Outline Today's Lecture

- finish Euler Equations and Transversality Condition
- Principle of Optimality: Bellman's Equation
- Study of Bellman equation with bounded F
- contraction mapping and theorem of the maximum

Infinite Horizon $T = \infty$

$$V^{*}\left(x_{0}\right)=\sup_{\left\{x_{t+1}\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^{t} F\left(x_{t}, x_{t+1}\right)$$

subject to,

$$x_{t+1} \in \Gamma\left(x_{t}\right) \tag{1}$$

with x_0 given

- $\sup\{\}$ instead of $\max\{\}$
- define $\left\{x_{t+1}^{\prime}\right\}_{t=0}^{\infty}$ as a plan
- define $\Pi\left(x_{0}\right) \equiv\left\{\left\{x_{t+1}^{\prime}\right\}_{t=0}^{\infty} \mid x_{t+1}^{\prime} \in \Gamma\left(x_{t}^{\prime}\right) \text{ and } x_{0}^{\prime}=x_{0}\right\}$

Assumptions

A1. $\Gamma(x)$ is non-empty for all $x \in X$
A2. $\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1})$ exists for all $x \in \Pi(x_0)$
then problem is well defined

Recursive Formulation: Bellman Equation

- value function satisfies

$$
\begin{aligned}
V^{*}(x_0) &= \max_{\substack{\{x_{t+1}\}_{t=0}^{\infty} \\ x_{t+1}\in\Gamma(x_t)}} \left\{ \sum_{t=0}^{\infty} \beta^{t} F(x_t, x_{t+1}) \right\} \\
&= \max_{x_1\in\Gamma(x_0)} \left\{ F(x_0, x_1) + \max_{\substack{\{x_{t+1}\}_{t=1}^{\infty} \\ x_{t+1}\in\Gamma(x_t)}} \sum_{t=1}^{\infty} \beta^{t} F(x_t, x_{t+1}) \right\} \\
&= \max_{x_1\in\Gamma(x_0)} \left\{ F(x_0, x_1) + \beta \max_{\substack{\{x_{t+1}\}_{t=1}^{\infty} \\ x_{t+1}\in\Gamma(x_t)}} \sum_{t=0}^{\infty} \beta^{t} F(x_{t+1}, x_{t+2}) \right\} \\
&= \max_{x_1\in\Gamma(x_0)} \left\{ F(x_0, x_1) + \beta V^{*}(x_1) \right\}
\end{aligned}
$$

continued...

- idea: use BE to find value function V^* and policy function g [Principle of Optimality]

Outline Today's Lecture

- housekeeping: ps#1 and recitation day/ theory general / web page
- finish Principle of Optimality:

  Sequence Problem $\underset{\text{(for values and plans)}}{\Longleftrightarrow}$ solution to Bellman Equation
- begin study of Bellman equation with bounded and continuous F
- tools: contraction mapping and theorem of the maximum

Sequence Problem vs. Functional Equation

- Sequence Problem: (SP)

$$\begin{aligned} V^*(x_0) &= \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\ &\text{s.t. } x_{t+1} \in \Gamma(x_t) \\ &x_0 \text{ given} \end{aligned}$$

- ... more succinctly

$$V^*(x_0) = \sup_{\tilde{x} \in \Pi(x_0)} u(\tilde{x}) \tag{SP}$$

- functional equation (FE) [this particular FE called Bellman Equation]

$$V(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta V(y)\} \tag{FE}$$

Principle of Optimality

IDEA: use BE to find value function V^* and optimal plan x^*

- **Thm 4.2.** V^* defined by SP $\Rightarrow V^*$ solves FE
- **Thm 4.3.** V solves FE and ...... $\Rightarrow V = V^*$
- **Thm 4.4.** $\tilde{x}^* \in \Pi\left(x_0\right)$ is optimal

  $\Rightarrow V^*\left(x_t^*\right) = F\left(x_t^*, x_{t+1}^*\right) + \beta V^*\left(x_{t+1}^*\right)$
- **Thm 4.5.** $\tilde{x}^* \in \Pi\left(x_0\right)$ satisfies $V^*\left(x_t^*\right) = F\left(x_t^*, x_{t+1}^*\right) + \beta V^*\left(x_{t+1}^*\right)$ and ......

  $\Rightarrow \tilde{x}^*$ is optimal

Why is this Progress?

