Outline Today’s Lecture

- finish Euler Equations and Transversality Condition
- Principle of Optimality: Bellman’s Equation
- Study of Bellman equation with bounded $F$
- contraction mapping and theorem of the maximum

Introduction to Dynamic Optimization Nr. 2
Infinite Horizon $T = \infty$

$$V^*(x_0) = \sup_{\{x_{t+1}\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t F(x_t, x_{t+1})$$

subject to,

$$x_{t+1} \in \Gamma(x_t)$$

(1)

with $x_0$ given

- $\sup \{}$ instead of $\max \{}$
- define $\{x'_{t+1}\}_{t=0}^\infty$ as a plan
- define $\Pi(x_0) \equiv \{\{x'_{t+1}\}_{t=0}^\infty | x'_{t+1} \in \Gamma(x'_t) \text{ and } x'_0 = x_0\}$
Assumptions

A1. $\Gamma (x)$ is non-empty for all $x \in X$
A2. $\lim_{T \to \infty} \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1})$ exists for all $x \in \Pi (x_0)$
then problem is well defined
Recursive Formulation: Bellman Equation

- value function satisfies

\[
V^*(x_0) = \max_{\{x_{t+1}\}_{t=0}^{\infty} \atop x_{t+1} \in \Gamma(x_t)} \left\{ \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \right\}
\]

\[
= \max_{x_1 \in \Gamma(x_0)} \left\{ F(x_0, x_1) + \max_{\{x_{t+1}\}_{t=1}^{\infty} \atop x_{t+1} \in \Gamma(x_t)} \sum_{t=1}^{\infty} \beta^t F(x_t, x_{t+1}) \right\}
\]

\[
= \max_{x_1 \in \Gamma(x_0)} \left\{ F(x_0, x_1) + \beta \max_{\{x_{t+1}\}_{t=1}^{\infty} \atop x_{t+1} \in \Gamma(x_t)} \sum_{t=0}^{\infty} \beta^t F(x_{t+1}, x_{t+2}) \right\}
\]

\[
= \max_{x_1 \in \Gamma(x_0)} \left\{ F(x_0, x_1) + \beta V^*(x_1) \right\}
\]

continued...
• Idea: use BE to find value function $V^*$ and policy function $g$ [Principle of Optimality]
Outline Today’s Lecture

• housekeeping: ps#1 and recitation day/ theory general / web page

• finish Principle of Optimality:
  Sequence Problem $\Longleftrightarrow$ solution to Bellman Equation
  (for values and plans)

• begin study of Bellman equation with bounded and continuous $F$

• tools: contraction mapping and theorem of the maximum
Sequence Problem vs. Functional Equation

- Sequence Problem: (SP)

\[ V^*(x_0) = \sup \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \]

s.t. \( x_{t+1} \in \Gamma(x_t) \)

\( x_0 \) given

- ... more succinctly

\[ V^*(x_0) = \sup_{\tilde{x} \in \Pi(x_0)} u(\tilde{x}) \] (SP)

- functional equation (FE) [this particular FE called Bellman Equation]

\[ V(x) = \max_{y \in \Gamma(x)} \{ F(x, y) + \beta V(y) \} \] (FE)
Principle of Optimality

**IDEA:** use BE to find value function $V^*$ and optimal plan $x^*$

- **Thm 4.2.** $V^*$ defined by SP $\Rightarrow V^*$ solves FE
- **Thm 4.3.** $V$ solves FE and ...... $\Rightarrow V = V^*$
- **Thm 4.4.** $\tilde{x}^* \in \Pi (x_0)$ is optimal
  $\Rightarrow V^* (x^*_t) = F (x^*_t, x^*_{t+1}) + \beta V^* (x^*_{t+1})$
- **Thm 4.5.** $\tilde{x}^* \in \Pi (x_0)$ satisfies $V^* (x^*_t) = F (x^*_t, x^*_{t+1}) + \beta V^* (x^*_{t+1})$ and ......
  $\Rightarrow \tilde{x}^*$ is optimal
Why is this Progress?

