one list <u>abbreviates</u> another if the second can be obtained from the first by replacing one or more formulas with the formulas that they abbreviate. Using this notion, we may extend some of our earlier definitions to the abbreviatory language.

definition *a.* If **A** and the members of Γ are in $\mathfrak{L}^+$ then:
(i) D is a derivation of **A** from Γ if D abbreviates a derivation of a formula that **A** abbreviates from assumptions that members of Γ abbreviate.
(ii) D is a proof of **A** if D abbreviates a proof of a formula that **A** abbreviates.

Like our extended truth definition (3.6), this is an inductive definition and, as in that case, we should show that the conditionals (of (i) and (ii)) can be strengthened to biconditionals. A rigorous proof is left to the problems. As before, **A** is a <u>theorem</u> if there is a proof of **A**, and **A** is derivable from Γ if there is a derivation of **A** from Γ.

Clearly a list of lines, each of which is either an assumption, an instance of one of the schemas AA1-AA5 (with appropriate annotation), or the result of applying modus ponens to formulas of previous lines (annotating appropriately), will constitute a derivation of **SL**. So also will "mixed" lists employing A1-A4 and R as well as AA1-AA4 and AR.

Let us look at an example.

b. $\mathbf{p \vee q}, \mathbf{p \supset r}, \mathbf{q \supset r} \vdash \mathbf{r}$.

1.(q⊃r) ⊃ (p⊃r) ⊃ (p∨q) ⊃ r   A5
2. q⊃r   assumption
3. (p⊃r) ⊃ (p∨q) ⊃ r   1,2 MP
4. p⊃r   assumption
5. (p∨q)⊃r   3,4 MP
6. p∨q   assumption
7. r   5,6 MP

The list above comprises an abbreviation of a derivation in **SL** of **r** from **p∨q**, **p⊃r**, and **q⊃r**. Lines 2, 4 and 6 are the assumptions of the derivation. The conclusion, in this case, rests on all three assumptions. Line 1 is an axiom. The remaining lines follow by modus ponens.

It is common to take derivations to be simply sequences of formulas. The annotations, on this view, serve to make it clear that the sequence is indeed a derivation and to help one understand the derivation as a piece of reasoning. Notice, however, that there is no way to tell from the formulas alone, whether the formula on line 1 above should be taken as an assumption or an axiom. In general there might be many ways to annotate a list of formulas so as to obtain a proof or derivation and each of the resulting lists would correspond to a different demonstration or deduction.

Our axiom system is very economical and that makes it convenient for proving general facts about theorems. It is less convenient, however, for showing that a particular formula is derivable from some particular assumptions. The next theorems alleviate this drawback.

Theorem c. (generalized transitivity). If $\Gamma \vdash \mathbf{A}$ and $\Delta, \mathbf{A} \vdash \mathbf{B}$ then $\Gamma, \Delta \vdash \mathbf{B}$.
Proof: The hypothesis of the theorem asserts that there is a derivation of **A** from Γ and a derivation of **B** from Δ and **A**. By appending the latter to the former (relabelling, where necessary) we obtain a derivation of **B** from $\Gamma \cup \Delta$.
Theorem d. (substitution).
If $\Gamma \vdash \mathbf{B}$ and $\Gamma', \mathbf{B}'$ is a substitution instance of $\Gamma, \mathbf{B}$ then $\Gamma' \vdash \mathbf{B}'$.
Proof. By making the appropriate substitution in each line of the derivation of **B** from Γ one obtains a derivation of **B'** from Γ'.

Theorems 5.3 and 5.4 make possible what may be regarded as a further device for the abbreviation of derivations. Suppose that $\{\mathbf{A}_1', \ldots, \mathbf{A}_n', \mathbf{B}'\}$ is a substitution instance of $\{\mathbf{A}_1, \ldots, \mathbf{A}_n, \mathbf{B}\}$ and suppose that we have a derivation of **B** from $\mathbf{A}_1, \ldots, \mathbf{A}_n$. Then if $\mathbf{A}_1', \ldots, \mathbf{A}_n'$ all occur in a derivation, we may append a line containing **B'** and an annotation that refers to the derivation of **B** from $\mathbf{A}_1, \ldots, \mathbf{A}_n$. For by substitution **B'** is derivable from $\mathbf{A}_1', \ldots, \mathbf{A}_n'$ and by transitivity **B'** must therefore be derivable from the assumptions used to derive $\mathbf{A}_1', \ldots, \mathbf{A}_n'$. To disabbreviate a derivation step that appeals to this device, one should insert all the "intermediate" formulas in the derivation of **B'** from $\mathbf{A_1}$**'**,...,$\mathbf{A_n}$**'**. As a special case this shortcut permits us to enter substitution instances of previously proved theorems anywhere in a derivation. Use of this device in derivations corresponds to the use of lemmas in informal reasoning. To fill a gap in a deductive chain, appeal is made to an independently established result.

Let us look at a couple of examples.

e. **p∨q ⊢ q∨p**.

