

Chapter 5. Completeness

This chapter establishes the completeness of PL, i.e., the result that a formula is derivable in PL from a set of formulas iff it is a consequence of that set and, consequently, that a formula is a theorem of PL iff it is valid. Some general remarks on completeness are contained in chapter I.5.

Let us first prove soundness, i.e., that every formula derivable from Γ is a consequence of Γ . As in the case of truth-functional logic, this is equivalent to the claim that every theorem is valid. We prove the latter claim by the now familiar principle of theorem induction.

Theorem a. (soundness). If $\vdash \mathbf{A}$ then $\models \mathbf{A}$.

Basis: (i) \mathbf{A} is an instance of one of A1 - A5. By the soundness of SL, \mathbf{A} is truth-functionally valid, and hence valid.

(ii) $\mathbf{A} = \forall \mathbf{x}(\mathbf{B} \supset \mathbf{C}) \supset (\forall \mathbf{x}\mathbf{B} \supset \forall \mathbf{x}\mathbf{C})$. Let (M, δ) be an interpretation for \mathbf{A} . Suppose $\models_M \forall \mathbf{x}(\mathbf{B} \supset \mathbf{C})[\delta]$ and $\models_M \forall \mathbf{x}\mathbf{B}[\delta]$ and let d be an arbitrary element of the domain of M . Then $\models_M \mathbf{B} \supset \mathbf{C}[\delta(d/x)]$ and $\models_M \mathbf{B}[\delta(d/x)]$, and consequently $\models_M \mathbf{C}[\delta(d/x)]$. Since d was arbitrary, $\models_M \forall \mathbf{x}\mathbf{C}[\delta]$. Hence $\models_M \forall \mathbf{x}(\mathbf{B} \supset \mathbf{C}) \supset (\forall \mathbf{x}\mathbf{B} \supset \forall \mathbf{x}\mathbf{C})$. Since (M, δ) was also arbitrary, $\models \forall \mathbf{x}(\mathbf{B} \supset \mathbf{C}) \supset (\forall \mathbf{x}\mathbf{B} \supset \forall \mathbf{x}\mathbf{C})$.

(iii) $\mathbf{A} = \forall \mathbf{x}\mathbf{B}(\mathbf{x}) \supset \mathbf{B}(\mathbf{y})$. Suppose $\models_M \forall \mathbf{x}\mathbf{B}(\mathbf{x})[\delta]$ where (M, δ) is an interpretation for \mathbf{A} . Then $\models_M \mathbf{B}(\mathbf{x})[\delta(d/x)]$ for all d in the domain of M . By theorem 2.5 $\models_M \mathbf{B}(\mathbf{y})[\delta(d/y)]$ for all such d and, in particular, $\models_M \mathbf{B}(\mathbf{y})[\delta]$. Hence $\models_M (\forall \mathbf{x}\mathbf{B}(\mathbf{x}) \supset \mathbf{B}(\mathbf{y}))[\delta]$ and, since (M, δ) was arbitrary, $\models \forall \mathbf{x}\mathbf{B}(\mathbf{x}) \supset \mathbf{B}(\mathbf{y})$.

(iv) $\mathbf{A} = \mathbf{B} \supset \forall \mathbf{x}\mathbf{B}$ where x is not free in \mathbf{B} . Suppose $\models_M \mathbf{B}[\delta]$ for (M, δ) an interpretation for \mathbf{A} . Since x is not free in \mathbf{B} theorem 2.5 implies that $\models_M \mathbf{B}[\delta(d/x)]$ for any d , i.e., $\models_M \forall \mathbf{x}\mathbf{B}[\delta]$. Hence $\models_M \mathbf{B} \supset \forall \mathbf{x}\mathbf{B}$ and therefore $\models \mathbf{B} \supset \forall \mathbf{x}\mathbf{B}$.