- **intuition:** breaks planning horizon into two: ‘now’ and ‘then’
- **notation:** reduces unnecessary notation (especially with uncertainty)
- **analysis:** prove existence, uniqueness and properties of optimal policy (e.g. continuity, monotonicity, etc...)
- **computation:** associated numerical algorithm are powerful for many applications
Proof of Theorem 4.3 (max case)

Assume for any \( \tilde{x} \in \Pi (x_0) \)

\[
\lim_{T \to \infty} \beta^T V(x_T) = 0.
\]

BE implies

\[
V(x_0) \geq F(x_0, x_1) + \beta V(x_1), \text{ all } x_1 \in \Gamma(x_0)
\]

\[
= F(x_0, x^*_1) + \beta V(x^*_1), \text{ some } x^*_1 \in \Gamma(x_0)
\]

Substituting \( V(x_1) \):

\[
V(x_0) \geq F(x_0, x_1) + \beta F(x_1, x_2) + \beta^2 V(x_2), \text{ all } x \in \Pi(x_0)
\]

\[
= F(x_0, x^*_1) + \beta F(x^*_1, x^*_2) + \beta^2 V(x^*_2), \text{ some } x^* \in \Pi(x_0)
\]
Continue this way

\begin{align*}
V(x_0) & \geq \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1}) + \beta^{n+1} V(x_{n+1}) \quad \text{for all } x \in \Pi(x_0) \\
& = \sum_{t=0}^{n} \beta^t F(x^*_t, x^*_{t+1}) + \beta^{n+1} V(x^*_{n+1}) \quad \text{for some } x^* \in \Pi(x_0)
\end{align*}

Since \( \beta^T V(x_T) \to 0 \), taking the limit \( n \to \infty \) on both sides of both expressions we conclude that:

\begin{align*}
V(x_0) & \geq u(\bar{x}) \quad \text{for all } \bar{x} \in \Pi(x_0) \\
V(x_0) & = u(\bar{x}^*) \quad \text{for some } \bar{x}^* \in \Pi(x_0)
\end{align*}
Bellman Equation as a Fixed Point

• define operator

\[ T(f)(x) = \max_{y \in \Gamma(x)} \{ F(x, y) + \beta f(y) \} \]

• \( V \) solution of BE \( \iff \) \( V \) fixed point of \( T \) [i.e. \( TV = V \)]

Bounded Returns:

• if \( \|F\| < B \) and \( F \) and \( \Gamma \) are continuous: \( T \) maps continuous bounded functions into continuous bounded functions

• bounded returns \( \Rightarrow T \) is a Contraction Mapping \( \Rightarrow \) unique fixed point

• many other bonuses
Needed Tools

- Basic Real Analysis (section 3.1):
  \{vector, metric, noSLP, complete\} spaces
  cauchy sequences
  closed, compact, bounded sets
- Contraction Mapping Theorem (section 3.2)
- Theorem of the Maximum: study of RHS of Bellman equation (equivalently of $T$) (section 3.3)
Bellman Equation: Principle of Optimality

- Principle of Optimality idea: use the functional equation

\[ V(x) = \max_{y \in \Gamma(x)} \{ F(x, y) + \beta V(y) \} \]

to find \( V^* \) and \( g \)

- note: nuisance subscripts \( t, t+1 \), dropped

- a solution is a function \( V(\cdot) \) the same on both sides

- IF BE has unique solution then \( V^* = V \)

- more generally the “right solution” to (BE) delivers \( V^* \)
Outline Today’s Lecture

- study Functional Equation (Bellman equation) with bounded and continuous $F$
- tools: contraction mapping and theorem of the maximum
Bellman Equation as a Fixed Point

- Define operator

\[ T(f)(x) = \max_{y \in \Gamma(x)} \{ F(x, y) + \beta f(y) \} \]

- \( V \) solution of BE \( \iff \) \( V \) fixed point of \( T \) [i.e. \( TV = V \)]

**Bounded Returns:**

- If \( \|F\| < B \) and \( F \) and \( \Gamma \) are continuous: \( T \) maps continuous bounded functions into continuous bounded functions

- Bounded returns \( \Rightarrow \) \( T \) is a Contraction Mapping \( \Rightarrow \) unique fixed point

- Many other bonuses
Our Favorite Metric Space

\[ S = \left\{ f : X \to R, \ f \text{ is continuous, and } \|f\| \equiv \sup_{x \in X} |f(x)| < \infty \right\} \]

with

\[ \rho (f, g) = \|f - g\| \equiv \sup_{x \in X} |f(x) - g(x)| \]

**Definition.** A linear space \( S \) is complete if any Cauchy sequence converges. For a definition of a Cauchy sequence and examples of complete metric spaces see SLP.

