

Chapter 4. Logical Notions

This chapter introduces various logical notions and investigates the relationship among them.

The first of these notions derives from logic's traditional concern with the assessment of arguments. An argument characteristically rests upon certain assumptions and leads to a certain conclusion. It is normally intended, in putting forward an argument, that the conclusion should follow from the assumptions. Accordingly, we say that an argument is intuitively valid if the conclusion does, in this intuitive sense, follow from the assumptions.

Two qualifications are in order. In any presentation of an argument, some of the assumptions will be explicit, but others may be implicit. For example in the argument from **All men are mortal** to **Socrates is mortal**, there is an implicit premiss to the effect that Socrates is a man. We shall assume, in the application of the notion of validity, that all implicit premisses have been made explicit.

Second, an argument may be either deductive or inductive. In a deductive argument, the conclusion is meant to follow conclusively from the premisses, while in an inductive argument something less is required. So the argument from the evidence in a trial to the guilt of the defendant may be inductive rather than deductive in character. We shall assume, in the application of the notion of validity, that all of the arguments are deductive.

It is sometimes difficult to say what the implicit premisses are in a particular case and hence to identify the argument under consideration. It might even be denied that there are any legitimate inductive arguments on the grounds that, once all of the implicit premisses are made explicit, such arguments will be seen to be deductive in nature. But these are not questions that we shall explore.

The intuitive notion of valid (deductive) argument is sometimes defined in modal terms: a valid argument is said to be one for which it is impossible that the premisses should be true and the conclusion false. This is a mistake. It may perhaps be granted that every valid argument has this modal feature (and so much may follow from their being deductive). But whether the converse holds is a matter for investigation, not stipulation.

Some arguments are logically valid, i.e., valid in virtue of their logical form. An example is the argument: **Either Santa is fish or Santa has wings; Santa is a not a fish; therefore Santa has wings**. The logical form in virtue of which it is valid is given by the scheme: **Either S or T; It is not the case that S; therefore T**. On the other hand the argument **John is a bachelor; therefore John is unmarried** is valid, but not in virtue of its logical form.

The second notion concerns not arguments, but sentences. Just as an argument may be logically valid, i.e. valid in virtue of its logical form, so may a sentence be logically true, i.e., true in virtue of its logical form. For example, the sentences **two is even or it is not the case that two is even** and **no number is both even and not even** are both logically true since they are true in virtue of their respective logical forms **P or not P** and **no P is both Q and not Q**. On the other hand, the plain sentence **two is even** is true, but not logically so. (On the platonic conception of logic, we should understand arguments as consisting of propositions rather than sentences and we should take logical truth to apply to propositions rather than sentences).

It is hard to say in general what logical form is. We take each formula of \mathcal{L} to represent a logical form. It is important, however, not to identify a formula with the logical form it represents. **p** and **q**, for example, should be taken to represent the same form, as should **(p∨q)** and **(r∨s)**. In general, two formulas will represent the same form when they have the same (concrete) instances; and it follows from this that the actual identity of the sentence letters in the formula will be irrelevant to the identity of the form. Thus as long as two formulas are "reletterings", i.e. substitution-instances of one another, their form will be the same.

The sense in which a formula "represents" a form must not be understood too strongly. Even though **p** and **q** represent the same form, **p∨p** does not represent the same form as **p∨q**. Thus, the form represented by **p∨q** cannot be regarded as the disjunction of the form represented by **p** and the form represented by **q**. More generally, formulas are not interchangeable representations of form: replacing an occurrence of a formula in **A** by another representing the same form may not preserve the form **A** represents.

Elsewhere, we shall consider languages in which the forms represented by formulas are not logical. One kind of case is illustrated by modal logics, in which the non-schematic expression, \Box , may stand in for an expression characteristic of a particular subject area. Another kind of case is perhaps illustrated by logics with term descriptions. For on certain views, these terms represent grammatical units that disappear in a more refined logical analyses of the sentences containing them.