Inductive step:

(i) \mathbf{A} is derived by modus ponens from \mathbf{B} and $\mathbf{B} \supset \mathbf{A}$. This case is easy and is left as an exercise.

(ii) $\mathbf{A} = \forall \mathbf{x}\mathbf{B}$ and is derived by Gen from the PL-theorem \mathbf{B} . Let (M, δ) be an arbitrary interpretation for \mathbf{A} and d an arbitrary member of the domain of M . By induction hypothesis, \mathbf{B} is valid, and so $\models_M \mathbf{B}[\delta(d/x)]$. Since d and (M, δ) were arbitrary, $\models \forall \mathbf{x}\mathbf{B}$.

The proof of sufficiency is harder than that for soundness, so let us begin

by reviewing the strategy. As with SL, strong sufficiency is equivalent to the condition that every consistent set of formulas is satisfiable. Suppose Γ is an arbitrary consistent set of formulas. Where Σ is the set of PL-theorems, $\Gamma' = \Gamma \cup \Sigma$ will at least be truth-functionally satisfiable. For otherwise \perp would be a truth-functional consequence of Γ' and by TFC, \perp would be derivable in PL from Γ' , contradicting the consistency of Γ' . So there is an extended sentential valuation γ under which Γ' is true. To prove sufficiency we need only to construct an interpretation (M, δ) that agrees with γ in the sense that $\models_M \mathbf{A}[\delta]$ iff $\gamma \models \mathbf{A}$. As long as δ assigns distinct objects to distinct variables, we can ensure agreement on atomic formulas by stipulating that $\cup P_i \delta(x_1), \dots, \delta(x_n)$ iff $P_i x_1 \dots x_n \in \gamma$. The simplest way to ensure that δ does assign distinct objects to each variable is just to let the domain of the model be the variables and to let $\delta(x) = x$. Since the \neg and \vee clauses in the definitions for truth under an assignment in a model and truth in a valuation coincide, the agreement between M, δ and γ will be preserved under negation and disjunction. The difficulty is to ensure that the agreement will be preserved under quantification, and especially to show that $\models_M \forall \mathbf{x} \mathbf{B}[\delta]$ implies $\gamma \models \forall \mathbf{x} \mathbf{B}$, when M, δ and γ agree on simpler formulas. Suppose this is false. Then $\gamma \not\models \forall \mathbf{x} \mathbf{B}$, i.e., $\gamma \models \exists \mathbf{x} \neg \mathbf{B}$. We would like to show that in this case \mathbf{B} is false in M under δ . If the existence of the object x for which γ verifies $\neg \mathbf{B}$ were witnessed by some variable y in the sense that $\gamma \models \neg \mathbf{B}(y/x)$, then our aim would be met. For since (M, δ) and γ agree on $\neg \mathbf{B}(y/x)$, $\not\models_M \neg \mathbf{B}(y/x)[\delta]$. By theorem 2.5, $\not\models_M \mathbf{B}[\delta']$ where δ' assigns y to x and is otherwise like δ and therefore $\not\models_M \forall \mathbf{x} \mathbf{B}[\delta]$. So the key to our proof will be to ensure that existential formulas verified by γ are witnessed by particular instances. We do this by adding to Γ a witnessing conditional $(\exists \mathbf{x}) \mathbf{A}(\mathbf{x}) \supset \mathbf{A}(\mathbf{y})$ for every existential formula $\exists \mathbf{x} \mathbf{A}(\mathbf{x})$ of $\mathcal{L}(\pi)$. If the witnessing variable \mathbf{y} already occurs in other formulas of Γ , it may not be possible to do this consistently. So to carry out this strategy there must be infinitely many new variables to serve as witnesses. For this reason our proof has two parts. First we prove that consistency implies satisfiability for sets that omit infinitely many variables. Then we prove that the general case can be reduced to this one. Let us look at the details.