**Theorem.** The set of bounded and continuous functions is Complete. See SLP.
Definition. Let $(S, \rho)$ be a metric space. Let $T : S \rightarrow S$ be an operator. $T$ is a contraction with modulus $\beta \in (0, 1)$

$$\rho (Tx, Ty) \leq \beta \rho (x, y)$$

for any $x, y$ in $S$. 
Contraction Mapping Theorem

**Theorem (CMThm).** If $T$ is a contraction in $(S, \rho)$ with modulus $\beta$, then (i) there is a unique fixed point $s^* \in S$,

$$s^* = Ts^*$$

and (ii) iterations of $T$ converge to the fixed point

$$\rho (T^n s_0, s^*) \leq \beta^n \rho (s_0, s^*)$$

for any $s_0 \in S$, where $T^{n+1} (s) = T(T^n (s))$. 

Introduction to Dynamic Optimization  
Nr. 6
CMThm – Proof

for (i) 1st step: construct fixed point \( s^* \)

take any \( s_0 \in S \) define \( \{s_n\} \) by \( s_{n+1} = Ts_n \) then

\[
\rho(s_2, s_1) = \rho(Ts_1, Ts_0) \leq \beta \rho(s_1, s_0)
\]

generalizing \( \rho(s_{n+1}, s_n) \leq \beta^n \rho(s_1, s_0) \) then, for \( m > n \)

\[
\rho(s_m, s_n) \leq \rho(s_m, s_{m-1}) + \rho(s_{m-1}, s_{m-2}) + \cdots + \rho(s_{n+1}, s_n)
\]
\[
\leq [\beta^{m-1} + \beta^{m-2} + \cdots + \beta^n] \rho(s_1, s_0)
\]
\[
\leq \beta^n [\beta^{m-n-1} + \beta^{m-n-2} + \cdots + 1] \rho(s_1, s_0)
\]
\[
\leq \frac{\beta^n}{1 - \beta} \rho(s_1, s_0)
\]

thus \( \{s_n\} \) is cauchy. hence \( s_n \to s^* \)
2nd step: show \( s^* = Ts^* \)

\[
\rho(Ts^*, s^*) \leq \rho(Ts^*, s_n) + \rho(s^*, s_n) \\
\leq \beta \rho(s^*, s_{n-1}) + \rho(s^*, s_n) \rightarrow 0
\]

3rd step: \( s^* \) is unique. \( Ts_1^* = s_1^* \) and \( s_2^* = Ts_2^* \)

\[
0 \leq a = \rho(s_1^*, s_2^*) = \rho(Ts_1^*, Ts_2^*) \leq \beta \rho(s_1^*, s_2^*) = \beta a
\]

only possible if \( a = 0 \Rightarrow s_1^* = s_2^* \).

Finally, as for (ii):

\[
\rho(T^n s_0, s^*) = \rho(T^n s_0, Ts^*) \leq \beta \rho(T^{n-1} s_0, s^*) \leq \cdots \leq \beta^n \rho(s_0, s^*)
\]
**Corollary.** Let $S$ be a complete metric space, let $S' \subset S$ and $S'$ close. Let $T$ be a contraction on $S$ and let $s^* = Ts^*$. Assume that

$$T(S') \subset S', \quad \text{i.e. if } s' \in S, \text{ then } T(s') \in S'$$

then $s^* \in S'$. Moreover, if $S'' \subset S'$ and

$$T(S') \subset S'', \quad \text{i.e. if } s' \in S', \text{ then } T(s') \in S''$$

then $s^* \in S''$. 
Blackwell’s sufficient conditions.
Let $S$ be the space of bounded functions on $X$, and $\| \cdot \|$ be given by the sup norm. Let $T : S \to S$. Assume that (i) $T$ is monotone, that is,

$$Tf(x) \leq Tg(x)$$

for any $x \in X$ and $g, f$ such that $f(x) \geq g(x)$ for all $x \in X$, and (ii) $T$ discounts, that is, there is a $\beta \in (0, 1)$ such that for any $a \in R_+$,

$$T(f + a)(x) \leq Tf(x) + a\beta$$

for all $x \in X$. Then $T$ is a contraction.
Proof. By definition

\[ f = g + f - g \]

and using the definition of \( \| \cdot \| \),

\[ f(x) \leq g(x) + \| f - g \| \]

then by monotonicity i)

\[ Tf \leq T(g + \| f - g \|) \]

and by discounting ii) setting \( a = \| f - g \| \)

\[ Tf \leq T(g) + \beta \| f - g \|. \]

Reversing the roles of \( f \) and \( g \) :

\[ Tg \leq T(f) + \beta \| f - g \| \]

\[ \Rightarrow \| Tf - Tg \| \leq \beta \| f - g \| \]
Bellman equation application

\[(Tv)(x) = \max_{y \in \Gamma(x)} \{ F(x, y) + \beta v(y) \}\]

Assume that \(F\) is bounded and continuous and that \(\Gamma\) is continuous and has compact range.