Formulas that are just like ordinary formulas except in containing any number of meta-linguistic variables for formulas in place of sentence-letters might be called "meta-formulas." Thus an ordinary formula, such as **(p₁∨p₂)**, is a meta-formula, as is **(A∨B)** or the mixed case **(A∨p₂)**. Just as formulas of \mathcal{L} represent the forms of ordinary sentences, metaformulas can be regarded as

representing the form of \mathcal{L} -formulas. Thus $(\mathbf{p}_1 \vee \mathbf{p}_2)$ (viewed now as a meta-formula) represents a form whose only instance is the formula $(\mathbf{p}_1 \vee \mathbf{p}_2)$ itself, while $(\mathbf{A} \vee \mathbf{B})$ represents a form whose instances are all the disjunctive formulas. Of course, these formula instances will themselves have 'ordinary' sentences as instances and so the metaformulas themselves can be said indirectly to have those sentences as their instances.

Since formulas of \mathcal{L} represent logical forms, it is appropriate to talk in a derivative way of the logical validity of schematic arguments constructed from formulas, and the logical truth of the formulas themselves. A schematic argument is logically valid if it represents a form in virtue of which any "concrete" instance would be valid. Similarly, a formula is logically true if it represents a form in virtue of which any concrete instance would be true.

The concrete instances in question belong to the hybrid language \mathcal{L}_1 , mentioned in chapter 2. In case the base language \mathcal{L}_0 is English, the sentences of the hybrid language may be rendered by sentences of \mathcal{L}_0 using the relevant constructions in **or** and **not**, and these English sentences will then also be of the given logical form under an appropriate use of the constructions (and similarly for other languages). On this extended usage, the schematic argument $\mathbf{P}, \neg \mathbf{P} \vee \mathbf{Q} \therefore \mathbf{Q}$ will be logically valid and the formula $\mathbf{P} \vee \neg \mathbf{P}$ will be logically true. Moreover, under an autonomous treatment, the hybrid language would also contain \supset, \wedge , and \equiv , understood in the appropriate way. English renditions of expressions of the hybrid language then require appropriately understood constructions in **only if**, **and**, and **if and only if**.

To *interpret* logic, we render logical formulas by sentences of ordinary language. To *apply* logic, we must proceed in the other direction. We represent sentences or propositions by logical formulas, we represent arguments by (formal) argument pairs, Γ, \mathbf{P} , and we represent demonstrations and deductions by proofs and derivations. (Strictly speaking, on the nominalistic conception of logic, we represent particular *readings* of the sentences, arguments, proofs and deductions, but for present purposes we may ignore that distinction.) The notion of representation involved may depend on our aims in applying logic. Let us suppose, for example, that we are interested in determining whether arguments are logically valid. We may say that $\{\mathbf{A}_1, \dots, \mathbf{A}_n\}, \mathbf{A}_0$ directly represents the argument from $\{S_1, \dots, S_n\}$ to S_0 if there is a rendition that assigns S_i (for $0 \leq i \leq n$) to A_i . (We sometimes call the simple rendition involved the "key" for the representation.) Direct representation allows logic to be applied only to arguments formulated in the rather stilted version of ordinary language into which formulas are rendered. In order to extend the reach of logic we may try to

paraphrase an argument of more normal language into a form that can be represented. For example, **Penelope is either faithful or optimistic; she isn't optimistic; so she is faithful** might be paraphrased as **It is the case that Penelope is faithful or that Penelope is optimistic. It is not the case that Penelope is optimistic. Therefore Penelope is faithful**, which can be represented by $\{p \vee q, \neg q\}, p$. We take the paraphrase to be equivalent, for our purposes, to the original, and we therefore accept an adequate representation of the paraphrase as an adequate representation of the original. There is some question as to how much leeway is permissible in paraphrase. On the platonist conception of logic it would be reasonable to require that the original sentences and their paraphrases express the same propositions. On the nominalist conception, however, the tightness required of the paraphrase may depend on the purposes at hand.

