Chapter 4. An Axiom System

In this chapter we define notions of proof and derivation for $\mathscr{L}(\pi)$. Our intention is that proofs should provide demonstrations of formal truths and that derivations should provide deductions of the formal consequences of assumptions. In addition we would like our system to be rich enough for all formally true formulas of $\mathscr{L}(\pi)$ to be provable and for **A** to be derivable from $\boldsymbol{\Gamma}$ whenever the argument from $\boldsymbol{\Gamma}$ to **A** is formally valid. Of course the notions of formal truth and formal validity are not mathematical notions. In chapter 5 we take up the strictly mathematical questions of completeness, i.e., the questions of whether all valid formulas are provable and whether any consequence of a set is derivable from it. The more philosophical questions of adequacy are not taken up until chapter 6.

The system **PL** (*predicate logic*) comprises the following axioms and rules.

Axioms

A1. $\mathbf{A \supset A}$
A2. $\mathbf{(A \supset B) \supset ((B \supset C) \supset (A \supset C))}$
A3. $\mathbf{A \supset (A \vee B)}$
A4. $\mathbf{A \supset (B \vee A)}$
A5. $\mathbf{(B \supset C) \supset ((A \supset C) \supset ((A \vee B) \supset C))}$
A6. $\mathbf{\forall x(A \supset B) \supset (\forall x A \supset \forall x B)}$
A7. $\mathbf{\forall x A(x) \supset A(y)}$
A8. $\mathbf{A \supset \forall x A}$ **where x is not free in A.**

Rules

MP $\mathbf{A, A \supset B \,/\, B}$
Gen $\mathbf{A \,/\, \forall x A}$.

Schemas A1 through A5 and rule MP are the same as they were in chapter I.4. A6 is called distribution; it indicates that the universal quantifier can be distributed over conditionals. A7 is called specification; it indicates that from a universal formula one can derive any specific instance. A8 is called vacuous quantification; together with A7 it implies that quantifying with respect to a variable not free in **A** produces a formula that is interderivable with **A**. In A7, **x** and **y** can be the same variable.

There is an important contrast between the rules MP and Gen. MP preserves truth under assignments, but Gen does not. So MP can be used in derivations with the assurance that it will never lead us from something true-under-δ to something false-under-δ. But Gen cannot. We do not expect to be able to argue from the claim that **x is as large as y** is true of Jupiter and earth to the claim that **all planets are as large as y** is true of earth.

What makes it legitimate to use Gen in derivations is that, although it does not take us from truths under δ to truths under δ, it does take us from general truths under δ to general truths under δ. If **x is as large as y** is true of every planet x and earth, then **all planets are as large as y** is likewise true of every planet **x** and earth (and, therefore, true of earth *simpliciter*). The axioms of **PL** are all general truths in the sense that they are true under every assignment of objects to free variables. The rule MP preserves general truth. We may therefore allow Gen to apply to formulas provable from the axioms of **PL** without leading to falsehood. We may not, however, apply it generally to formulas derivable from assumptions if our intention is that the assumptions are evaluated relative to an assignment.

The notions of proof and derivation in **PL** are defined accordingly, with Gen having application only to formulas that are preceded by a proof in **PL**. More specifically, derivations and proofs in **PL** are the records of generation associated with the following inductive definition of <u>derivability from assumptions</u> and <u>provability</u>.

(ia) Every axiom, i.e., every instance of one of $\mathbf{A}_1$ - $\mathbf{A}_8$, is derivable from Γ.

(ib) Every member of Γ is derivable from Γ

(iia) If **B** and **C** are derivable from Γ and **A** follows from **B** and **C** by modus ponens (i.e., **B** is the conditional $\mathbf{C}\supset\mathbf{A}$), then **A** is derivable from Γ.

(iib) If **B** is derivable from ϕ and **A** follows from **B** by Gen (i.e., **A** is $\forall\mathbf{x}\mathbf{B}$) then **A** is derivable from Γ.

(iii) **A** is provable if **A** is derivable from ϕ.

Provable formulas are, as usual, <u>theorems</u>. We continue to use the notation $\Gamma\vdash\mathbf{A}$ and $\vdash\mathbf{A}$ to indicate that **A** is derivable from Γ and that **A** is a theorem, adding a subscript to $\vdash$ if the axiom system is not clear from the context. The <u>assumptions on which a line of a derivation rests</u> are, as before, the formulas comprising the base from which L is generated.

We shall establish a number of results about provability and derivability in PL. As in Part I, some of these results justify methods for abbreviating derivations. In most cases the proofs of the results provide practicable procedures for recovering the derivations from their abbreviations.

Since PL contains all the axioms and rules of SL, any uniform substitution of formulas of $\mathscr{L}(\pi)$ for formulas of $\mathscr{L}$ in a SL-derivation will constitute a PL-derivation. By the completeness theorem for SL, then, $\mathbf{A}'$ is derivable from Γ', if $\Gamma',\mathbf{A}'$ is a substitution instance of $\Gamma,\mathbf{A}$ and **A** is a tautologous consequence of Γ. By theorem 1.44***, $\Gamma',\mathbf{A}'$ is such a pair iff **A** is a truth-functional consequence of Γ. Thus we obtain established the following result stating that PL extends SL.

Theorem *a.* (truth-functional consequence). If $\Gamma\vDash_{\mathbf{tf}}\mathbf{A}$ then $\Gamma\vdash\mathbf{A}$.