**Theorem.** \(T\) maps the set of continuous and bounded functions \(S\) into itself. Moreover \(T\) is a contraction.
Proof. That $T$ maps the set of continuous and bounded follow from the Theorem of Maximum (we do this next).
That $T$ is a contraction follows since $T$ satisfies the Blackwell sufficient conditions.
$T$ satisfies the Blackwell sufficient conditions. For monotonicity, notice that for $f \geq v$

\[
Tv(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\}
\]
\[
= F(x, g(x)) + \beta v(g(x))
\]
\[
\leq \{F(x, g(y)) + \beta f(g(x))\}
\]
\[
\leq \max_{y \in \Gamma(x)} \{F(x, y) + \beta f(y)\} = Tf(x)
\]

A similar argument follows for discounting: for $a > 0$

\[
T(v + a)(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta (v(y) + a)\}
\]
\[
= \max_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\} + \beta a = T(v)(x) + \beta a.
\]
Theorem of the Maximum

- want $T$ to map continuous function into continuous functions

$$ (Tv)(x) = \max_{y \in \Gamma(x)} \{ F(x, y) + \beta v(y) \} $$

- want to learn about optimal policy of RHS of Bellman

$$ G(x) = \arg \max_{y \in \Gamma(x)} \{ F(x, y) + \beta v(y) \} $$

- First, continuity concepts for correspondences
- ... then, a few example maximizations
- ... finally, Theorem of the Maximum
assume $\Gamma$ is non-empty and compact valued (the set $\Gamma(x)$ is non empty and compact for all $x \in X$)

**Upper Hemi Continuity (u.h.c.) at $x$:** for any pair of sequences $\{x_n\}$ and $\{y_n\}$ with $x_n \to x$ and $x_n \in \Gamma(y_n)$ there exists a subsequence of $\{y_n\}$ that converges to a point $y \in \Gamma(x)$.

**Lower Hemi Continuity (l.h.c.) at $x$:** for any sequence $\{x_n\}$ with $x_n \to x$ and for every $y \in \Gamma(x)$ there exists a sequence $\{y_n\}$ with $x_n \in \Gamma(y_n)$ such that $y_n \to y$.

**Continuous at $x$:** if $\Gamma$ is both upper and lower hemi continuous at $x$
Max Examples

\[ h(x) = \max_{y \in \Gamma(x)} f(x, y) \]

\[ G(x) = \arg \max_{y \in \Gamma(x)} f(x, y) \]

ex 1: \( f(x, y) = xy; \ X = [-1, 1]; \ \Gamma(x) = X. \)

\[ G(x) = \begin{cases} 
\{ -1 \} & x < 0 \\
[-1, 1] & x = 0 \\
\{ 1 \} & x > 0 
\end{cases} \]

\[ h(x) = |x| \]

continued...
ex 2: $f(x, y) = xy^2$, $X = [-1, 1]$; $\Gamma(x) = X$

\[
G(x) = \begin{cases} 
\{0\} & x < 0 \\
[-1, 1] & x = 0 \\
\{-1, 1\} & x > 0 
\end{cases}
\]

$h(x) = \max\{0, x\}$
Theorem of the Maximum

Define:

\[ h(x) = \max_{y \in \Gamma(x)} f(x, y) \]

\[ G(x) = \arg \max_{y \in \Gamma(x)} f(x, y) \]

\[ = \{ y \in \Gamma(x) : h(x) = f(x, y) \} \]

**Theorem.** (Berge) Let \( X \subset \mathbb{R}^l \) and \( Y \subset \mathbb{R}^m \). Let \( f : X \times Y \rightarrow \mathbb{R} \) be continuous and \( \Gamma : X \rightarrow Y \) be compact-valued and continuous. Then \( h : X \rightarrow \mathbb{R} \) is continuous and \( G : X \rightarrow Y \) is non-empty, compact valued, and u.h.c.
Theorem. Suppose \( \{ f_n(x,y) \} \) and \( f(x,y) \) are concave in \( y \) and \( f_n \rightarrow f \) in the sup-norm (uniformly). Define

\[
    g_n(x) = \arg \max_{y \in \Gamma(x)} f_n(x,y)
\]

\[
    g(x) = \arg \max_{y \in \Gamma(x)} f(x,y)
\]

then \( g_n(x) \rightarrow g(x) \) for all \( x \) (pointwise convergence); if \( X \) is compact then the convergence is uniform.

\[ \lim \max \rightarrow \max \lim \]
Monotonicity of $v^*$

**Theorem.** Assume that $F(\cdot, y)$ is increasing, that $\Gamma$ is increasing, i.e.

$$\Gamma(x) \subset \Gamma(x')$$

for $x \leq x'$. Then, the unique fixed point $v^*$ satisfying $v^* = Tv^*$ is increasing. If $F(\cdot, y)$ is strictly increasing, so is $v^*$. 