If we want to determine or explain, not merely *whether* an argument is logically valid, but *why* it is so, we might require that it and its paraphrase be both logically equivalent and of the same logical form. The original argument will then be valid in virtue of its logical form just case its paraphrase is valid in virtue of that same logical form. This standard allows "stylistic" changes. For example, pronouns can be replaced by the names they stand in for, as above, and clauses separated by semicolons can be replaced by sentences separated by periods. It does not allow paraphrasing **Penelope isn't optimistic** as **It is not the case that Penelope is optimistic**. For no form (logical or otherwise) has both of these sentences as instances. Similarly, it forbids replacing the simple term **bachelor**, with the conjunctive **unmarried male**, even if their meaning is the same.

If our interest is merely in determining whether a particular argument is logically valid, we may allow the paraphrased sentences to differ in logical form from the originals, provided the they are logically equivalent. That standard would permit paraphrasing **Penelope isn't optimistic** as **it is not the case that Penelope is optimistic**, for (at least on one reading) the sentence with the internal negation is logically equivalent to the one with the external negation. It might also permit us to paraphrase **Socrates is wise unless the Oracle is ignorant** as **It is not the case that the Oracle is ignorant only if Socrates is wise**, for the two sentences, though differing in form, appear to be logically equivalent.

Finally, if our interest is in determining whether an argument is formally valid, we may allow paraphrases that are only formally, and not logically, equivalent to the original. For example, in showing that the argument **There are**

at least two cats, so there is at least one cat is formally valid the logician may paraphrase **there are at least two cats** as **there is an x such that there is a y such that x is a cat and y is cat and it is not the case that x is identical to y**. The numerical sentence and its "identity" paraphrase are equivalent in virtue of their forms, but not in virtue of any *logical* forms.

On any of these sorts of applications, a good paraphrase will not depart from the original by more than is necessary for useful representation. For example, if our intention were to use SL to determine the logical validity of particular arguments, then it would not be appropriate to paraphrase **Socrates is wise** by **It is not the case that it is not the case that Socrates is wise**, or vice versa, even though the two sentences are logically equivalent.

Given a logical system, we may paraphrase ordinary sentences to the extent needed to represent them, noting that the looseness of the paraphrase limits the uses to which the representation may be put. In creating a logical system, we might wish to make it possible to represent many sentences with little paraphrasing. There is a constraint, however. One might say that logic has two aims: to formalize, i.e., to represent the forms of arguments and to systematize, i.e., to reduce the number of forms in virtue of which arguments are valid. In minimizing paraphrase, we would widen the scope of formalization at the cost of reducing systematization. The ideal logical system would be one that achieves the ideal balance between these competing aims.

The notion of an ideal logical system can be used to further clarify the notion of logical form. We might say that **A** directly represents the logical form of **S** when **A** directly represents **S**. Suppose, however, that no logical formula directly represents **S**. Then we might think that the logical form of **S** is (indirectly) represented by **A** if **A** directly represents **S'**, where **S'** is the closest paraphrase of **S** that can be directly represented by a formula. But the notion of closest representable paraphrase depends on the resources of the logical system. For example, in SL, the closest paraphrase to **Penelope isn't optimistic** that can be represented is **It is not the case that Penelope is optimistic**, whereas in a logical system with internal (i.e., predicate-modifying) negation, the closest paraphrase might be **Penelope is not optimistic**. The "genuine" logical form of an ordinary sentence **S** is the form represented by the formula that represents **S** in the ideal logical system. (Note that this remark does not constitute a reductive *analysis* of the notion of logical form, for whether **S'** is the closest representable paraphrase of **S** may already depend on whether **S'** has the same logical form as **S**.)

On the platonist conception of logic, this clarification is unnecessary, for

on that conception it is propositions, rather than sentences, to which the notion of form applies directly. The platonist can take the logical form of a sentence to be the logical form of the proposition expressed.

The idea that the appropriate logical representation of sentences depends on the needs at hand remains true even when the logical system and the representable paraphrase are fixed. For example, \mathbf{P} , $\mathbf{P} \vee \mathbf{Q}$ and $\mathbf{P} \vee \neg \mathbf{Q}$ all directly represent forms of **either Penelope is faithful or it is not the case that Penelope is pessimistic**. We can regard each of the three formulas as representing, in successively greater detail, a logical form of the English sentence. If our aim is to show that, or explain why, an argument is valid, there is no point in representing more detail than is necessary to do so. If an argument is not logically valid, however, we should represent as much of the logical form as we can.