Several theorems of chapter I.4 carry over in a straightforward way to the new axiom system. The argument for generalized transitivity carries over directly.

Theorem *b.* (generalized transitivity). If $\Gamma\vdash\mathbf{A}$ and $\Delta,\mathbf{A}\vdash\mathbf{B}$ then $\Gamma,\Delta\vdash\mathbf{B}$.

The deduction theorem and the replacement theorem also carry over to PL, although the proof in each case requires an extra step.

Theorem *c.* (deduction theorem). If $\Gamma,\mathbf{A}\vdash\mathbf{B}$ then $\Gamma\vdash\mathbf{A}\supset\mathbf{B}$.

<u>Proof</u>. As with theorem 1.19, the proof is by induction on the set of formulas derivable from $\Gamma\cup\{\mathbf{A}\}$. The only new case is that in which **B** is obtained from some formula **C** by rule Gen.

But in this case the derivation of $\mathbf{C}$ must contain a sublist that is a proof of $\mathbf{C}$ in SL. $\vdash\mathbf{C}$ implies $\vdash\forall\mathbf{x}\mathbf{C}$. By A4 $\vdash\neg\forall\mathbf{x}\mathbf{C}\vee(\neg\mathbf{A}\vee\forall\mathbf{x}\mathbf{C})$. By MP $\vdash\mathbf{A}\supset\forall\mathbf{x}\mathbf{C}$. This proves the new case. As in Part I, repeated application of the deduction theorem and the rule MP establishes a corollary relating consequence and the conditional.

Corollary. $\mathbf{A}_1,\dots,\mathbf{A}_n\vdash\mathbf{A}_{n+1}$ iff $\vdash\mathbf{A}_1\supset\dots\supset\mathbf{A}_{n+1}$.

Theorem d. (replacement of equivalents). Suppose $\mathbf{B}$ is a subformula of $\mathbf{A}$, $\mathbf{B}\dashv\vdash\mathbf{B}'$ and $\mathbf{A}'$ is the result of replacing an occurrence of $\mathbf{B}$ in $\mathbf{A}$ with $\mathbf{B}'$. Then $\mathbf{A}\vdash\mathbf{A}'$.

Proof. The proof is like that of theorem 5.14**. If $\mathbf{B}=\mathbf{A}$ there is nothing to prove, so we assume that $\mathbf{B}$ is distinct from $\mathbf{A}$ and proceed by induction on $\mathbf{A}$. The only new case is that in which $\mathbf{A}$ is $\forall\mathbf{x}\mathbf{C}$ for some formula $\mathbf{C}$. By induction hypothesis, $\mathbf{C}\vdash\mathbf{C}'$. By the deduction theorem, $\vdash\mathbf{C}\supset\mathbf{C}'$; by Gen, $\vdash\forall\mathbf{x}(\mathbf{C}\supset\mathbf{C}')$; by A6, $\vdash(\forall\mathbf{x}(\mathbf{C}\supset\mathbf{C}')\supset(\forall\mathbf{x}\mathbf{C}\supset\forall\mathbf{x}\mathbf{C}')$; by MP, $\vdash\forall\mathbf{x}\mathbf{C}\supset\forall\mathbf{x}\mathbf{C}'$; and so $\forall\mathbf{x}\mathbf{C}\vdash\forall\mathbf{x}\mathbf{C}'$ by MP.

The replacement theorem (together with generalized transitivity) implies that derivability of $\mathbf{A}$ from Γ is preserved when subformulas of $\mathbf{A}$ are replaced by equivalents. The replacement can occur within the scope of any connectives, including quantifier expressions, and it requires no modification to the replacement formula or to the context into which it is placed.

Next, we gather several useful results about derivability and quantifiers.

Theorem e.

(i) For n>0, $\forall\mathbf{x}(\mathbf{A}_1\supset\dots\supset\mathbf{A}_n)\vdash\forall\mathbf{x}\mathbf{A}_1\supset\dots\supset\forall\mathbf{x}\mathbf{A}_n$;
(ii) If $\Gamma\vdash\mathbf{A}$ then $\forall\mathbf{x}\Gamma\vdash\forall\mathbf{x}\mathbf{A}$;
(iii) If $\Gamma\vdash\mathbf{A}$ and $\mathbf{x}$ does not occur free in Γ then $\Gamma\vdash\forall\mathbf{x}\mathbf{A}$;
(iv) If $\Gamma\vdash\mathbf{A}(\mathbf{y})$, for $\mathbf{y}$ not free in Γ and $\mathbf{x}$ not free in $A(y)$, then $\Gamma\vdash\forall\mathbf{x}\mathbf{A}(\mathbf{x})$;
(v) If $\Gamma,\mathbf{A}(\mathbf{y})\vdash\mathbf{C}$, for $\mathbf{y}$ not free in Γ or $\mathbf{C}$ and $\mathbf{x}$ not free in $\mathbf{A}(\mathbf{y})$, then $\Gamma,\exists\mathbf{x}\mathbf{A}(\mathbf{x})\vdash\mathbf{C}$;
(vi) $\mathbf{A}({}^{\mathbf{y}}/_{\mathbf{x}})\vdash\exists\mathbf{x}\mathbf{A}$.