Lemma *b.* (single witness lemma). If Γ is consistent and \mathbf{y} does not occur in Γ then $\Gamma \cup \{ \exists \mathbf{x} \mathbf{A}(\mathbf{x}) \supset \mathbf{A}(\mathbf{y}) \}$ is consistent.

Proof. Suppose not. Then for some formulas $\mathbf{G}_1, \dots, \mathbf{G}_n$ in Γ ,

$G_1, \dots, G_n, \exists x A(x) \supset A(y) \vdash \perp$. By the deduction theorem
 $G_1, \dots, G_n \vdash (\exists x A(x) \supset \neg A(y)) \supset \perp$. By TFC, $G_1, \dots, G_n \vdash \exists x A(x) = \neg \forall x A(x)$ and
 $G_1, \dots, G_n \vdash \neg A(y)$. By universalization, $G_1, \dots, G_n \vdash \forall x \neg A(x)$, which contradicts
the consistency of Γ .

Γ is fully witnessed if, for every existential formula $\exists x A(x)$ in $\mathcal{L}(\pi)$, Γ contains
a witnessing conditional $\exists x A(x) \supset A(y)$ for some variable y . Γ omits infinitely
many variables if there are infinitely many variables that do not occur in any of
the formulas in Γ .

Lemma *c.* (witness lemma). Suppose Γ is consistent and omits
infinitely many variables. Then there is a set $\Gamma' \supseteq \Gamma$ that is
consistent and fully witnessed.

Proof. Let Γ be a consistent set not containing y_1, y_2, \dots and let A_1, A_2, \dots be an
enumeration of all the existential formulas of $\mathcal{L}(\pi)$. (As in Part I, we leave the
proof that such an enumeration exists to the exercises.) We build Γ' from Γ in
stages. At each stage i we add a witness for A_i , using as the witnessing variable
the first "new" y_i . The end result of this process will be a fully witnessed
extension of Γ .

More precisely, let B_1, B_2, \dots be an enumeration of all witnessing
conditionals in $\mathcal{L}(\pi)$. (Again the proof that such an enumeration exists is left as
an exercise.) Let $\Gamma_0 = \Gamma$. Now suppose Γ_i has been defined, $A_i = \exists x A(x)$ and B is
the first formula in the sequence B_1, B_2, \dots with antecedent A_i and consequent
 $A(y)$ for some y that does not occur in A_1, \dots, A_i or any formula of Γ . (There must
be such a B because Γ omits infinitely many variables and at most finitely many
of these can occur in A_1, \dots, A_i .) Define Γ_{i+1} as $\Gamma_i \cup \{B\}$. Finally, let
 $\Gamma' = \sqcup \{\Gamma_i; i \in \mathbb{N}\}$ (i.e., $A \in \Gamma'$ iff $A \in \Gamma_n$ for some $n=0, 1, 2, \dots$). Obviously Γ' , so
defined, is fully witnessed and contains Γ . All that needs to be checked is that
 Γ' is consistent. Observe first that each Γ_n is consistent. Γ_0 is consistent by
assumption; and, since any witnessing variable added at stage $n+1$ does not occur
in Γ_n , the single witness lemma ensures that the consistency of Γ_n implies the
consistency of Γ_{n+1} . But if each Γ_n is consistent then Γ' must also be consistent.
For if it were not then, for some $G_1, \dots, G_n \in \Gamma'$, $G_1, \dots, G_n \vdash \perp$. Since all of the

formulas G_i are in some Γ_n , this would contradict the assumption that each Γ_n was consistent.

definition d. Let α be a sentential $\mathcal{L}(\pi)$ -valuation.

- a. α is PL-acceptable if $\alpha \models A$ for every PL-theorem A .
- b. The canonical model on α is the model $M=(D, \nu)$ where D is the set $\{v_1, v_2, \dots\}$ of object variables of $\mathcal{L}(\pi)$ and $\nu Fx_1, \dots, x_n$ iff $\alpha \models Fx_1 \dots x_n$.
- c. A canonical model for Δ is the canonical model on some PL-acceptable valuation for a fully witnessed extension of Δ .
- d. A canonical interpretation on α (for Δ) is a pair (M, δ) where M is a canonical model on α (for Δ) and δ is the identity function on the set of variables.