1. p∨q assumption
2. p⊃(q∨p) A4
3. q⊃(q∨p) A3
4. q∨p 1,2,3 (5.2)

Here the strategy is to exploit the fact that example 5.2 makes possible a kind of "proof by cases". We have a disjunctive assumption, **p∨q.** So there are two cases to consider. First, we derive a conditional with our desired conclusion as consequent and **p** as antecedent; then, a similar conditional with **q** as antecedent. An appropriate substitution in 5.2 tells us that the conclusion is derivable. Notice that on line 4 we cite three line numbers corresponding to the three assumptions of the previous derivation to which we are making appeal. We also add, in brackets, a reference to that derivation. Again, strictly speaking lines 1-4 constitute only an abbreviation of a derivation in **SL**. In the derivation itself each of the formulas above would be disabbreviated and at least three additional steps would be inserted between the third and fourth lines.

In the next example we use the fact that conditionals abbreviate disjunctions to apply a similar strategy to conditional assumptions.

f. **p⊃q⊃r, p⊃q ⊢ p⊃r**

1. p⊃q⊃r assumption [¬p∨(q⊃r)]
2. ¬p⊃(¬p∨r) A3 [¬p⊃(p⊃r)]
3. (p⊃q) ⊃ (q⊃r) ⊃ (p⊃r) A2
4. p⊃q assumption
5. (q⊃r) ⊃ p⊃r 3,4 MP
6. p⊃r 1,2,5(5.2)

Here we have introduced an additional annotation. We want to use example 5.2 to show that the formula on line 6 follows from those on lines 1,2 and 5. The formulas on the latter lines are not themselves substitution instances of the assumptions of example 5.2. But there are substitution instances of the assumptions of example 5.2 that abbreviate the same formulas as do those of the formulas on lines 1,2 and 5. These we list in brackets at the end of the lines. We intend to abbreviate a derivation in 𝔏; interchanging two abbreviations of the same formula will not change the derivation that is abbreviated. In general we may at any stage of a derivation add to previous lines a formula in brackets whose unabbreviated form is the same as the formula on that line. In subsequent steps we may then assume that the formula on that line is the bracketed one.

We are now in a position to prove a very useful result.

Theorem g. (deduction theorem) If **Γ,A⊢B** then **Γ⊢A⊃B**.

Proof. The set of formulas derivable from **Γ**∪**{A}** is clearly a generated set, so we can proceed by induction. (See section *** of the appendix for a general discussion of this kind of argument.)

Basis. If **B** is an axiom or a member of **Γ** then a derivation whose only formula is B establishes that **Γ** ⊢ **B**. By A4, ⊢**B⊃(A⊃B)**. So by MP **A⊃B** can be derived from **Γ**. If **B=A** then A1 shows immediately that ⊢**A⊃B**.

<u>Inductive step</u>. Suppose that the theorem holds for **B=(C⊃D)** and for **B=C**. Then **Γ**⊢**(A⊃(C⊃D)** and **Γ**⊢**A⊃C**. It follows by example 5.6 that **Γ**⊢**A⊃D**, so the theorem holds for **D** as well.

Note that the converse of the deduction theorem is also true. For if a conditional is derivable from assumptions **Γ** then the consequent of the conditional can be derived from **Γ** and the antecedent by adding a single application of modus ponens. Thus we have a syntactic analog of theorem 4.3. By applying it repeatedly we obtain the following analog of the corollary to theorem 4.3.

<u>Corollary</u>. $\mathbf{A_1,...,A_n} \vdash \mathbf{A_{n+1}}$ iff $\vdash \mathbf{A_1 \supset ... \supset A_{n+1}}$.

The deduction theorem makes possible another shortcut for derivations. To show that a conditional **A⊃B** is derivable from some assumptions we need only show that **B** can be derived when **A** is added to those assumptions. A couple of examples will illustrate.

h. **p⊃p⊃q** ⊢ **p⊃q**

1. p⊃p⊃q assumption
    2.1. p assumption
    2.2. p⊃q 1,.1 MP
    2.3. q .1, .2 MP
3. p⊃q 2.1-2.3 Ded

Here the desired conclusion is a conditional. We list the given assumption on line one. On line 2.1 we begin another derivation, attempting to show that the consequent of our desired conditional can be derived from the antecedent together with the given assumption. This second derivation is indented to set it off from the main derivation, and it is numbered with a dot notation to indicate that the entire smaller derivation is regarded as the second step of the larger one. On line 2.3 the second derivation succeeds. The deduction theorem tells us the required conditional can be derived from the original assumptions alone, so we enter it on line three. It is important to keep in mind that in the indented derivation a new assumption is temporarily introduced. For this reason we cannot, in the main derivation, appeal to any lines of the indented derivation. On the other hand, in the indented derivation the original assumptions are still available to us, so it is perfectly reasonable to cite formulas derivable from these as we do in line 2.2.