Proof.(i) By induction on n. If n=1 this is just the trivial result that $\forall\mathbf{x}\mathbf{A}_1\vdash\forall\mathbf{x}\mathbf{A}_1$. Suppose n>1. By A6 and MP, $\forall\mathbf{x}(\mathbf{A}_1\supset\mathbf{A}_2\supset\dots\supset\mathbf{A}_n)\vdash\forall\mathbf{x}\mathbf{A}_1\supset\forall\mathbf{x}(\mathbf{A}_2\supset\dots\supset\mathbf{A}_n)$. By MP, $\forall\mathbf{x}(\mathbf{A}_1\supset\mathbf{A2}\supset\dots\supset\mathbf{A}_n),\forall\mathbf{x}\mathbf{A}_1\vdash\forall\mathbf{x}(\mathbf{A}_2\supset\dots\supset\mathbf{A}_n)$. By induction hypothesis $\forall\mathbf{x}(\mathbf{A}_2\supset\dots\supset\mathbf{A}_n)\vdash\forall\mathbf{x}\mathbf{A}_2\supset\dots\supset\forall\mathbf{x}\mathbf{A}_n$. By generalized transitivity $\forall\mathbf{x}(\mathbf{A}_1\supset\mathbf{A}_2\supset\dots\supset\mathbf{A}_n),\forall\mathbf{x}\mathbf{A}_1\vdash\forall\mathbf{x}\mathbf{A}_2\supset\dots\supset\forall\mathbf{x}\mathbf{A}_n$. The result then follows by the deduction theorem.

(ii) Suppose $\Gamma\vdash\mathbf{A}$. Since derivations are finite any derivation of $\mathbf{A}$ from Γ is a derivation of $\mathbf{A}$ from some finite subset $\{\mathbf{A}_1,\dots,\mathbf{A}_n\}$ of Γ. By the corollary to 4.3, this implies $\vdash\mathbf{A}_1\supset\dots\supset\mathbf{A}_{n+1}$, where $\mathbf{A}_{n+1}=\mathbf{A}$. By Gen, $\vdash\forall\mathbf{x}(\mathbf{A}_1\supset\dots\supset\mathbf{A}_{n+1})$. By (i), $\vdash\forall\mathbf{x}\mathbf{A}_1\supset\dots\supset\forall\mathbf{x}\mathbf{A}_{n+1}$. By the corollary to 4.3 again, $\{\forall\mathbf{x}\mathbf{A}_1,\dots,\forall\mathbf{x}\mathbf{A}_n\}\vdash\forall\mathbf{x}\mathbf{A}_{n+1}$, which implies $\forall\mathbf{x}\Gamma\vdash\forall\mathbf{x}\mathbf{A}$.

(iii) Suppose $\Gamma\vdash\mathbf{A}$ and $\mathbf{x}$ does not occur free in Γ. By (ii), $\forall\mathbf{x}\Gamma\vdash\forall\mathbf{x}\mathbf{A}$. By A8 and generalized transitivity $\Gamma\vdash\forall\mathbf{x}\mathbf{A}$.

(iv) Suppose $\Gamma\vdash\mathbf{A}(\mathbf{y})$, for $\mathbf{y}$ not free in Γ and x not free in A(y). By (iii), $\Gamma\vdash\forall\mathbf{y}\mathbf{A}(\mathbf{y})$. By specification, $\vdash\forall\mathbf{y}\mathbf{A}(\mathbf{y})\supset\mathbf{A}(\mathbf{x})$. By MP, $\forall\mathbf{y}\mathbf{A}(\mathbf{y})\vdash\mathbf{A}(\mathbf{x})$. By (iii), $\forall\mathbf{y}\mathbf{A}(\mathbf{y})\vdash\forall\mathbf{x}\mathbf{A}(\mathbf{x})$. By transitivity,

$\Gamma\vdash\forall\mathbf{xA(x)}$.

(v) Suppose $\Gamma,\mathbf{A(y)}\vdash\mathbf{C}$, for **y** not free in Γ or **C** and **x** not free in $\mathbf{A(y)}$. By the deduction theorem, $\Gamma\vdash\mathbf{A(y)}\supset\mathbf{C}$. By truth-functional consequence (4.1), $\mathbf{A(y)}\supset\mathbf{C} \vdash \neg\mathbf{C}\supset\neg\mathbf{A(y)}$. By transitivity, $\Gamma\vdash\neg\mathbf{C}\supset\neg\mathbf{A(y)}$. By MP, $\Gamma,\neg\mathbf{C} \vdash \neg\mathbf{A(y)}$. By (iv) (since **y** does not occur free in Γ or **C** and x is not free in $\mathbf{A(y)}$), $\Gamma,\neg\mathbf{C}\vdash\forall\mathbf{x}\neg\mathbf{A(x)}$. By the deduction theorem, $\Gamma\vdash\neg\mathbf{C}\supset\forall\mathbf{x}\neg\mathbf{C}$. By truth-functional consequence and transitivity again, $\Gamma\vdash\neg\forall\mathbf{x}\neg\mathbf{A(x)}\supset\mathbf{C}$; and hence $\Gamma,\exists\mathbf{xA(x)}\vdash\mathbf{C}$ by MP.