The account of logical validity and of logical truth that has been given is of course still informal even though the application of these notions is to certain formal objects, schematic arguments in the one case and formulas in the other. It is therefore natural to attempt to give a direct and formal account of how these notions operate within the language \mathcal{L} . This is standardly done as follows.

It is said that \mathbf{A} is a truth-functional consequence of set Γ of formulas (or that Γ truth-functionally implies \mathbf{A} or that Γ, \mathbf{A} is a truth-functionally valid argument pair) if there is no valuation that makes all the formulas of Γ true and \mathbf{A} false. For example, $\neg(\mathbf{p} \vee \mathbf{r})$ is a truth-functional consequence of the set $\{\neg \mathbf{p}, \neg \mathbf{r}\}$. For suppose that $\alpha \models \neg \mathbf{p}$ and $\alpha \models \neg \mathbf{r}$. Then $\mathbf{p} \notin \alpha$ and $\mathbf{r} \notin \alpha$ by clauses (i) and (ii) of the truth definition; and so $\alpha \models \neg(\mathbf{p} \vee \mathbf{r})$ by the example after the truth definition. We say that \mathbf{A} is a truth-functional consequence of the formula \mathbf{B} (or that the formula \mathbf{B} truth-functionally implies \mathbf{A}) if \mathbf{A} is a truth-functional consequence of $\{\mathbf{B}\}$. Note that our definition implies that, if there is no valuation that makes all the formulas of Γ true, then Γ truth-functionally implies any formula. Thus \mathbf{p}_5 is a logical consequence of $\{\mathbf{p}, \neg \mathbf{p}\}$. We extend the use of the symbol ' \models ' to include truth-functional consequence. If Γ is a set of formulas and \mathbf{A} is a formula then, $\alpha \models \Gamma$ means that every formula in Γ is true in α . $\Gamma \models \mathbf{A}$ means that \mathbf{A} is a truth-functional consequence of Γ and $\Gamma \not\models \mathbf{A}$ means that \mathbf{A} is not a

truth-functional consequence of Γ . The new definition, then, can be rewritten as follows.

definition a. $\Gamma \models A$ iff, for any valuation α , if $\alpha \models \Gamma$ then $\alpha \models A$.

Corresponding to the intuitive notion of logical truth is the notion of truth-functional validity. Say that a formula A is truth-functionally valid if it is true in any valuation. We write $\models A$ or $\not\models A$ to indicate that A is or is not truth-functionally valid. Truth-functionally valid formulas are sometimes said to be tautologies. For example, $p \supset p$ is a tautology, but $\neg(p \supset \neg p)$ is not because it is falsified by any valuation that does not contain p .

definition b. $\models A$ iff, for any valuation α , $\alpha \models A$

' \models ' is now being used for three slightly different concepts: truth in a valuation, truth-functional consequence, and truth-functional validity. Understanding its meaning in a given context requires knowing what the symbol to its left denotes. Since we use different notation for sets of formulas and valuations there should be no confusion. Note also that definitions 4.1 and 4.2 mesh in that $\phi \models A$ according to definition 4.1 iff $\models A$ according to definition 4.2. $\Gamma \cup \{A_1, \dots, A_n\} \models B$ will usually be written as $\Gamma, A_1, \dots, A_n \models B$, and, if $\Gamma = \phi$, simply as $A_1, \dots, A_n \models B$. Thus, if $\Gamma = \phi$ and $n=0$, the notations of definitions 4.1 and 4.2 coincide. (We remind the reader who is unfamiliar with the symbols ' ϕ ' '{...}' and ' \cup ' to consult the appendix on set theory.)

The wary reader will notice that it is not evident that the formal notions are faithful to their informal counterparts. The question of the adequacy of the definitions of truth-functional validity and of truth-functional consequence will be taken up in chapter 5.

The following result connects consequence and the conditional.

Theorem c. $\Gamma, A \models B$ iff $\Gamma \models (A \supset B)$

Proof $\Gamma, A \models B$ iff for any valuation in which Γ is true either A is not true or B is true. But by Theorem 3.8 (vi), this holds iff $(A \supset B)$ is true in any valuation in which Γ is true, i.e., iff $\Gamma \models (A \supset B)$.