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Proof

By the corollary of the CMThm, it suffices to show \( Tf \) is increasing if \( f \) is increasing. Let \( x \leq x' \):

\[
Tf (x) = \max_{y \in \Gamma (x)} \{ F (x, y) + \beta f (y) \} \\
= F (x, y^*) + \beta f (y^*) \text{ for some } y^* \in \Gamma (x) \\
\leq F (x', y^*) + \beta f (y^*)
\]

since \( y^* \in \Gamma (x) \subset \Gamma (x') \)

\[
\leq \max_{y \in \Gamma (x')} \{ F (x, y) + \beta f (y) \} = Tf (x')
\]

If \( F (\cdot, y) \) is strictly increasing

\[
F (x, y^*) + \beta f (y^*) < F (x', y^*) + \beta f (y^*).
\]
Concavity (or strict) concavity of \( v^* \)

**Theorem.** Assume that \( X \) is convex, \( \Gamma \) is concave, i.e. \( y \in \Gamma (x) \), \( y' \in \Gamma (x') \) implies that

\[
y^\theta \equiv \theta y' + (1 - \theta) y \in \Gamma (\theta x' + (1 - \theta) x) \equiv \Gamma (x^\theta)
\]

for any \( x, x' \in X \) and \( \theta \in (0, 1) \). Finally assume that \( F \) is concave in \( (x, y) \). Then, the fixed point \( v^* \) satisfying \( v^* = Tv^* \) is concave in \( x \). Moreover, if \( F (\cdot, y) \) is strictly concave, so is \( v^* \).
Differentiability

• can’t use same strategy: space of differentiable functions is not closed
• many envelope theorems
• Formula: if $h(x)$ is differentiable and $y$ is interior then

$$h'(x) = f_x(x, y)$$

right value... but is $h$ differentiable?

• one answer (Demand Theory) relies on f.o.c. and assuming twice differentiability of $f$

• won’t work for us since $f = F(x,y) + \beta V(y)$ and we don’t even know if $f$ is once differentiable! → going in circles
First a Lemma...

**Lemma.** Suppose $v(x)$ is concave and that there exists $w(x)$ such that $w(x) \leq v(x)$ and $v(x_0) = w(x_0)$ in some neighborhood $D$ of $x_0$ and $w$ is differentiable at $x_0$ ($w'(x_0)$ exists) then $v$ is differentiable at $x_0$ and $v'(x_0) = w'(x_0)$.

**Proof.** Since $v$ is concave it has at least one subgradient $p$ at $x_0$:

$$w(x) - w(x_0) \leq v(x) - v(x_0) \leq p \cdot (x - x_0)$$

Thus a subgradient of $v$ is also a subgradient of $w$. But $w$ has a unique subgradient equal to $w'(x_0)$. $\square$
Now a Theorem

**Theorem.** Suppose $F$ is strictly concave and $\Gamma$ is convex. If $x_0 \in \text{int}(X)$ and $g(x_0) \in \text{int}(\Gamma(x_0))$ then the fixed point of $T, V$, is differentiable at $x$ and

$$V'(x) = F_x(x, g(x))$$

**Proof.** We know $V$ is concave. Since $x_0 \in \text{int}(X)$ and $g(x_0) \in \text{int}(\Gamma(x_0))$ then $g(x_0) \in \text{int}(\Gamma(x))$ for $x \in D$ a neighborhood of $x_0$ then

$$W(x) = F(x, g(x_0)) + \beta V(g(x_0))$$

and $W(x) \leq V(x)$ and $W(x_0) = V(x_0)$ and $W'(x_0) = F_x(x_0, g(x_0))$ so the result follows from the lemma. \[\square\]
Recursive Methods
Outline Today’s Lecture

- finish off: theorem of the maximum
- Bellman equation with bounded and continuous $F$
- differentiability of value function
- application: neoclassical growth model
- homogenous and unbounded returns, more applications
Our Favorite Metric Space

\[ S = \left\{ f : X \to \mathbb{R}, \ f \text{ is continuous, and } \| f \| \equiv \sup_{x \in X} |f(x)| < \infty \right\} \]

with

\[ \rho (f, g) = \| f - g \| \equiv \sup_{x \in X} |f(x) - g(x)| \]

\[ (Tv)(x) = \max_{y \in \Gamma(x)} \left\{ F(x, y) + \beta v(y) \right\} \]

Assume that \( F \) is bounded and continuous and that \( \Gamma \) is continuous and has compact range.