(vi) Note that **A** may have free occurrences of both **x** and **y**, but $\mathbf{A}(^{y}/_{x})$ has no free occurrences of **x**. Let $\mathbf{A}=\mathbf{A}(\mathbf{x})$. By A7, $\vdash\forall\mathbf{x}\neg\mathbf{A}(\mathbf{x})\supset\neg\mathbf{A}(\mathbf{y})$, i.e, $\vdash\forall\mathbf{x}\neg\mathbf{A}(\mathbf{x})\supset\neg\mathbf{A}(^{y}/_{x})$. By truth-functional consequence, $\forall\mathbf{x}\neg\mathbf{A}(\mathbf{x})\supset\neg\mathbf{A}(^{y}/_{x})\vdash\mathbf{A}(^{y}/_{x})\supset\exists\mathbf{xA}(\mathbf{x})$. By transitivity, $\vdash\mathbf{A}(^{y}/_{x})\supset\exists\mathbf{xA}(\mathbf{x})$, and by MP, $\mathbf{A}(^{y}/_{x})\vdash\exists\mathbf{xA}(\mathbf{x})$.

Notice that the substitution notation used to express clause vi is different than that in the other clauses. Clause vi does imply the more perspicuous result that if $\Gamma\vdash\mathbf{B}(\mathbf{x})$ then $\Gamma\vdash\exists\mathbf{yB}(\mathbf{y})$, but it also includes the case "if $\Gamma\vdash\mathbf{Pvv}$ then $\Gamma\vdash\exists\mathbf{uPvu}$, which is not an instance of that result.

Theorem 4.5 can be used to show that alphabetical variants are interderivable.

Theorem *f.* If **A** is an alphabetical variant of **B** then $\mathbf{A}\vdash\mathbf{B}$.

Proof. By induction on **A**. If **A** is atomic, then **A** an alphabetical variant of **B** implies $\mathbf{A=B}$, which implies $\mathbf{A}\vdash\mathbf{B}$. If $\mathbf{A}=\neg\mathbf{C}$, then $\mathbf{B}=\neg\mathbf{D}$, where **D** is an alphabetical variant of **C**. By induction hypothesis $\mathbf{C}\dashv\vdash\mathbf{D}$ and, by the replacement theorem, $\mathbf{A}\dashv\vdash\mathbf{B}$. Similarly, if $\mathbf{A}=\mathbf{C}\vee\mathbf{D}$ then $\mathbf{B}=\mathbf{C}'\vee\mathbf{D}'$ where $\mathbf{C}'$ and $\mathbf{D}'$ are alphabetical variants of **C** and **D**, so by induction hypothesis $\mathbf{C}\dashv\vdash\mathbf{C}'$ and $\mathbf{D}\dashv\vdash\mathbf{D}'$, and by replacement $\mathbf{A}\dashv\vdash\mathbf{B}$.

It remains only to consider the case $\mathbf{A}=\forall\mathbf{xC}$. Since **A** is an alphabetical variant of **B**, $\mathbf{B}=\forall\mathbf{yD}$ where **D** is an "almost alphabetic variant" of **C** in the sense that, if we choose a variable **z** that does not occur in **A** or **B**, then $\mathbf{C}(^{z}/_{x})$ is an alphabetic variant of $\mathbf{D}(^{z}/_{y})$. By induction hypothesis, $\mathbf{C}(^{z}/_{x}) \vdash \mathbf{D}(^{z}/_{y})$. By specification, $\vdash\mathbf{A}\supset\mathbf{C}(^{z}/_{x})$. By modus ponens $\mathbf{A}\vdash\mathbf{C}(^{z}/_{x})$. By transitivity, $\mathbf{A}\vdash\mathbf{D}(^{z}/_{y})$. By part (iii) of 4.5, $\mathbf{A}\vdash\forall\mathbf{zD}(^{z}/_{y})$. Similarly, by specification and modus ponens, $\forall\mathbf{zD}(^{z}/_{y})\vdash\mathbf{D}$. By part (iii) of 4.5,
$\forall\mathbf{zD}(^{z}/_{y})\vdash\forall\mathbf{yD}$, i.e., $\forall\mathbf{zD}(^{z}/_{y})\vdash\mathbf{B}$. By transitivity, then, $\mathbf{A}\vdash\mathbf{B}$.

Theorems 2.5 and 4.6, make it possible to extend the use of our notations for substitutions. Suppose, for some i, $1\leq i\leq n$, $\mathbf{y}_i$ is not free for $\mathbf{x}_i$ in $\mathbf{A}(\mathbf{x}_1,\ldots,\mathbf{x}_n)$. We would like $(\mathbf{A})(\mathbf{y}_1,\ldots,\mathbf{y}_n)$ to refer to a similar formula in which $\mathbf{y}_1,\ldots,\mathbf{y}_n$ play the role of $\mathbf{x}_1,\ldots,\mathbf{x}_n$. If $\mathbf{y}_i$ is free for $\mathbf{x}_i$ we can, as before, just substitute the former for the latter. If not then we need to replace the offending family of occurrences of the bound variable $\mathbf{y}_i$ before the substitution is made. So let us say that, when $(\mathbf{A})(\mathbf{y}_1,\ldots,\mathbf{y}_n)$ occurs in the same context and subsequent to $\mathbf{A}(\mathbf{x}_1,\ldots,\mathbf{x}_n)$, it refers to the result of uniformly substituting $\mathbf{y}_1,\ldots,\mathbf{y}_n$ for $\mathbf{x}_1,\ldots,\mathbf{x}_n$ in any alphabetic variant of $\mathbf{A}(\mathbf{x}_1,\ldots,\mathbf{x}_n)$ for which $\mathbf{y}_1,\ldots,\mathbf{y}_n$ are free for $\mathbf{x}_1,\ldots,\mathbf{x}_n$. Since every formula contains only finitely many variables, there will always be such an alphabetic variant. By theorem 2.5, all such variants will be true under the same assignments in the same models and by theorem 4.6 any two will be interderivable. Hence the ambiguity in the notation will not matter in any proof- or model-theoretic context. For example, axiom schema A7 remains valid when the notation is extended in this way. Alternatively, when we wish to make clear that the resulting formula is a function of both $\mathbf{y}_1,\ldots,\mathbf{y}_n$