If Δ is consistent and omits infinitely many variables, then, by the previous lemma, Δ can be extended to a fully witnessed consistent set Δ' . By the argument in our strategy review, $\Delta' \cup \{A: \vdash A\}$ is truth-functionally satisfiable. Hence we have proved the following:

Lemma e. If Δ is consistent and omits infinitely many variables, then there is a canonical model for Δ .

To prove the special case of sufficiency, it remains only to prove that when Δ satisfies the special condition, the canonical interpretation for Δ agrees with the valuation on which it is based.

Lemma f. (agreement lemma). Suppose (M, δ) is the canonical interpretation on α . Then, for all A , $\models_M A[\delta]$ iff $\alpha \models A$.

Proof. By induction on A .

(i) If A is an atomic formula $Fx_1 \dots x_n$ then $\alpha \models A$ iff $Fx_1 \dots x_n \in \alpha$ (by the definition of truth in a valuation) iff $\nu Fx_1, \dots, x_n$ (by the definitions of M and δ) iff $\models_M Fx_1 \dots x_n[\delta]$ (by clause i of the truth definition).

(ii) $\alpha \models \neg B$ iff $\alpha \not\models B$ iff $\not\models_M B[\delta]$ (by induction hypothesis) iff $\models_M \neg B[\delta]$ (by clause ii of the truth definition).

(iii) $\alpha \models B \vee C$ iff $\alpha \models B$ or $\alpha \models C$ iff $\models_M B[\delta]$ or $\alpha \models_M C[\delta]$ (by induction hypothesis) iff $\models_M (B \vee C)[\delta]$ (by clause iii of the truth definition).

(iv) Suppose $A = \forall x B(x)$. If $\alpha \models A$ then, since α is PL-acceptable, $\alpha \models B(y)$

for all variables y . By induction hypothesis, $\models_M \mathbf{B}(y)[\delta]$ for all variables y . By theorem 2.5, $\models_M \mathbf{B}(x)[\gamma]$ for all assignments γ that are x -variants of δ . By clause (iv) of the truth definition, $\models_M \mathbf{A}[\delta]$. If $\alpha \neq \mathbf{A}$ then $\alpha \models \neg \forall x \mathbf{B}x$ and, since α is PL-acceptable, $\alpha \models \exists x \neg \mathbf{B}(x)$. Since $\alpha \models \Gamma'$ and Γ' is fully witnessed, $\alpha \models \neg \mathbf{B}(y)$ for some variable y , i.e., $\alpha \neq \mathbf{B}(y)$. By induction hypothesis $\neq_M \mathbf{B}(y)[\delta]$. By corollary 2 of 2.5 $\neq_M \forall x \mathbf{B}(x)[\delta]$.

We have now established all that is needed to prove the special case of sufficiency. Before proceeding to the general case, we briefly review this result.

Lemma g. (special-case sufficiency). If Γ is consistent and omits infinitely many variables then Γ is satisfiable.

Proof. Suppose Γ is consistent and omits infinitely many variables. By lemma 5.5 there is a canonical model for Γ , i.e., there is a PL-acceptable valuation α for a fully witnessed extension of Γ and there is a canonical model M on α . By lemma 5.6 α assigns the same truth values to formulas as does the model M under the identity assignment δ . Since α satisfies an extension of Γ , M and δ do so as well.

Lemma h. (sufficiency). If Γ is consistent then it is satisfiable.

Proof. Suppose Γ is consistent. Let 2Γ be the result of doubling the subscripts of every variable that occurs in Γ . By the corollary to 4.7 2Γ is consistent. Since the variables v_1, v_3, v_5, \dots do not occur in 2Γ , special-case sufficiency implies that 2Γ is satisfiable. By corollary 3 to 2.5, Γ is also satisfiable.