More precisely, the "numbers" of lines are now of the form $n_1.n_2. ... .n_k$, where $k>0$ and each n_i is a positive integer. There are three orderings on these numbers that constrain derivations. One is the "lexicographic" ordering that orders the words in a dictionary. The lexicographic order on our line numbers (which we represent by $\prec$) can be defined by induction: $m_1.m_2. ... .m_k \prec n_1.n_2. ... .n_{k'}$ if $m_1<n_1$; or if k=1 and $m_1=n_1$ and k'>1; or if k>1, $m_1=n_1$, and $m_2. ... .m_k \prec n_2. ... .n_{k'}$. This ordering constrains the linear sequence of lines, numbers of later lines being required to lexicographically succeed those of earlier ones. The second is the ordering by length (i.e., by the size of k). The line numbers of a subderivation must be longer than those of the host derivation. The third ordering constrains the permissible annotations. Say that $n_1.n_2. ... .n_k$ is *subordinate to* $m_1.m_2. ... .m_j$ iff $k \leq j$, $n_h=m_h$ for all $h<k$, and $n_k<m_k$. The sources of any numbered line must be subordinate to that number. To make the annotation easier, when j = k we will indicate the line number cited by just writing its last coordinate preceded by a dot. Thus, in line 2.2 of the derivation above the '.1' on the right side refers to line 2.1.

This definition allows the numbering to 'jump'; we can go directly from line 1 to line 3, for example. To avoid jumps, we may demand that the successor of given line $m_1.m_2. ... .m_j$ be either $m_1.m_2. ... .(m_j + 1)$ if the line is not indented or $m_1.m_2. ... .m_j.1$ if it is.

Let us look at another example.

i. **p⊃q⊃r ⊢ q⊃p⊃r**

1.p⊃q⊃r assumption

  2.1. q assumption

    2.2.1. p assumption

    2.2.2. q⊃r 1, .1 MP

    2.2.3. r 2.1, .2 MP

  2.3. p⊃r 2.2.1-2.2.3 Ded

3.q⊃p⊃r 2.1-2.3 Ded

In this example the deduction theorem is used twice. The formula to be derived is a conditional, so we assume its antecedent in line 2.1 with the idea of deriving its consequent. The consequent is itself a conditional, so in line 2.2.1 we assume its antecedent. Again assumptions of a derivation are available in subderivations, but not vice versa.

The use of the deduction theorem in this way may also be regarded as an abbreviatory device, though one that is a little more complex than those that have been employed before. The derivation of **SL** that is abbreviated by examples like the one above is determined by the proof of theorem 5.7. But it is unimportant, for the purposes of knowing that there is a derivation, to know exactly how the abbreviatory derivations are to be filled out.

We now give several examples that will provide additional practice in finding derivations and contribute to the proof of the theorem that follows them.

j. **p ⊢ ¬¬p**

1. p assumption
2. ¬p⊃¬p A1 [¬¬p∨¬p]
3. ¬p∨¬¬p 2 (5.5) [p⊃¬¬p]
4. ¬¬p 1,3 MP

k. **p⊃q ⊢ ¬q⊃¬p**

1. p⊃q assumption

  2.1 p assumption

  2.2 q 1, .1 MP

  2.3 ¬¬q .2 (5.10)

3. p⊃¬¬q 2.1-2.3 Ded [¬p∨¬¬q]
4. ¬¬q∨¬p 3 (5.5) [¬q⊃¬p]

l. **p⊃q ⊢ (p∨r)⊃(q∨r)**

1. p⊃q assumption

  2.1 p∨r assumption

    2.2.1 p assumption

    2.2.2 q 1, .1 MP

    2.2.3 q⊃(q∨r) A3

    2.2.4 q∨r .2, .3 MP

  2.3 p⊃(q∨r) 2.2.1-2.2.4 Ded

  2.4 r⊃(q∨r) A4

  2.5 q∨r .2, .3, .4 (5.2)

3. (p∨r)⊃(q∨r) 2.1-2.5 Ded

m. $\mathbf{p\supset q \vdash (r\lor p)\supset(r\lor q)}$

1. $p\supset q$ assumption	
2. $(p\lor r)\supset(q\lor r)$	1 (5.12)
3.1 $r\lor p$	assumption
3.2 $p\lor r$	.1 (5.5)
3.3 $q\lor r$	2, .2 MP
3.4 $r\lor q$	.3 (5.5)
4. $(r\lor p)\supset(r\lor q)$	3.1-3.4 Ded

We say that **A** and **B** are provably equivalent (in symbols $\mathbf{A}\dashv\vdash\mathbf{B}$) if $\mathbf{A}\vdash\mathbf{B}$ and $\mathbf{B}\vdash\mathbf{A}$.

Theorem *n.* (Replacement of equivalents) Suppose **A** (which we also write as **A(B)**) contains an occurrence of **B** as a subformula, $\mathbf{B}\dashv\vdash\mathbf{B'}$ and **A'** (which we also write as **A(B')**) is the result of replacing that occurrence of **B** in **A** by **B'**. Then $\mathbf{A}\vdash\mathbf{A'}$.