and $\mathbf{x}_1,\ldots,\mathbf{x}_n$, we may use the notation $\mathbf{(A)}(^{y_1,\ldots,y_n}/_{\mathbf{x}_1,\ldots,\mathbf{x}_n})$ to indicate a formula that results from this "substitution-in-a-variant".

Theorem 4.6 implies that the derivability relation does not distinguish among alphabetic variants: if $\mathbf{A}'$ and $\mathbf{B}_1',\ldots,\mathbf{B}_n'$ are alphabetic variants of $\mathbf{A}$ and $\mathbf{B}_1,\ldots,\mathbf{B}_n$ respectively then $\mathbf{B}_1',\ldots,\mathbf{B}_n' \vdash \mathbf{A}'$ iff $\mathbf{B}_1,\ldots,\mathbf{B}_n \vdash \mathbf{A}$. Just as truth under an assignment can be regarded as a property of the statement expressed by a formula rather than the formula itself, so derivability can be regarded as a relation among statements expressed by formulas, rather than among the formulas themselves. There is some difficulty, however, in regarding the formulas of the derivations themselves as statements. $\forall\mathbf{xFx}\supset\forall\mathbf{yFy}$, for example, is an alphabetic variant of an instance of A1, but it is not itself an instance of A1. For this reason, it is natural to consider a modification of the system PL, in which axiom and rule schemas do not distinguish among alphabetic variants. Each axiom schema $\mathbf{A}$ of PL is replaced by the set of axioms $|\mathbf{A}'|$ for $\mathbf{A}'$ an instance of $\mathbf{A}$. Each rule schema $\mathbf{B_1,\ldots,B_n}$ **/** $\mathbf{A}$ is similarly replaced by the "schema" $|\mathbf{B}_1'|, \ldots, |\mathbf{B}_n'|\ /\ |\mathbf{A}'|$ for $\mathbf{A}'$ and $\mathbf{B}_1',\ldots,\mathbf{B}_n'$ instances of $\mathbf{A}$ and $\mathbf{B}_1,\ldots,\mathbf{B}_n$ respectively. We call a derivation in this system a statement derivation in PL. Every PL-derivation translates into a statement derivation. However, the distinction between many of the PL-derivations and between many of the PL-theorems will disappear. In particular, the theorem $\mathbf{(A \equiv A')}$ will translate into the triviality $|\mathbf{A \equiv A}|$. Any statement derivation will correspond to different abridged PL-derivation, obtained by choosing a representative from each equivalence class in the derivation. The proof of theorem 4.6 provides a procedure for expanding the abridged derivation into a proper derivation of PL.

Statement derivations correspond more closely to deductions in ordinary language or in the "language" of propositions; and it would be possible to develop predicate logic using statements in place of sentences. The awkwardness of dealing with equivalence classes could to some extent be removed by a notational sleight of hand. Thus $\vdash|\mathbf{A}|$ could be rewritten as $\vdash\mathbf{A}$ under a suitable reinterpretation of '$\vdash$'; and similarly for the truth-predicate and other cases. However, we have largely followed standard practice in working directly with formulas.

Recall that, because the axioms and rules of SL were schematic, the result of uniformly substituting formulas for sentence letters in a derivation of $\mathbf{A}$ from $\mathbf{\Gamma}$ was itself a derivation. This allowed us to show easily that $\mathbf{\Gamma}\vdash_{\mathbf{SL}}\mathbf{A}$ implies $\mathbf{\Gamma}'\vdash_{\mathbf{SL}}\mathbf{A}'$ when $\mathbf{\Gamma}',\mathbf{A}'$ is a substitution instance of $\mathbf{\Gamma},\mathbf{A}$. Now consider how one might make a substitution for atoms in a derivation showing $\forall\mathbf{u}\forall\mathbf{v}(\mathbf{Puv}\supset\neg\mathbf{Pvu})\vdash\forall\mathbf{u}\neg\mathbf{Puu}$. It would clearly be inappropriate to substitute distinct formulas for the three atomic subformulas, **Puv**, **Pvu** and **Puu**, that occur here. Two results that should follow by substitution from $\forall\mathbf{u}\forall\mathbf{v}(\mathbf{Puv}\supset\neg\mathbf{Pvu})\vdash\forall\mathbf{u}\neg\mathbf{Puu}$ are $\forall\mathbf{u}\forall\mathbf{w}(\mathbf{Puw}\supset\neg\mathbf{Pwu})\vdash\forall\mathbf{u}\neg\mathbf{Puu}$ and $\forall\mathbf{u}\forall\mathbf{v}(\mathbf{Quv}\supset\neg\mathbf{Qvu})\vdash\forall\mathbf{u}\neg\mathbf{Quu}$. In the first case the variable **w** is substituted for the variable **v**. In the second case the predicate **Q** is substituted for the predicate letter **P**. Substitution of variables is fairly straightforward:

Theorem *g.* (substitution of variables). Let r be a relettering of the variables (free and bound) that occur in formulas of $\mathbf{\Gamma},\mathbf{A}$. Then $\mathbf{\Gamma}\vdash\mathbf{A}$ implies $\mathbf{r(\Gamma)}\vdash\mathbf{r(A)}$.