As with SL, there is a noteworthy corollary to strong completeness.

Corollary (Compactness) If Γ is finitely satisfiable, it is satisfiable.

The proof is the same as that for SL given in chapter 4. The proof of strong completeness given here yields a second important result:

Corollary (Löwenheim-Skolem theorem) If Γ is satisfiable, it is satisfiable by a countably infinite model.

Proof. Suppose Γ is satisfiable. By soundness, it is consistent. By the corollary to 4.7, 2Γ is consistent. By the proof above 2Γ is satisfied by its canonical model, whose domain is the countably infinite set $\{v_1, v_2, \dots\}$. By corollary 3 to 2.5, Γ is satisfiable by a model of the same size.

In Part I, the proof that SL is complete was followed by a proof that it is decidable. To prove that SL was decidable, we exhibited a procedure for answering the question **is A a theorem?**, which could clearly be seen to be effective. Similar proofs establishing the decidability of the monadic fragment of PL are outlined in the problems below. PL, by contrast, is undecidable. However, in order to establish the nonexistence of an effective procedure, we can no longer rely on our informal conception of an effective procedure but must submit it to a more precise characterization. Providing such a characterization would take us beyond the themes of the book. The interested reader may consult [Boolos and Jeffrey**].

Problems

1[p]. (Schematic predicate logic). We define a scheme inductively:

- (i) each of the meta-linguistic variables ' A_1 ', ' A_2 ', ... is a scheme;
- (ii) if Σ_1 and Σ_2 are schemes, then so are $\neg\Sigma_1$, $(\Sigma_1 \vee \Sigma_2)$, and $\forall x_i \Sigma_1$.

We may define the instance of a scheme in the obvious way. A (strictly) schematic formulation of predicate logic is one in which the axioms and rules are all presented in the form of schemes (an infinite number of either will be allowed) and whose theorems are theorems of PL. The (most) refined schemes of a formula may be obtained by replacing distinct atomic subformulas with distinct meta-linguistic variables. Call a formula strictly valid if all instances of any of its most refined scheme is valid. a. Show that $\forall v P v \supset P u$ is a theorem of PL but is not strictly valid and that any theorem of a schematic formulation of predicate logic is strictly valid. It follows that predicate logic cannot have a strictly schematic formulation. b. Let $PL(\Sigma)$ be the axiom system obtained from PL by replacing A7-A8 with the following:

$$A7'. \forall x A \supset A$$

$$A8'. \forall x A \supset \forall x \forall x A$$

$$A9. \neg \forall x A \supset \forall x \neg \forall x A$$

$$A10. \forall x \forall y A \supset \forall y \forall x A$$

A11. $\exists x \forall y A \supset \forall y \exists x A$

Show that $PL(\Sigma)$ axiomatizes the strictly valid formulas of PL. [Hint:***] [[I moved this from Chapter 4, since it's a completeness argument. As you can see, my original conjecture was false. With the new axioms there is a straightforward modal-logic completeness proof, but at the moment I don't see how to get from worlds and the appropriate accessibility relations to sequences and "cylindrifications". If we can't do this, I'd recommend keeping this here as an open problem—we'd be giving people some incentive to read.]]

2.[p] (elimination of nesting in monadic formulas). Let $\mathcal{L}(\pi^1)$ be the language obtained from $\mathcal{L}(\pi)$ by dropping all predicate letters except those of degree one. Say that a formula of $\mathcal{L}(\pi^1)$ is non-iterative if none of its quantifiers occurs within the scope of another quantifier. Show that any formula of $\mathcal{L}(\pi^1)$ is provably equivalent to a noniterative formula of $\mathcal{L}(\pi^1)$. [Hint: Use the fact that $\forall x B(C)$ is equivalent to $\odot \wedge B(\top) \vee (\neg C \wedge B(\perp))$, where x is not free in C to remove subformulas within the scope of $\forall x$ that do not contain x free]