**Theorem 4.6.** \( T \) maps the set of continuous and bounded functions \( S \) into itself. Moreover \( T \) is a contraction.
**Proof.** That $T$ maps the set of continuous and bounded follow from the Theorem of Maximum (we do this next)
That $T$ is a contraction $\rightarrow$ Blackwell sufficient conditions
$\rightarrow$monotonicity, notice that for $f \geq v$

\[
Tv(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\}
\]

\[
= F(x, g(x)) + \beta v(g(x))
\]

\[
\leq \{F(x, g(y)) + \beta f(g(x))\}
\]

\[
\leq \max_{y \in \Gamma(x)} \{F(x, y) + \beta f(y)\} = Tf(x)
\]

$\rightarrow$discounting: for $a > 0$

\[
T(v + a)(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta(v(y) + a)\}
\]

\[
= \max_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\} + \beta a = T(v)(x) + \beta a.
\]
Theorem of the Maximum

• want $T$ to map continuous functions into continuous functions

\[(Tv)(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\}\]

• want to learn about optimal policy of RHS of Bellman

\[G(x) = \arg \max_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\}\]

• First, continuity concepts for correspondences
• ... then, a few example maximizations
• ... finally, Theorem of the Maximum
Continuity Notions for Correspondences

assume $\Gamma$ is non-empty and compact valued (the set $\Gamma(x)$ is non empty and compact for all $x \in X$)

**Upper Hemi Continuity (u.h.c.) at** $x$: for any pair of sequences $\{x_n\}$ and $\{y_n\}$ with $x_n \to x$ and $x_n \in \Gamma(y_n)$ there exists a subsequence of $\{y_n\}$ that converges to a point $y \in \Gamma(x)$.

**Lower Hemi Continuity (l.h.c.) at** $x$: for any sequence $\{x_n\}$ with $x_n \to x$ and for every $y \in \Gamma(x)$ there exists a sequence $\{y_n\}$ with $x_n \in \Gamma(y_n)$ such that $y_n \to y$.

**Continuous at** $x$: if $\Gamma$ is both upper and lower hemi continuous at $x$
Max Examples

\[ h(x) = \max_{y \in \Gamma(x)} f(x, y) \]
\[ G(x) = \arg \max_{y \in \Gamma(x)} f(x, y) \]

**ex 1:** \( f(x, y) = xy; \; X = [-1, 1]; \; \Gamma(x) = X. \)

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\{1\} & x > 0 
\end{cases} \]
\[ h(x) = |x| \]

continued...
ex 2: \( f(x, y) = xy^2; \ X = [-1, 1]; \ \Gamma(x) = X \)

\[
G(x) = \begin{cases} 
\{0\} & x < 0 \\
[-1, 1] & x = 0 \\
\{-1, 1\} & x > 0 
\end{cases}
\]

\[
h(x) = \max \{0, x\}
\]
Theorem of the Maximum

Define:

\[ h(x) = \max_{y \in \Gamma(x)} f(x, y) \]

\[ G(x) = \arg \max_{y \in \Gamma(x)} f(x, y) \]

\[ = \{ y \in \Gamma(x) : h(x) = f(x, y) \} \]

**Theorem 3.6.** (Berge) Let \( X \subset \mathbb{R}^l \) and \( Y \subset \mathbb{R}^m \). Let \( f : X \times Y \to \mathbb{R} \) be continuous and \( \Gamma : X \to Y \) be compact-valued and continuous. Then \( h : X \to \mathbb{R} \) is continuous and \( G : X \to Y \) is non-empty, compact valued, and u.h.c.
Theorem 3.8. Suppose \( \{f_n(x, y)\} \) and \( f(x, y) \) are concave in \( y \) that and \( \Gamma \) is convex and compact valued. Then if \( f_n \to f \) in the sup-norm (uniformly). Define

\[
\begin{align*}
  g_n(x) &= \operatorname{arg\ max}_{y \in \Gamma(x)} f_n(x, y) \\
  g(x) &= \operatorname{arg\ max}_{y \in \Gamma(x)} f(x, y)
\end{align*}
\]

then \( g_n(x) \to g(x) \) for all \( x \) (pointwise convergence); if \( X \) is compact then the convergence is uniform.
Uses of Corollary of CMThm

