

Chapter 2. Semantics

This chapter complements the previous chapter by providing a semantics for the language L . We shall provide a semantics for this, and for the other languages we consider, in two somewhat different ways - one informal and the other formal. On the informal approach, we simply explain the use of the terms of the language under consideration in much the same kind of way that we might explain the use of terms of ordinary language. On the formal approach, we provide a rigorous account of the conditions under which the sentences of the language are to be deemed true or false.

One informal account, in the present case, might go as follows. The sentence letters are to stand in for declarative sentences, i.e., for sentences that are true or false. (Later we shall have reason to question the assumption that all of the stand-in sentences must be true or false). In order for them to be true or false, they must belong to a language, such as English, that is already understood. The connective \vee is to be understood in the same way as **or** in English or **oder** in German, i.e. disjunctively; and the connective \neg is to be understood in the same way as **not** in English and **nicht** in German, i.e. as a negation. Because of certain ambiguities in the use of **or** and **not**, it would be more accurate to say that $(S\vee T)$, for any declarative sentences S and T , is to be understood in the same way as **it is the case that S or that T** and that $\neg S$, for any declarative sentence S , is to be understood in the same way as **it is not the case that S** . However, even this may not be accurate enough, since the ordinary language construction in **or** may have several uses and it is our intention that \vee should only correspond in sense to one of them. This special use is illustrated by the sentence **it is the case that the number of planets is less than ten or that the number of planets is greater than eight**. It corresponds to the so-called inclusive use, which permits the possibility that either one or both of the clauses connected by the **or** are true. By associating expressions of ordinary language, as used in the appropriate way, with the symbols of L , the informal semantics allows one to interpret L -formulas by means of sentences of ordinary-language.

This account of the informal semantics can itself be made more precise. We suppose that we are given a language L_0 , such as English, which is already interpreted. By a rendition of L in L_0 is meant an assignment of sentences from L_0 to sentence letters of L . A rendition of L in L_0 can be extended to a function that assigns to each expression E of L to whose sentence letters the rendition makes an assignment the "hybrid" expression obtained by replacing each sentence letter in E by the assigned sentence. The resulting expressions, under the different renditions, form a language L_1 that combines the logical means of expression in L with the "ordinary" means of expression in L_0 . When it is necessary to distinguish them, we shall label the initial assignment of sentences to sentence letters a simple rendition, and the resulting assignment of L_1 -expressions to L -expressions an extended rendition. An expression of L_1 that is assigned to an expression E

of L by means of a (simple or extended) rendition is said to be rendering of or to be rendered by E (relative to the rendition). Renderings of formulas are said to be (hybrid) sentences. Note that the hybrid language L_1 may itself be used to provide a rendition of L . However, when this is done, the resulting sentences are the same as those of L_1 .

The interpretation of L_0 can be extended in the obvious way to L_1 by taking advantage of our understanding of \neg and \vee . Indeed, if L_0 already contains the means for expressing negation and disjunction, the sentences of L_1 can be rendered (in an indirect sense) by sentences of L_0 . For example, in the case of English, sentences of the form $\neg S$ and $(S \vee T)$ can be rendered by the sentences **it is not the case that S** and **it is either the case that S or that T** , respectively.

It is possible to provide a somewhat different informal account, according to which sentence letters are not regarded as standing in for sentences, but rather as standing for propositions. Propositions are understood to be the kinds of things that are *expressed* by declarative sentences, but it need not be supposed that any language is capable of expressing all propositions or even that every proposition is expressible in some language or other. On this approach, negation and disjunction are regarded as *operations* on propositions, the first transforming a proposition into its negation and the second transforming a pair of propositions into their disjunction.

The choice between the two informal accounts reflects a general issue that will come up in other contexts. The first approach can be regarded as embodying a *nominalistic* conception of logic, and the second a *platonistic* conception. Those who favor a platonistic conception may speak of "propositional variables" where we speak of "sentence letters"; and they take logic to be about the nonlinguistic realm of objects, properties and propositions rather than about the linguistic realm of names, predicates and sentences. Although the platonistic conception has much to recommend it, we shall, for the most part, adopt the nominalistic conception. On topics where the difference is important (which include the nature of derivations and the issue of whether names denote) we will sometimes consider both conceptions. This will be the case, for example, when we discuss how derivations are to be thought of and whether names in predicate logic always denote.

We turn to the formal account of the semantics. This provides truth-conditions for the formulas of L . Now a formula by itself is neither true nor false; for even given our informal understanding of the connectives, the truth or falsity of a formula will in general depend upon the identity of its constituents. So the truth-conditions should be given relative to some information about the constituents.

But what information? Note that the connectives are truth-functional in the sense that the truth value (truth or falsity) of a negation or disjunction depends only upon the truth values of its immediate components. To be specific, if S and T are any English sentences, then: **it is not the case that S** is true if and only if S is not true; **and it is the**

case that S or that T is true iff S and T are not both false. Therefore the truth or falsity of any formula can be determined from the truth value of its sentence letters.