3.[p](decidability of monadic predicate logic via normal forms)Monadic predicate logic can be identified with the set of valid closed formulas of $\mathcal{L}(\pi^1)$. Let Q_1, \dots, Q_n , $n \geq 0$, be a list of predicate letters of $\mathcal{L}(\pi^1)$. An object description Q_1, \dots, Q_n is a formula $(B_1 \wedge \dots \wedge B_n)$ such that each B_i is either $Q_i v$ or $\neg Q_i v$ for $i=1, \dots, n$. There are 2^n such formulas. Call them D_1, \dots, D_m . A model description in Q_1, \dots, Q_n is a conjunction $E_1 \wedge \dots \wedge E_m$ such that, for $1 \leq i \leq m$, E_i is either $\exists v D_i$ or $\neg \exists v D_i$. A disjunctive normal form in Q_1, \dots, Q_n is a disjunction of model descriptions in Q_1, \dots, Q_n with no repetitions. (Recall again that a conjunction or disjunction of a single formula is to be taken as that formula itself. Take a conjunction of zero formulas to be \top and a disjunction of zero formulas to be \perp .)

a. Show that every closed formula A of $\mathcal{L}(\pi^1)$ can be effectively transformed into an equivalent that is a disjunctive normal form in Q_1, \dots, Q_n . [Hint: By problem 2, A is equivalent to a formula B with no nested quantifiers. B can easily be transformed into an equivalent formula C whose quantifiers are all existential (and still unnested). Since there is no nesting, the scopes of distinct quantifier occurrences do not overlap. So they can all be "relettered" to the variable v . By problem 15 of chapter 5, Part I ** and the rule of replacement we can put the matrices of the quantifiers in C into disjunctive normal form. By 4.12 j, the existential quantifiers can be distributed across the disjunctions, leaving a

truth functional formula combination \mathbf{D} of object descriptions. Putting \mathbf{D} into disjunctive normal form yields a formula of the desired form.]

b. Use a to show that monadic predicate logic is decidable. [Hint: Show that a disjunctive normal form is inconsistent iff it is \perp . To test \mathbf{A} for validity, put $\neg\mathbf{A}$ in disjunctive normal form.]

4.[p](decidability of monadic predicate logic via finite model property) Let $\mathcal{L}(\pi^1)$ be as described in the previous problem, let $M=(D,\nu)$ be a model, and let $\mathbf{Q}_1, \dots, \mathbf{Q}_n$, $n \geq 0$, be a list of predicate letters of $\mathcal{L}(\pi^1)$. Say that elements d and e of D are indistinguishable in model M relative to $\mathbf{Q}_1, \dots, \mathbf{Q}_n$ if, for $1 \leq i \leq n$, $\nu\mathbf{Q}_i d$ iff $\nu\mathbf{Q}_i e$. a. Suppose that δ and δ' are M -assignments such that, for all variables \mathbf{x} , $\delta(\mathbf{x})$ and $\delta'(\mathbf{x})$ are indistinguishable relative to the predicate letters that occur in \mathbf{A} . Show that $\models_M \mathbf{A}[\delta]$ iff $\models_M \mathbf{A}[\delta']$. b. Use part a to show that monadic predicate logic has the finite model property, i.e., that every satisfiable formula is true in some model whose domain is finite. [Hint: Let M be a model such that $\models_M \mathbf{A}$. Construct a finite model from M by identifying the objects in M 's domain that are indistinguishable with respect to the predicate letters of \mathbf{A} , and show that it verifies the same formulas as M in the language of these letters.] c. Use the proof of part b to prove that monadic predicate logic is decidable. [Hint: To determine whether \mathbf{A} is valid, it is sufficient to calculate a value of n such that if $\neg\mathbf{A}$ is satisfiable it is verified by a model of at most n elements and then to check whether \mathbf{A} is verified by any such model.]