Proof. **A** is generated from **B** by successively forming negations of previously generated formulas or forming disjunctions of previously generated formulas and arbitrary formulas. We may therefore prove the theorem by induction on **A** as so generated. The basis is the case in which **A**=**B** and, consequently, **B'**=**A'**. In this case the hypothesis of the theorem contains the condition to be proved. Now suppose that **A** is obtained by negating a previously generated formula **C** and that the theorem holds for **C**. Then the occurrence of **B** is part of **C**, which can be written **C(B)**, and $\mathbf{A'} = \neg\mathbf{C(B')}$. We must show $\neg\mathbf{C(B)} \vdash \neg\mathbf{C(B')}$. By induction hypothesis $\mathbf{C(B')}\vdash\mathbf{C(B)}$. By the deduction theorem this implies $\vdash\mathbf{C(B')} \supset \mathbf{C(B)}$. By example 5.11 (and substitution) $\vdash \neg\mathbf{C(B)}\supset\neg\mathbf{C(B')}$. By adding one application of MP to the derivation of $\neg\mathbf{C(B)}\supset\neg\mathbf{C(B')}$ we obtain a derivation of $\neg\mathbf{C(B')}$ from $\neg\mathbf{C(B)}$. Finally, suppose **A** is obtained by disjoining a previously generated formula **C** with an arbitrary formula **D**. There are two subcases to consider according whether **C** is the left disjunct of **A** or the right disjunct. In the former subcase $\mathbf{A'}=\mathbf{C(B')}\lor\mathbf{D}$ and we must show $\mathbf{(C(B)\lor D)} \vdash \mathbf{(C(B')\lor D)}$. By induction hypothesis, $\mathbf{C(B)}\vdash\mathbf{C(B')}$. By the deduction theorem $\vdash\mathbf{C(B)}\supset\mathbf{C(B')}$. By example 5.12 $\vdash \mathbf{(C(B)\lor D)\supset(C(B')\lor D)}$, and therefore $\mathbf{(C(B)\lor D)}\vdash\mathbf{(C(B')\lor D)}$. The latter subcase is similar, with example 5.13 playing the role of example 5.12.

The relation between **A** and **A'** described in the replacement theorem is completely symmetric: If **A'** is the result of replacing one occurrence <u>**B**</u> of a formula **B** in **A** by another formula **B'**, then **A** is the result of replacing the corresponding occurrence <u>**B'**</u> of **B'** in **A'** by **B**. It follows that the last '$\vdash$' in the theorem can be replaced by a '$\dashv\vdash$': Replacing a subformula with a provable equivalent results in a formula provably equivalent to the original.

Notice that the <u>replacement theorem</u> has to do with replacing <u>one</u> occurrence of a formula by another, whereas the <u>substitution theorem</u> has to do with replacing <u>every</u> occurrence of a sentence letter by a formula. It is possible to apply the replacement theorem repeatedly until all the occurrences of a certain subformula are replaced, but it is not possible to apply the substitution theorem in such a way that only some occurrences of a particular sentence letter are replaced.

The replacement theorem makes it possible to replace a subformula of a formula in a derivation by a provable equivalent, no matter how deeply embedded that subformula might be. For example, 5.5 shows that $\mathbf{A}\lor\mathbf{B}$ and $\mathbf{B}\lor\mathbf{A}$ are provably equivalent, so the replacement theorem allows us to reverse the order of disjuncts of a subformula anywhere in a derivation. To do so, of course, is to use what may be regarded as yet another device for abbreviating derivations.

SL actually possesses a property stronger than that expressed by theorem 5.14.

Theorem o. (strong replacement) Let **A'** be the result of replacing one occurrence of **B** in **A** with **B'**. Then ⊢ **(B≡B')⊃(A≡A')**.

Theorem 5.14 says that the formula obtained by replacing part of a formula with a provably equivalent formula is provably equivalent to the original. Theorem 5.15 says that it can be proved <u>in **SL**</u> that the formula obtained by replacing part of a formula with a (merely) equivalent formula is equivalent to the original. The proof that the latter result subsumes the former is left as an exercise. The proof of theorem 5.15 itself is left as a problem.

Let us summarize the three new devices for abbreviating derivations:

1 (previous results). We may appeal to substitution instances of previously proved formulas or argument pairs.

2 (Ded). We may append a line containing the conditional **A⊃B** to a derivation from Γ if **B** has been derived from Γ∪**{A}**.

3. (Rep). If **A** has been shown to be provably equivalent to **B**, we may append to a derivation a line containing any formula obtained by replacing some occurrences of **A** in the formula of a previous line by **B**.

Just as the abbreviatory formulas can be regarded as formulas of a richer object language $\mathfrak{L}^+$, these abbreviatory devices can be regarded as rules in a richer axiom system SL^+. The task of defining precisely the notions of proof and derivation for this axiom system is not trivial and is left as a problem. The task of defining the abbreviation relation between derivations of SL^+ is also nontrivial. The deduction theorem guarantees that a formula derivable by the rule Ded is derivable in SL, but the <u>procedure</u> for converting applications of Ded to applications of the rules of SL must be extracted from the <u>proof</u> of the deduction theorem. In fact, reasonably efficient disambiguation procedures can be given for our three rules, though the proof of this is left as a problem. To regard the new rules as abbreviations, it is sufficient to observe that <u>some</u> disabbreviation procedure exists.