Let us make these ideas precise. We first of all need to know which constituents are true and which false. This information is embodied in a valuation α which is taken to be a set of constituents: membership of a constituent in α is taken to indicate that the constituent is true and non-membership is taken to indicate that it is false. We then need to ascertain, given the information embodied in a valuation, which formulas are true and which false. This further information is embodied in the truth and falsity predicates \models and $\not\models$, which are taken to be relations between valuations and formulas. If α is a valuation, $\alpha \models \mathbf{A}$ is taken to indicate that \mathbf{A} is true given α , i.e., that α verifies \mathbf{A} , and $\alpha \not\models \mathbf{A}$ is taken to indicate that \mathbf{A} is false given α , i.e., that α falsifies \mathbf{A} . ' \models ' and ' $\not\models$ ' are defined by the following clauses:

definition a. (i) for \mathbf{p}_i a constituent, $\alpha \models \mathbf{p}_i$ if $\mathbf{p}_i \in \alpha$, and $\alpha \not\models \mathbf{p}_i$ if $\mathbf{p}_i \notin \alpha$;
 (ii) $\alpha \models \neg \mathbf{A}$ if $\alpha \not\models \mathbf{A}$, and $\alpha \not\models \neg \mathbf{A}$ if $\alpha \models \mathbf{A}$;
 (iii) $\alpha \models (\mathbf{A} \vee \mathbf{B})$ if either $\alpha \models \mathbf{A}$ or $\alpha \models \mathbf{B}$, and $\alpha \not\models (\mathbf{A} \vee \mathbf{B})$ if $\alpha \not\models \mathbf{A}$
 and $\alpha \not\models \mathbf{B}$.

This formulation of the truth definition for \mathbf{L} invokes some conventions that will be used throughout the book. ' \in ' and ' \notin ' are used for **is a member of** and **is not a member of**. **or** is used in the inclusive sense: **S or T** is true if at least one of S and T is true. (We use **or else** to indicate the exclusive sense). Note that it is for constituents, and not for formulas in general, that truth in a valuation corresponds to membership in the valuation.

It seems clear that these clauses uniquely determine the truth value of each formula relative to a valuation α . For by following the construction of a formula according to the formation rules (i)-(iii), one can determine whether it is true or false relative to α by applying the corresponding clauses (i)-(iii) in the definition of \models above. For example, suppose that $\mathbf{p}_1 \notin \alpha$ and $\mathbf{p}_2 \notin \alpha$. Then $\alpha \not\models \mathbf{p}_1$ and $\alpha \not\models \mathbf{p}_2$ by (i); so $\alpha \not\models (\mathbf{p}_1 \vee \mathbf{p}_2)$ by (iii). By (ii) $\alpha \models \neg(\mathbf{p}_1 \vee \mathbf{p}_2)$ and $\alpha \models \neg \mathbf{p}_1$ so, by (iii) again, $\alpha \models (\neg(\mathbf{p}_1 \vee \mathbf{p}_2) \vee \neg \mathbf{p}_1)$. The general statement of the result is as follows:

Lemma b. For any formula \mathbf{A} and valuation α , $\alpha \models \mathbf{A}$ or else $\alpha \not\models \mathbf{A}$.

A rigorous proof of this claim requires the unique readability theorem (see exercise 1.21).

The truth definition above is equivalent to the standard one in terms of assignments or truth tables. A valuation corresponds to an assignment of \mathbf{T} to the sentence letters that belong to it and of \mathbf{F} to those that do not. A formula is then true for a valuation iff it receives \mathbf{T} for the corresponding assignment.

Although the informal and formal accounts of the semantics are different, the

latter respects the former. For suppose that we are given a rendition of L (i.e. an assignment of sentences from some interpreted language L_1) to the sentence letters of L . α is said to be the intended valuation relative to that rendition if, for any sentence letter p , p is a member of α iff the sentence rendered by p is true. (For example, if the translations of p_1 and p_2 are **snow is white** and **pigs fly**, p_1 is a member of α but p_2 is not.) Then for any formula A , $\alpha \models A$ holds iff the sentence rendered by A is true. The rendition delivers the same truths as the corresponding valuation. Similarly in case propositions, rather than sentences, are assigned to the sentence letters.

The truth definition ensures that the truth-value of a formula depends solely on the truth-values of its constituents. More precisely, the following result obtains:

Theorem c. Let $\underline{C}_1, \dots, \underline{C}_n$ be a list of all the occurrences of the constituents of A (we may take them in the order in which they appear), and let A' be the result of replacing each constituent occurrence \underline{C}_i in A by a possibly different constituent C_i' . Suppose $C_i \in \alpha$ iff $C_i' \in \alpha'$ for $1 \leq i \leq n$. Then $\alpha \models A$ iff $\alpha' \models A'$.

This theorem can be proved in a straightforward way by formula induction. The details are left as an exercise. Note that in the case $A=A'$ the theorem yields the following special result.