By applying theorem 4.3 repeatedly until Γ is empty we obtain the following result, which connects the notions of consequence and tautology.

Corollary. $A_1, \dots, A_n \models A_{n+1}$ iff $\models A_1 \supset \dots \supset A_{n+1}$.

(Recall that conditional--or a conjunction or a disjunction--of only one formula is to be understood as the formula itself.)

There are at least two other logical notions of interest. A set Γ of formulas is said to be truth-functionally satisfiable if there is some valuation α in which all the formulas of Γ are true. For example, $\{p_1, \neg p_2, p_3, \neg p_4, \dots\}$ is truth functionally satisfiable because it is verified by $\{p_1, p_3, \dots\}$ but $\{p \vee q, \neg p, \neg q\}$ is not truth functionally satisfiable.

Definition d. Γ is truth-functionally satisfiable if $\alpha \models \Gamma$ for some valuation α .

There are the following connections between satisfiability and consequence.

Theorem e. (i) $\Gamma \models A$ iff $\Gamma \cup \{\neg A\}$ is not satisfiable;
(ii) Γ is satisfiable iff $\Gamma \not\models \perp$.

Proof (i) The following sentences are equivalent: $\Gamma \models A$; $\alpha \models A$ whenever $\alpha \models \Gamma$ (by definition 4.1); there is no α such that $\alpha \models \Gamma$ and $\alpha \models \neg A$ (by the truth definition); $\Gamma \cup \{\neg A\}$ is not satisfiable (by definition 4.4).

(ii) The proof of this is left as an exercise.

Formulas A and B are said to be truth-functionally equivalent or semantically equivalent ($A \dashv\vdash B$) if they are true under exactly the same valuations. In other words:

definition f. $A \dashv\vdash B$ if $A \models B$ and $B \models A$

Just as theorem 4.3 connects consequence with the conditional the following result connects equivalence with the biconditional.

Theorem g. $A \dashv\vdash B$ iff $\models A \equiv B$

This follows directly from theorem 4.3. The details are left as an exercise.

Drills, exercises and problems

1[e]. Prove part (ii) of theorem 4.5.

2[d]. Say whether the following are true or false and give reasons.

a. $((p \supset p) \supset p) \supset p$ is a tautology.

b. $\neg(((p \supset p) \supset p) \supset p)$ is truth functionally satisfiable.

c. $p \models p \vee r$

d. $p \models r \supset p$

e. $\models p \vee (\neg p \wedge p)$

f. $A \vee \neg A$ is a tautology.

g. $A \vee A$ is truth-functionally satisfiable.

h. $\Gamma, A \models A$

i. If $\Gamma, \neg A \models B$ then $\Gamma, \neg B \models \neg A$

j. $p \supset \neg p$ has a substitution instance that is truth functionally valid.

k. $p \equiv \neg p$ has a substitution instance that is truth functionally valid.
3[e]. Prove theorem 4.7

4[d]. For each of the following formulas, specify a simple rendition, and state the sentence rendered by the formula. Then provide the closest "natural" English paraphrase you can find that preserves logical equivalence.

a. $\neg p$

b. $(p \vee \neg \neg p)$

c. $q \vee (\neg p \vee q)$

d. $((p \vee q) \vee (r \vee p))$

5[e]. Give two representations in \mathcal{L} for the following arguments, and specify the keys. Make the first as detailed as you can, and the second no more detailed than necessary to show that the argument is valid.

a. Helen was not unattractive. Therefore, Helen was only unattractive if she was attractive.

b. Achilles and Patroclus are Greek heroes. Patroclus kills Hector if neither Achilles nor Agamemnon does. If Patroclus kills Hector then Achilles is no hero. Thus, either Achilles kills Hector or Patroclus does.

c. Hector, unlike Helen, Achilles, Patroclus, or Homer himself, was born in Troy. Therefore Achilles was not born in Troy.

6[e]. Give an example to show that there is not always a unique "minimally detailed" way to represent the form of a logically true sentence as a tautology.