It is worth noting that the additional rules, like the additional connectives, need <u>not</u> be regarded as abbreviations in a notationally richer system. One alternative would be to consider them to be convenient metalinguistic devices for referring to derivations in SL. A second alternative, which we explore briefly here, is to view them as part of an autonomous system of rules that determines the same derivability relation and the same class of theorems (in $\mathfrak{L}^+$) as does SL. It was noted earlier that SL, despite its theoretical advantages, does not provide a very convenient method of showing that for showing that particular formulas are derivable from particular assumptions.

A related disadvantage is that derivations of the system do not correspond very well to how we actually reason with the connectives. For example, to demonstrate a sentence of the form **if S and T then S** it would be natural to assume **S and T**, to infer **S** from this assumption, and then, from the fact that **S** was inferred from **S and T**, to conclude **if S and T then S**. No proof in SL corresponds very closely to such a demonstration.

We would like the autonomous system of derivability to take care of these difficulties. Systems of "natural deduction" have been formulated to do just this; and, under a judicious choice of derived rules, our own system may be regarded in this way.

The following theorem collects some properties of derivability that have already been established and some additional ones that will be useful in characterizing such a system.

Theorem p.

i (∨I). $\mathbf{A}\vdash\mathbf{A}\vee\mathbf{B}$ and $\mathbf{B}\vdash\mathbf{A}\vee\mathbf{B}$.

ii (∨E). If $\Gamma,\mathbf{A}\vdash\mathbf{C}$, $\Delta,\mathbf{B}\vdash\mathbf{C}$ and $\Sigma\vdash(\mathbf{A}\vee\mathbf{B})$ then $\Gamma\cup\Delta\cup\Sigma\vdash\mathbf{A}\vee\mathbf{B}$.

iii (¬I). If $\Gamma,\mathbf{A}\vdash\mathbf{B}$ and $\Delta,\mathbf{A}\vdash\neg\mathbf{B}$ then $\Gamma\cup\Delta\vdash\neg\mathbf{A}$.

iv (¬E). $\neg\neg\mathbf{A}\vdash\mathbf{A}$.

v (⊃I). If $\Gamma,\mathbf{A}\vdash\mathbf{B}$ then $\Gamma\vdash\mathbf{A}\supset\mathbf{B}$.

vi (⊃E). $\mathbf{A}\supset\mathbf{B},\mathbf{A}\vdash\mathbf{B}$

vii (∧I). $\mathbf{A},\mathbf{B}\vdash\mathbf{A}\wedge\mathbf{B}$

viii (∧E). $\mathbf{A}\wedge\mathbf{B}\vdash\mathbf{A}$ and $\mathbf{A}\wedge\mathbf{B}\vdash\mathbf{B}$

The proof of the theorem is left as an exercise.

The clauses of 5.16 correspond to rules in a system we shall label **ND**. Unlike **SL** and **SL+**, **ND** contains no axioms. Nevertheless, derivations are represented in a similar way. Lines are numbered with dot notation and subderivations are indented. In a given derivation, one may only cite formulas or subderivations that occur previously in that derivation or in some larger derivation in which that derivation is embedded. There are eight rules. Clauses i,iv,vi,vii, and viii, correspond to what may be called *proper* rules. The clauses are of the form $\mathbf{A}_1,...,\mathbf{A}_n\vdash\mathbf{B}$ and the corresponding rule permits one to enter **B** on any line after $\mathbf{A}_1,...,\mathbf{A}_n$ have been derived. Such rules correspond to the most straightforward case of inference, those that do not result in the dropping or 'discharge' of assumptions that have already been made. Clauses ii, iii and v correspond to *improper* rules. The clauses are of the form **if** $\Gamma_1\vdash\mathbf{A}_1,...,\Gamma_n\vdash\mathbf{A}_n$ **then** $\Delta\vdash\mathbf{B}$ and the corresponding rule allows one to enter **B** on any line after the members of Δ have all been derived, provided that, for $1\leq i\leq n$, the derivation contains a subderivation establishing that $\Gamma_i\vdash\mathbf{A}_i$. Such rules correspond to a less straightforward case of inference, those which may result in the discharge of assumptions that have been made. Notice that, for each of the connectives ¬,∨,⊃, and ∧, ND contains a rule for "introducing" an occurrence of the connective and a rule for "eliminating" such an occurrence. (In the cases of ∨-introduction and ∧-elimination, there are actually two such rules, with a common label.)

Theorem 5.16 ensures that any formula derivable from Γ in ND is so derivable in SL (and consequently that the theorems of ND are theorems of SL). Since the rule ⊃E of ND is identical to MP, establishing the converse can be established by providing natural deduction proofs of A1-A5. We leave the details as an exercise.