Corollary 1 If valuations α and β agree on the truth-functional constituents of A , i.e., $\alpha \cap \text{CST}(A) = \beta \cap \text{CST}(A)$, then $\alpha \models A$ iff $\beta \models A$.

In the case $\alpha=\beta$ the theorem implies a result about equivalence under replacements.

Corollary 2 Suppose A' is the result of replacing an occurrence of a constituent C by a possibly different constituent C' . Then $\alpha \models C \equiv C'$ implies $\alpha \models A \equiv A'$.

We leave as problem the task of showing that these two corollaries jointly imply the result itself.

Drills, exercises and problems

1[d]. Suppose $\alpha = \{p_1, p_3, p_5, p_7\}$. Determine whether each of the following is true or false in α .

- a. $(p_1 \vee p_2)$
- b. $\neg(\neg p_1 \vee \neg p_3)$
- c. $((p_2 \vee p_4) \vee p_6)$

2[d]. Suppose p_1 stands for **It rains in Spain** and p_2 , **It snows in Iceland**. Render the

following in ordinary language in accordance with our informal semantics.

- a. $(p_2 \vee \neg p_1)$
- b. $(p_1 \vee (p_2 \vee p_1))$

3[e]. Show that the formal semantics for **L** does not respect an informal semantics according to which \vee is understood as **because**.

4[e]. Prove theorem 2.c.

*5[e]. (syntax and semantics). Show that it is possible to determine syntax and semantics simultaneously by replacing definitions 1.a and 2.a by an inductive definition that defines truth and falsity of expressions relative to assignments. The formulas can then be defined as the expressions which have truth values under some assignments. Prove that this definition picks out the same class of expressions as definition.

6[p]. Suppose we allow formulas of **L** to be assigned, in addition to the values true and false, a third value, "undefined." For example, a sentence letter standing in for **Nano is deaf** is to be considered undefined if the name **Nano** has no reference, or if it refers to a number. a. What truth-conditions might be reasonable in this case? (Note: there are several reasonable alternatives.) b. Let a tautology be a formula that is not false under any assignment of truth values to sentence letters. Show that the set of tautologies in three-valued logic is the same as the set of tautologies in two valued logic (with only true designated) as long as (i) the truth-tables agree on the definite truth values and (ii) the three-valued semantics is stable in the sense that changing an assignment of an undefined value to a "definite" one never changes the truth value of a formula from true to false or from false to true. This result shows how our logic might serve to identify the logical non-falsehoods under a nominalistic semantics in which sentences are allowed to be neither true nor false. Its relevance to the platonist is less clear. For he can take one of two views: (i) sentences that lack a truth-value express propositions that lack a truth-value; (ii) all propositions have a truth-value and sentences that lack a truth-value fail to express any proposition at all. In the former case, there is a separate question as to the logic of non-falsehood, to which our result might be taken to provide an answer. But in the latter case, all propositions are true or false and so the question of logical non-falsehood reduces to the question of logical truth.

7[p]. Suppose we take formulas of **L** to be assigned the values N ("necessarily true") I

("impossible") and C ("contingent," i.e., possibly true and possibly false." a. Complete as much of the resulting three-valued truth tables for \neg and \vee as can be reasonably determined. b. Now add three "undetermined" truth values: NC for formulas that might have values N or C; IC for those that might have I or C; and NIC, for those that might have N,I, or C. Give reasonable six-valued truth tables for \neg and \vee for this understanding of the truth values. c. Show that, with an appropriate reinterpretation of the terms **true**, **false**, and **undefined**, conditions i and ii of the previous problem are satisfied, and so this system has the same tautologies as classical logic. But note that a "tautology" is now merely a formula that never has truth value I. For example, $\mathbf{p} \vee \neg \mathbf{p}$ is a tautology not, as we might expect, because it is always necessary, but because it takes the value N when \mathbf{p} is assigned N or I, the value NC when \mathbf{p} is assigned C or NC, and the value NIC when \mathbf{p} is assigned NIC. Hence this many-valued approach to necessity, contingency and impossibility does not seem particularly informative. A more promising approach will be considered elsewhere.

8[p]. Define precisely a transformation of formulas of \mathbf{L} into English sentences on the basis of the informal semantics of this chapter. What assumptions are required to show that the truth conditions respect the transformation?

9[p]. (Analysis of theorem). Prove 2.c from its two corollaries. [Hint: Generalize corollary 2 to include the replacement of n constituent occurrences. Pick constituents $\mathbf{D}_1, \dots, \mathbf{D}_n$ distinct from both $\mathbf{C}_1, \dots, \mathbf{C}_n$ and $\mathbf{C}'_1, \dots, \mathbf{C}'_n$. Consider the assignment that differs from α only in assigning $\alpha(\mathbf{C}_1), \dots, \alpha(\mathbf{C}_n)$ to $\mathbf{D}_1, \dots, \mathbf{D}_n$.]