The result is evident since nothing turns on the particular identity of the variables in the formulation of the axioms and rules. The strict proof is by induction on the formulas derivable from $\mathbf{\Gamma}$ and is left as an exercise. The result is a kind of syntactic counterpart of the third corollary to 2.5. A simple corollary will be useful in the completeness proof of the next chapter.

Corollary. Let r be a relettering of the variables of $\mathbf{\Gamma}$. If $\mathbf{\Gamma}$ is consistent, so is r($\mathbf{\Gamma}$).
Proof. If r($\mathbf{\Gamma}$) is not consistent then $\mathbf{r(\Gamma)\vdash\bot}$. By the theorem $\mathbf{r^{-1}r(\Gamma)\vdash r^{-1}(\bot)}$. But, since $\bot$ contains no variables and $\mathbf{r^{-1}r(\Gamma)=\Gamma}$, this implies $\mathbf{\Gamma\vdash\bot}$, i.e., that $\mathbf{\Gamma}$ is not consistent.

Substitution of predicates for predicate letters can be more subtle. In the example cited above we merely substituted one predicate letter for another of the same degree. Let us call a substitution (of zero or more predicates) of this kind a simple substitution. By performing such a substitution on every line of a derivation of **A** from $\mathbf{\Gamma}$ we obtain a derivation of $\mathbf{A'}$ from $\mathbf{\Gamma'}$. More complicated forms of predicate substitution will be considered later.

Let us now examine a few derivations and describe the abbreviatory rules made possible by the preceding results.

h. $\forall\mathbf{u}\forall\mathbf{vPuv}\vdash\forall\mathbf{uPuu}$

1.$\forall u\forall vPuv$	assumption
2.$\forall u\forall vPuv\supset\forall vPuv$	A7
3.$\forall vPuv\supset Puu$	A7
4.Puu	1,2,3 TFC
5.$\forall uPuu$	4 UI

This derivation contains two lines that are neither assumptions nor lines justified by the axioms and rules of PL. The rule **TFC**, or **truth-functional consequence** is justified by theorem 4.1. On any line in a derivation, we may enter truth-functional consequences of formulas on the lines above, citing those lines as sources. In particular **TFC** permits tautologies to be entered anywhere in a derivation. **TFC** has a somewhat different character than the devices of abbreviation we considered previously. 4.1 guarantees that there <u>is</u> an appropriate derivation, but it does not directly provide a means of generating of it. So it may appear that we have no procedure for disabbreviating derivations that appeal to TFC. In fact, it is quite easy to state such a procedure. (See exercise *** below.) The procedure is not one that would be practical to actually carry out, but it does show that TFC can, in principle, be regarded as an abbreviatory device. For present purposes, it is sufficient that any line justified by TFC is derivable from the preceding ones.

The rule **UI**, or **universal introduction**, cited on line 5 is a strengthening of the rule **Gen** made possible by part (iv) of theorem 4.5. Suppose $\mathbf{A(y)}$ occurs on some line of a derivation, none of the assumptions on which this line rests contain a free occurrence of $\mathbf{y}$ and $\mathbf{x}$ does not occur free in $\mathbf{A(y)}$. Then we may append a line with formula $\forall\mathbf{xA(x)}$. Theorem 4.5 (iv) ensures that the formula added is indeed derivable. In this case line 5 could not have been obtained by Gen because the derivation contains no <u>proof</u> of **Puu**. Since the sole assumption on which line five rests contains no free occurrences of **u**, however, line four is a legitimate application of **UI**.

i. $\exists\mathbf{vPvv}\vdash\exists\mathbf{u}\exists\mathbf{vPuv}$

1.$\exists vPvv$	assumption
2.1 Puu	assumption
2.2 $\exists vPuv$	.1 (EI)
2.3 $\exists u\exists vPuv$	.2 (EI)
3. $\exists u\exists vPuv$	2.1-2.3 (EE)

The citation **EI** on lines 2.2 and 2.3 refers to the rule of **existential introduction**. If a formula **A** occurs on any line of a derivation, it is permissible to enter ∃**xA′** on a subsequent line, where **A′** is obtained by replacing zero or more free occurrences of some variable **y** by **x**, provided **x** is free for **y** in **A**. This is justified by part vi of theorem 4.5.

The new abbreviatory device cited on line 3 is justified by part (v) of theorem 4.5. If some sublist of the previous lines constitutes a derivation of **C** from **A(y)** and other assumptions **Γ** and **y** does not occur free in **C** or the formulas of **Γ**, then **C** is derivable from **Γ** and ∃**xA(x)**. As before, we indent the subderivation to show that it includes a new assumption. We do not indent the conclusion of the rule (which in this case has the same formula as the last line of the subderivation) to show that it does not rest on the new assumption. We call this the **existential elimination** rule and use the letters **EE** to cite it in a derivation.

j. **¬∀uPu⊣⊢∃u¬Pu**

a.(left to right)

1.¬∀uPu	assumption	
2.1 ∀u¬¬Pu	assumption	
2.2 ∀uPu	.1 Rep (TFC)	
2.3 ⊥	1,.2 TFC	
3.∀u¬¬Pu⊃⊥	2.1-2.3 Ded	
4.∃u¬Pu	3 TFC	[¬∀u¬¬Pu]

b.(right to left)