We close this chapter with a number of examples that will be useful subsequently. Those needed for the completeness theorem of the next chapter are numbered separately and proved in ND. The others are listed together and the proofs are left as exercises.

q. $\mathbf{p},\neg\mathbf{p}\vdash\mathbf{q}$.

1. p assumption
2. ¬p assumption

4.¬¬q ¬I

5. q 4 ¬E

Note that, in the application of ¬I on line 4, we take line 1 to constitute a proof that $\mathbf{p},\neg\mathbf{q}\vdash\mathbf{p}$, and line 2 to constitute a proof that **¬p**,**¬q**⊢**¬q**. This is perfectly legitimate according to our definitions. **Γ**⊢**A** does not require that there be a derivation in which **A** rests on every member of **Γ**. (Indeed, if **Γ** were infinite there could be no such derivation.) Nevertheless, some logicians would balk, not only at this derivation, but at what is derived. Some of them might question the correctness of ∧I in contexts like that of line 4 above, and others, while accepting each step, might question the derivation as a whole. These worries about the system itself do not undermine the observation that, within the system, the derivation is legitimate.

r. **¬(p∨q)** ⊢ **¬p**.
1. ¬(p∨q) assumption
    2.1 p assumption
    2.2 p∨q .1 ∨I
3. ¬p 1-1, 2.1-2.2 ¬I

s. **¬(p∨q)** ⊢ **¬q**.
(Similar to 5.18.)

t. ⊥ ⊢ **q**.
1. ⊥ assumption [¬(¬p∨p)]
2. ¬p∨p A1
3. ¬¬q 1-1, 2-2 ¬I
4. q 3 ¬E

u. **p∨q** ⊢ **¬¬p∨q**
1. p∨q assumption
    2.1 p assumption
        2.2.1 ¬p assumption
    2.3 ¬¬p 2.1-2.1, .1-.1 ¬I
    2.4 ¬¬p∨q .3 ∨I
    3.1 q assumption
    3.2 ¬¬p∨q .1 ∨I
4. ¬¬p∨q 1, 2.1-2.4, 3.1-3.2 ∨E

v. **¬¬p**⊢**p**
1. ¬¬p assumption
2. ¬p∨p A1
3. ¬¬¬p∨p 2 (5.21) [¬¬p⊃p]
4. p 1,3 ⊃E

w. **¬p⊃⊥ ⊢ p**

1. ¬p⊃⊥ assumption
   2.1 ¬p
   2.2 ⊥ 1, 2.1 ⊃E [¬(¬p∨p)]
   2.3 ¬p∨p A1
3. ¬¬p 2.1-2.2, 2.1-2.3 ¬I
4. p 3 ¬E

x.

a. **p⊃q,q⊃r⊢p⊃r**
b. **p⊢p∨q**, **p⊢q∨p**
c. **p∧q⊢p**, **p∧q⊢q**
d. **p,q⊢p∧q**
e. **p∧q** ⊣⊢ **q∧p**
f. **p∨q∨r** ⊣⊢ **p∨(q∨r)**
g. **¬(p∨q)** ⊣⊢ **¬p∧¬q**
h. **¬(p∧q)** ⊣⊢ **¬p∨¬q**
i. **p⊃(p≡⊤)**
j. **¬p⊃(p≡⊥)**
k. ⊢**p⊃q⊃p∧q**
l. **p⊃q⊃r** ⊣⊢ **(p∧q)⊃r**
m. **¬(p∧q)** ⊣⊢ **p⊃¬q**
n. **¬p⊃¬q** ⊣⊢ **q⊃p**

<u>Exercises and problems</u>

1**[e]**. Prove Consis(Γ) iff there is some formula not derivable from Γ.
2**[e]**. Write out in full the derivation abbreviated by 5.2 and 5.5.
*3**[e]**. a. Write out in full the derivation abbreviated by 5.8. b. Give a derivation of the result in 5.19, that makes no appeal to Rep, Ded and previous results. In what ways, if any, does your derivation differ from one that might be obtained by applying a general disabbreviation procedure to 1.31.
*4**[e]**. Consider the following (ordered) list of formulas: ¬p∨p,¬p∨r, ¬q∨r, p, q, r. Turn this into a derivation in **SL** by adding annotations. Now find a second annotation of the same list in which the lines justified by assumption are the same ones as before. How many additional derivations can be made by reannotating the original list of formulas? If derivations were defined as lists of formulas, what would be a good definition for the assumptions of a derivation? What, according to this definition, would be the assumptions of the derivation comprised by the list of formulas in this exercise.
5**[e]**. Sketch a proof of theorem 5.14 by formula induction on **A** as generated from sentence letters by disjunction and negation. Compare the number of cases that must be considered in your proof with the number that were considered in the proof given in the text.
6**[e]**. Prove that if **Γ,¬A** ⊢ **B∧¬B** then **Γ**⊢**A**. Formulate a device for abbreviating derivations based on this result.
7**[e]**. Write abbreviations of derivations to establish each of the results in 5.24. You may cite any

of the numbered derivations in the text and as well as solutions to exercises already done.