- **intuition:** breaks planning horizon into two: ‘now’ and ‘then’
- **notation:** reduces unnecessary notation (especially with uncertainty)
- **analysis:** prove existence, uniqueness and properties of optimal policy (e.g. continuity, montonicity, etc...)
- **computation:** associated numerical algorithm are powerful for many applications

Proof of Theorem 4.3 (max case)

Assume for any $\tilde{x} \in \Pi(x_0)$

$$\lim_{T \to \infty} \beta^T V(x_T) = 0.$$

BE implies

$$\begin{aligned} V(x_0) &\geq F(x_0, x_1) + \beta V(x_1), \text{ all } x_1 \in \Gamma(x_0) \\ &= F(x_0, x_1^*) + \beta V(x_1^*), \text{ some } x_1^* \in \Gamma(x_0) \end{aligned}$$

Substituting $V(x_1)$:

$$\begin{aligned} V(x_0) &\geq F(x_0, x_1) + \beta F(x_1, x_2) + \beta^2 V(x_2), \text{ all } x \in \Pi(x_0) \\ &= F(x_0, x_1^*) + \beta F(x_1^*, x_2^*) + \beta^2 V(x_2^*), \text{ some } x^* \in \Pi(x_0) \end{aligned}$$

Continue this way

$$\begin{aligned} V(x_0) &\geq \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1}) + \beta^{n+1} V(x_{n+1}) \text{ for all } x \in \Pi(x_0) \\ &= \sum_{t=0}^{n} \beta^t F\left(x_t^*, x_{t+1}^*\right) + \beta^{n+1} V\left(x_{n+1}^*\right) \text{ for some } x^* \in \Pi(x_0) \end{aligned}$$

Since $\beta^T V(x_T) \to 0$, taking the limit $n \to \infty$ on both sides of both expressions we conclude that:

$$\begin{aligned} V(x_0) &\geq u(\tilde{x}) \text{ for all } \tilde{x} \in \Pi(x_0) \\ V(x_0) &= u(\tilde{x}^*) \text{ for some } \tilde{x}^* \in \Pi(x_0) \end{aligned}$$

Bellman Equation as a Fixed Point

- define operator

$$T(f)(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta f(y)\}$$

- V solution of BE $\iff$ V fixed point of T [i.e. $TV = V$]

Bounded Returns:

- if $\|F\| < B$ and F and Γ are continuos: T maps continuous bounded functions into continuous bounded functions
- bounded returns $\Rightarrow$ T is a Contraction Mapping $\Rightarrow$ unique fixed point
- many other bonuses

Needed Tools

- Basic Real Analysis (section 3.1):

  $\{$vector, metric, noSLP, complete$\}$ spaces

  cauchy sequences

  closed, compact, bounded sets

- Contraction Mapping Theorem (section 3.2)

- Theorem of the Maximum: study of RHS of Bellman equation (equivalently of T) (section 3.3)

Bellman Equation: Principle of Optimality

- Principle of Optimality idea: use the functional equation

$$V(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta V(y)\}$$

  to find V^* and g

- note: nuisance subscripts t, $t+1$, dropped
- a solution is a function $V(\cdot)$ the same on both sides
- **IF** BE has unique solution then $V^* = V$
- more generally the “right solution” to (BE) delivers V^*

Outline Today's Lecture

- study Functional Equation (Bellman equation) with bounded and continuous F
- tools: contraction mapping and theorem of the maximum

Bellman Equation as a Fixed Point

- define operator

$$T\left(f\right)\left(x\right)=\max_{y\in\Gamma(x)}\left\{F\left(x,y\right)+\beta f\left(y\right)\right\}$$

- V solution of BE $\iff$ V fixed point of T [i.e. $TV=V$]

Bounded Returns:

- if $\|F\|<B$ and F and Γ are continuous: T maps continuous bounded functions into continuous bounded functions
- bounded returns $\Rightarrow$ T is a Contraction Mapping $\Rightarrow$ unique fixed point
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Our Favorite Metric Space

$$S = \left\{ f : X \to R,\ f \text{ is continuous, and } \|f\| \equiv \sup_{x \in X} |f(x)| < \infty \right\}$$

with

$$\rho(f, g) = \|f - g\| \equiv \sup_{x \in X} |f(x) - g(x)|$$

Definition. A linear space S is complete if any Cauchy sequence converges. For a definition of a Cauchy sequence and examples of complete metric spaces see SLP.

Theorem. The set of bounded and continuous functions is Complete. See SLP.