1. ∃u¬Pu	assumption
2.1 ∀uPu	assumption
2.2 Pu	.1 UE
2.3.1 ¬Pu	assumption
2.3.2 ⊥	2.2,.1 TFC
2.4 ⊥	2.3.1-2.3.2 EE
3. ∀uPu⊃⊥	2.1-2.4 Ded
4. ¬∀uPu	3 TFC

On line 2.2 of the a derivation above, appeal is made to the **replacement** rule which, by theorem 4.4, can be carried over from Part I. Since Pu and ¬¬Pu are truth functionally equivalent TFC ensures that they are provably equivalent. Replacement allows us to replace any occurrence of a subformula by a provable equivalent. As in part I, when citing replacement we include a parenthetical reference to the derivation or rule that establishes the relevant equivalence. (The equivalence actually used may be a relettering or a simple predicate substitution instance of the one cited. After the general result of substitution is proved, we can allow the use of arbitrary predicate substitution instances of previously proved biconditionals.)

Line 3 is justified by the abbreviatory rule **Ded**, which, by 4.3, can be carried over from part I. **A**⊃**B** may be derived from **Γ** if we have a derivation of **B** from **Γ**∪**{A}**. As before, we indent the subderivations and permit appeal to previous lines of the outer derivation.

In the b derivation above, note that the existential elimination rule is applied within a subderivation. Since **u** does not occur free in ⊥ or in ∀**uPu**, this application of **EE** is legitimate. On line 2.2 we cite the citation **UE** refers to a rule of **universal elimination** that permits the entry of $\mathbf{A(y)}$ when $\forall\mathbf{xA(x)}$ occurs on previous lines (whether $\mathbf{y}$ and $\mathbf{x}$ are distinct or identical). The justification of this rule is very similar to that of **EI** and is left as an exercise.

Let us look at one more example.

k. $\neg\exists\mathbf{vQv}\dashv\vdash\forall\mathbf{w}\neg\mathbf{Qw}$

1. $\neg\exists vQv$	assumption		
2.1 $\neg\forall w\neg Qw$	assumption		
2.2 $\exists w\neg\neg Qw$	.1	[4.10a]	
2.3 $\exists v\neg\neg Qv$	.2	AV	
2.4 $\exists vQv$		.3	Rep (TFC)
3. $\neg\forall w\neg Qw\supset\exists vQv$	2.1-2.4		Ded
4. $\neg\exists vQv\supset\forall w\neg Qw$		3	TFC
5. $\forall w\neg Qw$		1,4	MP

On line 2.2 appeal is made, as it was in Part I, to previous derivations. If **A** has been derived from $\mathbf{B_1,\ldots,B_n}$ and some relettering or simple substitution instance of $\mathbf{B_1,\ldots,B_n}$ occur in a derivation, then it is permissible to enter the corresponding substitution instance of **A**. In this case the step from line 2.1 to line 2.2 is justified by the substitution of **w** and **Q** for **u** and **P** in 4.10a: $\neg\forall\mathbf{uPu}\vdash\exists\mathbf{u}\neg\mathbf{Pu}$. $\neg\forall\mathbf{wQw}\vdash\exists\mathbf{v}\neg\neg\mathbf{Qv}$ is not a substitution instance of $\neg\forall\mathbf{uPu}\vdash\exists\mathbf{u}\neg\mathbf{Pu}$, and so line 2.3 cannot be obtained directly from line 2.1. But since the formula on line 2.3 is an alphabetical variant of that on line 2.2, theorem 4.6 ensures that the formula on line three is derivable (and the proof of that theorem shows how a derivation can be constructed). In general the rule AV permits entry of alphabetical variants of formulas of previous lines.

As in part I other abbreviatory "rules" can be added as needed. Indeed, every metalogical result of the form **If** $\mathbf{\Gamma_1\vdash A_1,\ldots,\Gamma_n\vdash A_n}$ **then** $\mathbf{f(\Gamma_1,\ldots,\Gamma_n)\vdash g(A_1,\ldots,A_n)}$ corresponds to a possible rule for abbreviating derivations. The rules illustrated above--TFC, Ded, Rep, AV, UI, EI, UE, EE, and appeal to previous derivations--will be sufficient for our needs. Let us call a derivation-presentation that avails itself of no more than the axioms and rules of PL and these nine devices a PL(1)-derivation. Every fully-presented derivation in PL is, of course, a PL(1)-derivation, as is every derivation-presentation in the previous examples. Using extended derivations, it is not difficult to prove the result below. The details are left as exercises.