Monotonicity of $v^*$

**Theorem 4.7.** Assume that $F(\cdot, y)$ is increasing, that $\Gamma$ is increasing, i.e.

$$\Gamma(x) \subset \Gamma(x')$$

for $x \leq x'$. Then, the unique fixed point $v^*$ satisfying $v^* = Tv^*$ is increasing. If $F(\cdot, y)$ is strictly increasing, so is $v^*$. 

Introduction to Dynamic Optimization

Nr. 10
Proof

By the corollary of the CMThm, it suffices to show $Tf$ is increasing if $f$ is increasing. Let $x \leq x'$:

$$Tf(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta f(y)\}$$

$$= F(x, y^*) + \beta f(y^*) \text{ for some } y^* \in \Gamma(x)$$

$$\leq F(x', y^*) + \beta f(y^*)$$

since $y^* \in \Gamma(x) \subset \Gamma(x')$

$$\leq \max_{y \in \Gamma(x')} \{F(x, y) + \beta f(y)\} = Tf(x')$$

If $F(\cdot, y)$ is strictly increasing

$$F(x, y^*) + \beta f(y^*) < F(x', y^*) + \beta f(y^*).$$
Concavity (or strict) concavity of $v^*$

**Theorem 4.8.** Assume that $X$ is convex, $\Gamma$ is concave, i.e. $y \in \Gamma (x)$, $y' \in \Gamma (x')$ implies that

$$y^\theta \equiv \theta y' + (1 - \theta) y \in \Gamma (\theta x' + (1 - \theta) x) \equiv \Gamma (x^\theta)$$

for any $x, x' \in X$ and $\theta \in (0, 1)$. Finally assume that $F$ is concave in $(x, y)$. Then, the fixed point $v^*$ satisfying $v^* = Tv^*$ is concave in $x$. Moreover, if $F (\cdot, y)$ is strictly concave, so is $v^*$.
convergence of policy functions

• with concavity of $F$ and convexity of $\Gamma \rightarrow \text{optimal policy correspondence } G(x)$ is actually a continuous function $g(x)$

• since $v_n \rightarrow v$ uniformly $\Rightarrow g_n \rightarrow g$
  (Theorem 4.8)

• we can use this to derive comparative statics
Differentiability

- can’t use same strategy as with monotonicity or concavity: space of differentiable functions is *not* closed
- many envelope theorems, imply differentiability of $h$

\[ h(x) = \max_{y \in \Gamma(x)} f(x, y) \]

- always if formula: if $h(x)$ is differentiable and there exists a $y^* \in \text{int}(\Gamma(x))$ then

\[ h'(x) = f_x(x, y) \]

...but is $h$ differentiable?

continued...
• one approach (e.g. Demand Theory) relies on smoothness of $\Gamma$ and $f$ (twice differentiability) → use f.o.c. and implicit function theorem

• won’t work for us since $f(x, y) = F(x, y) + \beta V(y)$ → don’t know if $f$ is once differentiable yet! → going in circles...
First a Lemma...

**Lemma.** Suppose $v(x)$ is concave and that there exists $w(x)$ such that $w(x) \leq v(x)$ and $v(x_0) = w(x_0)$ in some neighborhood $D$ of $x_0$ and $w$ is differentiable at $x_0$ ($w'(x_0)$ exists) then $v$ is differentiable at $x_0$ and $v'(x_0) = w'(x_0)$.

**Proof.** Since $v$ is concave it has at least one subgradient $p$ at $x_0$:

$$w(x) - w(x_0) \leq v(x) - v(x_0) \leq p \cdot (x - x_0)$$

Thus a subgradient of $v$ is also a subgradient of $w$. But $w$ has a unique subgradient equal to $w'(x_0)$.
Benveniste and Sheinkman