8**[e]**. Prove that theorem 5.15 implies theorem 5.14. (Hint: Suppose that theorem 5.15 holds and that B is provably equivalent to B'. By strong replacement $\vdash \mathbf{(B\equiv B')\supset(A\equiv A')}$. Use the deduction theorem and other facts to show $\vdash\mathbf{B\equiv B'}$. By appending the derivation establishing the latter fact to that establishing the former and adding an additional step, one obtains a derivation establishing that $\vdash\mathbf{A\equiv A'}$. It remains only to show that this implies that **A** and **A'** are provably equivalent.)

9**[p]**. (Strong replacement) Prove theorem 5.15. Hint: Use induction on **A** as generated from **B** by forming negations and disjoining with arbitrary formulas.

10**[p]**. Prove the statements that result from replacing **if** in (i) and (ii) of definition 5.1 by **iff**. [Hint. It is sufficient to prove case (i). A derivation in $\mathscr{L}^+$, like a formula, has a unique unabbreviated form.]
11**[p]**. (i) We may abstract from the linear format of derivations in SL and define a derivation to be to be an arbitrary <u>set</u> of (annotated) lines. What condition on the annotations makes it possible for a derivation to have a standard linear format? Give an example of a set-derivation that is "pathological" in the sense that it contains unprovable formulas even though no line is justified by assumption. (ii) We have not been rigorous in our account of derivations in SL^+. Give a precise definition of derivations in SL^+ as sets of lines and a concomitant definition of the assumptions on which each line rests.

12**[p]**. Give a definition for abbreviation of derivations in SL^+ like the definition of abbreviation for formulas of $\mathscr{L}^+$ given in the text. First, define what it is for a derivation to <u>subdirectly</u> abbreviate a derivation. This will require separate clauses for eliminating steps justified by previous derivation, by Ded and by Rep. Use this notion to define direct abbreviation and abbreviation.

13**[p]**. Prove that every abbreviation of a derivation in $\mathbf{SL}^+$ is an abbreviation of exactly one such derivation.

14**[p]**. (Negation normal form.) A formula is in <u>negation normal form</u> if every negation sign it contains immediately precedes a sentence letter. Let $\mathscr{L}^*$ be the language built from the sentence letters and the truth-functional connectives $\neg$, $\vee$, and $\wedge$. (a) Show (without using completeness) that every formula of $\mathscr{L}^*$ is provably equivalent to a formula of $\mathscr{L}^*$ in negation normal form. Hint: Define a well-founded relation of "having narrower negations" so that sentence letters and their negations are minimal elements and the negations of the disjuncts and conjuncts of **A** have narrower negations than does the negation of **A** itself. Proceed by induction on this relation. (See appendix for a discussion of this kind of argument.) (b) Deduce from (a) that every formula of $\mathscr{L}^+$ is provably equivalent to a formula of $\mathscr{L}^+$ in negation normal form. (c) Prove that $\neg\mathbf{(p\vee q)}$ does not have a provably equivalent negation normal form in $\mathscr{L}$. Hint: Find a simple semantic property that is shared by all formulas in negation normal form, but not by $\neg\mathbf{(p\vee q)}$.

15**[p]**. (Disjunctive and conjunctive normal forms.) Let $\mathbf{Q_1},\ldots,\mathbf{Q_n}$, $n\geq 0$, be a list of sentence letters. A <u>state description</u> in $\mathbf{Q_1,...,Q_n}$ is a conjunction $\mathbf{B_1 \wedge ... \wedge B_n}$ such that each $\mathbf{B_i}$ is either $\mathbf{Q_i}$ or $\neg\mathbf{Q_i}$ for i=1,...,n. A <u>disjunctive normal form in $\mathbf{Q_1,...Q_n}$</u> is a disjunction of state descriptions in $\mathbf{Q_1,...,Q_n}$ with no repetitions. (Recall that a conjunction or disjunction of a single formula is to be taken as that formula itself. Take a conjunction of zero formulas to be $\top$ and a disjunction of zero formulas to be $\bot$.) a. Without using completeness, show that every formula **A** is provably equivalent to some disjunctive normal form in $\mathbf{Q_1,...,Q_n}$. (We call such a formula a disjunctive normal form of **A**.) Hint: First put **A** into the negation normal form described in problem 10. Then define a well-founded relation on formulas in negation normal form of "having narrower conjunctions" and proceed by induction on this relation. b. Formulate and prove a "dual" result to a to the effect that every formula is provably equivalent to a conjunctive normal form. Hint: Use the fact that A is provably equivalent to the negation of the disjunctive normal form of ¬A.

16**[p]**. (Alternate proof of conjunctive and disjunctive normal forms.) Prove by formula induction on **A** that **A** is provably equivalent to a conjunctive normal form and a disjunctive normal form. Hint: Show that **¬A** is provably equivalent to a disjunctive normal form if **A** is provably equivalent to a conjunctive normal form and that **¬A** is provably equivalent to a conjunctive normal form if **A** is provably equivalent to a disjunctive normal form. Use this information and the fact that $\mathbf{A \vee B}$ is provably equivalent to $\mathbf{\neg(\neg A \wedge \neg B)}$ to prove that the property of having provably equivalent conjunctive and disjunctive normal forms is preserved under disjunctions.