Theorem *l.* . The following formulas are theorems of PL:

a. $\forall\mathbf{v}\neg\mathbf{Pv}\equiv\neg\exists\mathbf{vPv}$
b. $\forall\mathbf{v(Pv\wedge Qv)}\equiv\forall\mathbf{vPv}\wedge\forall\mathbf{vQv}$
c. $\forall\mathbf{v(Pv\vee Q)}\equiv\forall\mathbf{vPv}\vee\mathbf{Q}$
d. $\forall\mathbf{v(Pv\vee Qv)}\supset\forall\mathbf{vPv}\vee\forall\mathbf{vQv}$
e. $\forall\mathbf{v(Pv\supset Q)}\equiv(\exists\mathbf{vPv}\supset\mathbf{Q})$
f. $\forall\mathbf{v(Q\supset Pv)}\equiv(\mathbf{Q}\supset\forall\mathbf{vPv})$
g. $\exists\mathbf{v}\neg\mathbf{Pv}\equiv\forall\mathbf{v}\neg\mathbf{Pv}$
h. $\exists\mathbf{v(Pv\wedge Qv)}\supset\exists\mathbf{vPv}\wedge\exists\mathbf{vQv}$
i. $\exists\mathbf{v(Pv\wedge Q)}\equiv\exists\mathbf{vPv}\wedge\mathbf{Q}$
j. $\exists\mathbf{v(Pv\vee Qv)}\equiv\exists\mathbf{vPv}\vee\exists\mathbf{vQv}$
k. $\exists\mathbf{v(Pv\supset Q)}\equiv(\forall\mathbf{vPv}\supset\mathbf{Q})$
l. $\exists\mathbf{v(Q\supset Pv)}\equiv(\mathbf{Q}\supset\exists\mathbf{vPv})$
m. $\forall\mathbf{u}\forall\mathbf{vPuv}\equiv\forall\mathbf{v}\forall\mathbf{uPuv}$

n. $\exists \mathbf{u} \exists \mathbf{v} \mathbf{Puv} \equiv \exists \mathbf{v} \exists \mathbf{u} \mathbf{Puv}$
o. $\exists \mathbf{u} \forall \mathbf{v} \mathbf{Puv} \supset \forall \mathbf{v} \exists \mathbf{u} \mathbf{Puv}$
p. $\forall \mathbf{u} \mathbf{Pu} \supset \exists \mathbf{u} \mathbf{Pu}$
q. $\forall \mathbf{u} \forall \mathbf{v} \mathbf{Puv} \supset \forall \mathbf{u} \mathbf{Puu}$

Exercises and Problems

1**[e]**. State and prove the result about simple predicate substitutions appealed to in the proof of 4.11.

2**[e]**. a. Prove a-q of 4.12 by proving a-q within a convenient system PL(*) of derivation-presentations. List and justify every rule of PL(*) that is foreign to PL(1). b. For theorem a of 4.12, provide a PL(1)-derivation and a fully-presented PL-derivation.

3**[e]**. Give an example to show that not every alphabetic variant of an axiom is an axiom.

4**[e]**. Complete the proof of theorem 4.7.

4'[p]**(continuation of problem ***. Redo the whole of the proof theory for predicate logic using linked formulas in place of ordinary formulas. (This is no small task. Consider, for example, the question of reformulating the axiom of Specification (A7). We no longer make a substitution of **y** for **x** in the formula **A(x)**. Instead we must consider how the links in the formula $\forall$**xA(x)** might be modified once the quantifier prefix is dropped.)

5**[p]**. (proof of substitution for predicate letters via language enlargement). [[This language is now treated below]] Consider the language $\mathfrak{L}(\lambda)$ in which the λ-terms are not mere abbreviations but full-fledged expressions. Add to the axiom system **PL** the scheme:

A) $\lambda \mathbf{x}_1 ... \mathbf{x}_n \mathbf{B} \mathbf{t}_1 ... \mathbf{t}_n \equiv \mathbf{B}[^{\mathbf{t}1}/_{\mathbf{x}1} ... ^{\mathbf{t}n}/_{\mathbf{x}n}]$ if, for $1 \leq i \leq n$, $\mathbf{t}_i$ is free for $\mathbf{x}_i$

a. Show that the relettering result (theorem 4.7) still obtains.
b. Show that substitution for predicate letters holds in the new axiom system.
c. Show that the new system is a conservative extension of the old, i.e., that any theorem of the new system that has no complex predicates is provable in **PL**. Hint: show how any formula of the form $\lambda \mathbf{x}_1 ... \mathbf{x}_n \mathbf{B} \mathbf{t}_1 ... \mathbf{t}_n$ can be eliminated from a derivation in favor of a formula containing one fewer occurrence of a λ-term.

6**[p]**. Consider the system obtained from PL by replacing A7 by a more "schematic" kind of specification: $\forall \mathbf{xA} \supset \mathbf{A}$. Show that not all the theorems of PL are provable in this system. [Hint for a model-theoretic solution: let a "differentiated" assignment be an assignment that never assigns the same object to distinct variables (such a restriction on the interpretation of the variables was proposed by Wittgenstein in the Tractatus). Modify clause (iv) the truth definition to say: If $\mathbf{A} = \forall \mathbf{xB}$ then $\vDash_M \mathbf{A}[\delta]$ if $\vDash_M \mathbf{B}[\delta']$ for all δ' that are differentiated x-variants of δ, and $\nvDash_M \mathbf{A}[\delta]$ if $\nvDash_M \mathbf{B}[\delta']$ for some differentiated x-variant δ' of δ, and modify the definition of truth in a model so that $\vDash_M \mathbf{A}$ iff $\vDash_M \mathbf{A}[\delta]$ for all differentiated M-assignments δ. Use induction to show that every theorem of the new axiom system is valid according to the new semantics. Then find a countermodel for $\forall$vPvu$\supset$Puu.] [Hint for a proof-theoretic solution: Drop all the quantifier expressions. Show that every theorem of the modified system is tautologous even though

$\forall v Pv \supset Pu$ transforms into $Pv \supset Pu$, which is not a tautology.