Now a Theorem

**Theorem.** Suppose $F$ is strictly concave and $\Gamma$ is convex. If $x_0 \in \text{int} \,(X)$ and $g(x_0) \in \text{int} \,(\Gamma \,(x_0))$ then the fixed point of $T$, $V$, is differentiable at $x$ and

$$ V' \,(x) = F_x \,(x, g(x)) $$

**Proof.** We know $V$ is concave. Since $x_0 \in \text{int} \,(X)$ and $g(x_0) \in \text{int} \,(\Gamma \,(x_0))$ then $g(x_0) \in \text{int} \,(\Gamma \,(x))$ for $x \in D$ a neighborhood of $x_0$ then

$$ W \,(x) = F \,(x, g(x_0)) + \beta V \,(g(x_0)) $$

and then $W(x) \leq V(x)$ and $W(x_0) = V(x_0)$ and $W'(x_0) = F_x \,(x_0, g(x_0))$ so the result follows from the lemma.
Recursive Methods
Outline Today’s Lecture

- “Anything goes”: Boldrin Montrucchio
- Global Stability: Liapunov functions
- Linear Dynamics
- Local Stability: Linear Approximation of Euler Equations
treat $X = [0, 1] \in \mathbb{R}$ case for simplicity

- take any $g(x) : [0, 1] \rightarrow [0, 1]$ that is twice continuously differentiable on $[0, 1]$
  $\Rightarrow g'(x)$ and $g''(x)$ exists and are bounded

- define

$$W(x, y) = -\frac{1}{2}y^2 + yg(x) - \frac{L}{2}x^2$$

- Lemma: $W$ is strictly concave for large enough $L$
Proof

\[ W(x, y) = -\frac{1}{2}y^2 + yg(x) - \frac{L}{2}x^2 \]

\[
\begin{align*}
W_1 &= yg'(x) - Lx \\
W_2 &= -y + g(x)
\end{align*}
\]

\[
\begin{align*}
W_{11} &= yg''(x) - L \\
W_{22} &= -1 \\
W_{12} &= g'(x)
\end{align*}
\]

Thus \( W_{22} < 0; \, W_{11} < 0 \) is satisfied if \( L \geq \max_x |g''(x)| \)

\[
W_{11}W_{22} - W_{12}W_{21} = -yg''(x) + L - g'(x)^2 > 0
\]

\[ \Rightarrow \quad L > g'(x)^2 + yg''(x) \]

Then \( L > \left[ \max_x |g'(x)| \right]^2 + \max_x |g''(x)| \) will do.
Decomposing \( W \) (in a concave way)

- define \( V (x) = W (x, g(x)) \) and \( F \) so that
  \[
  W (x, y) = F (x, y) + \beta V (y)
  \]
  that is \( F (x, y) = W (x, y) - \beta V (y) \).

- Lemma: \( V \) is strictly concave
  
  Proof: immediate since \( W \) is concave and \( X \) is convex. Computing the second derivative is useful anyway:
  \[
  V'' (x) = g'' (x) g (x) + g' (x)^2 - L
  \]
  since \( g \in [0, 1] \) then clearly our bound on \( L \) implies \( V'' (x) < 0 \).
Concavity of $F$

- Lemma: $F$ is concave for $\beta \in [0, \tilde{\beta}]$ for some $\tilde{\beta} > 0$

\[
F_{11}(x, y) = W_{11}(x, y) = yg''(x) - L
\]
\[
F_{12}(x, y) = W_{12}(x, y) = -1
\]
\[
F_{22}(x, y) = W_{22} - \beta V_{22} = -1 - \beta \left[g''(x)g(x) + g'(x)^2 - L\right]
\]
\[
F_{11}F_{22} - F_{12}^2 > 0
\]
\[
\Rightarrow (yg''(x) - L) \left(-1 - \beta \left[g''(x)g(x) + g'(x)^2 - L\right]\right) - g'(x)^2 > 0
\]
... concavity of $F$

- Let

$$
\eta_1(\beta) = \min_{x,y} (-F_{22}) \\
\eta_2(\beta) = \min_{x,y} [F_{11}F_{22} - F_{12}^2] \\
\eta(\beta) = \min \{ H_1(\beta), H_2(\beta) \} \geq 0
$$

- for $\beta = 0 \eta(\beta) > 0$. $\eta$ is continuous (Theorem of the Maximum) $\Rightarrow$ exists $\tilde{\beta} > 0$ such that $H(\beta) \geq 0$ for all $\beta \in [0, \tilde{\beta}]$. 
Monotonicity

- Use

\[ W(x, y) = -\frac{1}{2} y^2 + yg(x) - \frac{L_1}{2} x^2 + L_2 x \]

- \( L_2 \) does not affect second derivatives

- claim: \( F \) is monotone for large enough \( L_2 \)