17**[e]** Give natural deduction proofs of A1-A5.

18**[p]**. (Natural deduction) i)Prove theorem 5.16.
ii) Prove that the following "structural" rules hold of derivability in SL:

reflexivity. $\mathbf{A \vdash A}$
expansion. If $\mathbf{\Gamma \vdash A}$ then $\mathbf{\Gamma \cup \Delta \vdash A}$.
cut. If $\mathbf{\Gamma \vdash A}$ and $\mathbf{A, \Delta \vdash B}$ then $\mathbf{\Gamma \cup \Delta \vdash B}$.

19**[p]**. (Sequent calculus). Let $\Rightarrow$ be a binary relation on sets of formulas. Consider the following conditions on $\Rightarrow$:

reflexivity. $\mathbf{A \Rightarrow A}$.
expansion. If $\mathbf{\Gamma \Rightarrow \Delta}$ then $\mathbf{\Gamma \cup \Gamma' \Rightarrow \Delta \cup \Delta'}$.
cut. If $\mathbf{\Gamma \Rightarrow A, \Delta}$ and $\mathbf{A, \Gamma' \Rightarrow \Delta'}$ then $\mathbf{\Gamma \cup \Gamma' \Rightarrow \Delta \cup \Delta'}$.
$\vee$left. If $\mathbf{\Gamma, A \Rightarrow \Delta}$ and $\mathbf{\Gamma', B \Rightarrow \Delta'}$ then $\mathbf{\Gamma \cup \Gamma', A \vee B \Rightarrow \Delta \cup \Delta'}$.
$\vee$right. If $\mathbf{\Gamma \Rightarrow A, \Delta}$ then $\mathbf{\Gamma \Rightarrow A \vee B, \Delta}$;
If $\mathbf{\Gamma \Rightarrow B, \Delta}$ then $\mathbf{\Gamma \Rightarrow A \vee B, \Delta}$.
$\neg$left. If $\mathbf{\Gamma \Rightarrow A, \Delta}$ then $\mathbf{\Gamma, \neg A \Rightarrow \Delta}$.
$\neg$right. If $\mathbf{\Gamma, A \Rightarrow \Delta}$ then $\mathbf{\Gamma \Rightarrow \neg A, \Delta}$.
$\wedge$left. If $\mathbf{\Gamma, A \Rightarrow \Delta}$ then $\mathbf{\Gamma, A \wedge B \Rightarrow \Delta}$.
If $\mathbf{\Gamma, B \Rightarrow \Delta}$ then $\mathbf{\Gamma \Rightarrow A \wedge B, \Delta}$.
$\wedge$right. If $\mathbf{\Gamma \Rightarrow A, \Delta}$ and $\mathbf{\Gamma' \Rightarrow B, \Delta'}$ then $\mathbf{\Gamma \cup \Gamma' \Rightarrow A \wedge B, \Delta \cup \Delta'}$.
$\supset$left. If $\mathbf{\Gamma \Rightarrow A, \Delta}$ and $\mathbf{\Gamma', B \Rightarrow \Delta'}$ then $\mathbf{\Gamma \cup \Gamma', A \supset B \Rightarrow \Delta \cup \Delta'}$
$\supset$right. If $\mathbf{\Gamma, A \Rightarrow B, \Delta}$ then $\mathbf{\Gamma \Rightarrow A \supset B, \Delta}$.

a. Suppose $\vdash$ is a relation between sets of formulas and formulas satisfying the conditions of theorem 5.16 and problem 18, part ii. Let $\Rightarrow$ be defined: $\mathbf{\Gamma \Rightarrow \Delta}$ iff $\mathbf{\Gamma \vdash (D_1 \vee ... \vee D_n)}$ for some

$\mathbf{D_1},\ldots,\mathbf{D_n}$ which are members of Δ. Prove that $\Rightarrow$ satisfies each of the above conditions.

b. Suppose $\Rightarrow$ satisfies the conditions above. Prove that if $\mathbf{D_1} \vee \ldots \vee \mathbf{D_n}$ is derivable from Γ in **SL** then $\Gamma \Rightarrow \{\mathbf{D_1},\ldots,\mathbf{D_n}\}$.

c. Let **SLD** be the relation of derivability in **SL**; let **NDD** be the intersection of all relations satisfying the conditions listed in theorem 5.16 and problem 18, part ii; let **SCD** be the intersection of all relations satisfying the conditions listed in this problem; and let **SC1** be the relation that holds between Γ and **D** iff **SC** holds between Γ and **{D}**. Using the results of problem 18 and parts (a) and (b) of this problem, show that **SLD**, **NDD** and **SC1** are identical. Thus the conditions of theorem 5.16 and problem 18, part ii (or those listed above in the statement of this problem) could have been used to characterize derivability in sentential logic.