Contraction Mapping

Definition. Let (S, ρ) be a metric space. Let $T : S \to S$ be an operator. T is a contraction with modulus $\beta \in (0, 1)$

$$\rho\left(Tx, Ty\right) \leq \beta\rho\left(x, y\right)$$

for any x, y in S.

Contraction Mapping Theorem

Theorem (CMThm). If T is a contraction in (S, ρ) with modulus β, then (i) there is a unique fixed point $s^* \in S$,

$$s^* = Ts^*$$

and (ii) iterations of T converge to the fixed point

$$\rho\left(T^n s_0, s^*\right) \leq \beta^n \rho\left(s_0, s^*\right)$$

for any $s_0 \in S$, where $T^{n+1}(s) = T\left(T^n(s)\right).$

CMThm – Proof

for (i) **1st step:** construct fixed point s^*
take any $s_0 \in S$ define $\{s_n\}$ by $s_{n+1} = Ts_n$ then

$$\rho(s_2, s_1) = \rho(Ts_1, Ts_0) \leq \beta\rho(s_1, s_0)$$

generalizing $\rho(s_{n+1}, s_n) \leq \beta^n \rho(s_1, s_0)$ then, for $m > n$

$$\begin{aligned}
\rho(s_m, s_n) &\leq \rho(s_m, s_{m-1}) + \rho(s_{m-1}, s_{m-2}) + ... + \rho(s_{n+1}, s_n) \\
&\leq \left[\beta^{m-1} + \beta^{m-2} + ... + \beta^n\right] \rho(s_1, s_0) \\
&\leq \beta^n \left[\beta^{m-n-1} + \beta^{m-n-2} + ... + 1\right] \rho(s_1, s_0) \\
&\leq \frac{\beta^n}{1-\beta} \rho(s_1, s_0)
\end{aligned}$$

thus $\{s_n\}$ is cauchy. hence $s_n \to s^*$

2nd step: show $s^* = Ts^*$

$$\begin{aligned} \rho\left(Ts^*, s^*\right) &\leq \rho\left(Ts^*, s_n\right) + \rho\left(s^*, s_n\right) \\ &\leq \beta\rho\left(s^*, s_{n-1}\right) + \rho\left(s^*, s_n\right) \to 0 \end{aligned}$$

3nd step: s^* is unique. $Ts_1^* = s_1^*$ and $s_2^* = Ts_2^*$

$$0 \leq a = \rho\left(s_1^*, s_2^*\right) = \rho\left(Ts_1^*, Ts_2^*\right) \leq \beta\rho\left(s_1^*, s_2^*\right) = \beta a$$

only possible if $a = 0 \Rightarrow s_1^* = s_2^*$.
Finally, as for (ii):

$$\rho\left(T^n s_0, s^*\right) = \rho\left(T^n s_0, Ts^*\right) \leq \beta\rho\left(T^{n-1} s_0, s^*\right) \leq \cdots \leq \beta^n \rho\left(s_0, s^*\right)$$

Corollary. Let S be a complete metric space, let $S' \subset S$ and S' close. Let T be a contraction on S and let $s^* = Ts^*$. Assume that

$$T\left(S'\right) \subset S', \quad \text{i.e. if } s' \in S, \text{ then } T\left(s'\right) \in S'$$

then $s^* \in S'$. Moreover, if $S'' \subset S'$ and

$$T\left(S'\right) \subset S'', \ \text{ i.e. if } s' \in S', \text{ then } T\left(s'\right) \in S''$$

then $s^* \in S''$.

Blackwell's sufficient conditions.
Let S be the space of bounded functions on X, and $\|\cdot\|$ be given by the sup norm. Let $T : S \rightarrow S$. Assume that (i) T is monotone, that is,

$$Tf\left(x\right) \leq Tg\left(x\right)$$

for any $x \in X$ and g, f such that $f\left(x\right) \geq g\left(x\right)$ for all $x \in X$, and (ii) T discounts, that is, there is a $\beta \in \left(0, 1\right)$ such that for any $a \in R_{+}$,

$$T\left(f + a\right)\left(x\right) \leq Tf\left(x\right) + a\beta$$

for all $x \in X$. Then T is a contraction.

Proof. By definition

$$f = g + f - g$$

and using the definition of $\|\cdot\|$,

$$f(x) \leq g(x) + \|f - g\|$$

then by monotonicity i)

$$Tf \leq T(g + \|f - g\|)$$

and by discounting ii) setting $a = \|f - g\|$

$$Tf \leq T(g) + \beta \ \|f - g\| .$$

Reversing the roles of f and g :

$$Tg \leq T(f) + \beta \ \|f - g\|$$

$$\Rightarrow \|Tf - Tg\| \leq \beta \|f - g\|$$

Bellman equation application

$$(Tv)(x) = \max_{y \in \Gamma(x)} \{F(x,y) + \beta v(y)\}$$

Assume that F is bounded and continuous and that Γ is continuous and has compact range.

Theorem. T maps the set of continuous and bounded functions S into itself. Moreover T is a contraction.

Proof. That T maps the set of continuos and bounded follow from the Theorem of Maximum (we do this next)
That T is a contraction follows since T satisfies the Blackwell sufficient conditions.
T satisfies the Blackwell sufficient conditions. For monotonicity, notice that for $f \geq v$

$$\begin{aligned} Tv(x) &= \max_{y \in \Gamma(x)} \{F(x,y) + \beta v(y)\} \\ &= F(x, g(x)) + \beta v(g(x)) \\ &\leq \{F(x, g(y)) + \beta f(g(x))\} \\ &\leq \max_{y \in \Gamma(x)} \{F(x,y) + \beta f(y)\} = Tf(x) \end{aligned}$$

A similar argument follows for discounting: for $a > 0$

$$T(v+a)(x) = \max_{y \in \Gamma(x)} \{F(x,y) + \beta(v(y) + a)\}$$
$$= \max_{y \in \Gamma(x)} \{F(x,y) + \beta v(y)\} + \beta a = T(v)(x) + \beta a.$$

Theorem of the Maximum

- want T to map continuous function into continuous functions

$$(Tv)(x) = \max_{y \in \Gamma(x)} \{F(x,y) + \beta v(y)\}$$

- want to learn about optimal policy of RHS of Bellman

$$G(x) = \arg\max_{y \in \Gamma(x)} \{F(x,y) + \beta v(y)\}$$

- First, continuity concepts for correspondences
- ... then, a few example maximizations
- ... finally, Theorem of the Maximum

Continuity Notions for Correspondences

assume Γ is non-empty and compact valued (the set $\Gamma(x)$ is non empty and compact for all $x \in X$)
Upper Hemi Continuity (u.h.c.) at x: for any pair of sequences $\{x_n\}$ and $\{y_n\}$ with $x_n \to x$ and $x_n \in \Gamma(y_n)$ there exists a subsequence of $\{y_n\}$ that converges to a point $y \in \Gamma(x)$.

Lower Hemi Continuity (l.h.c.) at x: for any sequence $\{x_n\}$ with $x_n \to x$ and for every $y \in \Gamma(x)$ there exists a sequence $\{y_n\}$ with $x_n \in \Gamma(y_n)$ such that $y_n \to y$.

Continuous at x: if Γ is both upper and lower hemi continuous at x

Max Examples

$$\begin{aligned} h(x) &= \max_{y \in \Gamma(x)} f(x,y) \\ G(x) &= \arg \max_{y \in \Gamma(x)} f(x,y) \end{aligned}$$

ex 1: $f(x,y) = xy$; $X = [-1,1]$; $\Gamma(x) = X$.

$$\begin{aligned} G(x) &= \begin{cases} \{-1\} & x < 0 \\ [-1,1] & x = 0 \\ \{1\} & x > 0 \end{cases} \\ h(x) &= |x| \end{aligned}$$

continued...

ex 2: $f(x,y) = xy^2$; $X = [-1,1]$; $\Gamma(x) = X$

$$\begin{aligned} G(x) &= \begin{cases} \{0\} & x<0 \\ [-1,1] & x=0 \\ \{-1,1\} & x>0 \end{cases} \\ h(x) &= \max\{0,x\} \end{aligned}$$

Theorem of the Maximum

Define:

$$\begin{aligned} h(x) &= \max_{y \in \Gamma(x)} f(x,y) \\ G(x) &= \arg\max_{y \in \Gamma(x)} f(x,y) \\ &= \{y \in \Gamma(x) : \; h(x) = f(x,y)\} \end{aligned}$$

Theorem. (Berge) Let $X \subset \mathbb{R}^l$ and $Y \subset \mathbb{R}^m$. Let $f : X \times Y \to \mathbb{R}$ be continuous and $\Gamma : X \to Y$ be compact-valued and continuous. Then $h : X \to \mathbb{R}$ is continuous and $G : X \to Y$ is non-empty, compact valued, and u.h.c.

$\lim \max \rightarrow \max \lim$

Theorem. Suppose $\{f_n(x,y)\}$ and $f(x,y)$ are concave in y and $f_n \rightarrow f$ in the sup-norm (uniformly). Define

$$\begin{aligned} g_n(x) &= \arg \max_{y \in \Gamma(x)} f_n(x,y) \\ g(x) &= \arg \max_{y \in \Gamma(x)} f(x,y) \end{aligned}$$

then $g_n(x) \rightarrow g(x)$ for all x (pointwise convergence); if X is compact then the convergence is uniform.

Uses of Corollary of CMThm

Monotonicity of v^*
Theorem. Assume that $F(\cdot, y)$ is increasing, that Γ is increasing, i.e.

$$\Gamma(x) \subset \Gamma(x')$$

for $x \leq x'$. Then, the unique fixed point v^* satisfying $v^* = Tv^*$ is increasing. If $F(\cdot, y)$ is strictly increasing, so is v^*.

Proof

By the corollary of the CMThm, it suffices to show Tf is increasing if f is increasing. Let $x \leq x'$:

$$\begin{aligned} Tf(x) &= \max_{y \in \Gamma(x)} \{F(x,y) + \beta f(y)\} \\ &= F(x,y^*) + \beta f(y^*) \text{ for some } y^* \in \Gamma(x) \\ &\leq F(x',y^*) + \beta f(y^*) \end{aligned}$$

since $y^* \in \Gamma(x) \subset \Gamma(x')$

$$\leq \max_{y \in \Gamma(x')} \{F(x,y) + \beta f(y)\} = Tf(x')$$

If $F(\cdot, y)$ is strictly increasing

$$F(x,y^*) + \beta f(y^*) < F(x',y^*) + \beta f(y^*).$$

Concavity (or strict) concavity of v^*

Theorem. Assume that X is convex, Γ is concave, i.e. $y \in \Gamma(x)$, $y' \in \Gamma(x')$ implies that

$$y^\theta \equiv \theta y' + (1-\theta) y \in \Gamma(\theta x' + (1-\theta) x) \equiv \Gamma\left(x^\theta\right)$$

for any $x, x' \in X$ and $\theta \in (0,1)$. Finally assume that F is concave in (x, y). Then, the fixed point v^* satisfying $v^* = Tv^*$ is concave in x. Moreover, if $F(\cdot, y)$ is strictly concave, so is v^*.

Differentiability

- can't use same strategy: space of differentiable functions is not closed
- many envelope theorems
- Formula: if $h\left(x\right)$ is differentiable and y is interior then

  $$h'\left(x\right) = f_x\left(x, y\right)$$

  right value... but is h differentiable?
- one answer (Demand Theory) relies on f.o.c. and assuming twice differentiability of f
- won't work for us since $f = F(x, y) + \beta V\left(y\right)$ and we don't even know if f is once differentiable! $\rightarrow$ going in circles

Benveniste and Sheinkman

First a Lemma...

Lemma. Suppose $v(x)$ is concave and that there exists $w(x)$ such that $w(x) \leq v(x)$ and $v(x_0) = w(x_0)$ in some neighborhood D of x_0 and w is differentiable at x_0 ($w'(x_0)$ exists) then v is differentiable at x_0 and $v'(x_0) = w'(x_0)$.

Proof. Since v is concave it has at least one subgradient p at x_0 :

$$w(x) - w(x_0) \leq v(x) - v(x_0) \leq p \cdot (x - x_0)$$

Thus a subgradient of v is also a subgradient of w. But w has a unique subgradient equal to $w'(x_0)$.$\square$

Benveniste and Sheinkman

Now a Theorem

Theorem. Suppose F is strictly concave and Γ is convex. If $x_0 \in int\,(X)$ and $g\,(x_0) \in int\,(\Gamma\,(x_0))$ then the fixed point of T, V, is differentiable at x and

$$V'\,(x) = F_x\,(x, g\,(x))$$

Proof. We know V is concave. Since $x_0 \in int\,(X)$ and $g\,(x_0) \in int\,(\Gamma\,(x_0))$ then $g\,(x_0) \in int\,(\Gamma\,(x))$ for $x \in D$ a neighborhood of x_0 then

$$W\,(x) = F\,(x, g\,(x_0)) + \beta V\,(g\,(x_0))$$

and then $W\,(x) \leq V\,(x)$ and $W\,(x_0) = V\,(x_0)$ and $W'\,(x_0) = F_x\,(x_0, g\,(x_0))$ so the result follows from the lemma.