Investment Options and the Business Cycle

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Abstract

This paper extends Lucas (1978) to a production economy with two capital goods. It is an RBC model in which each unit of investment requires a new idea, an “option”. When options are scarce, new capital is harder to put in place and the value of old capital rises. Thus the stock market and Tobin’s Q are negative indexes of intangibles. During a boom, Q rises gradually, as options are used up. Because investment represents an exercise of options, the model has a larger intertemporal substitution incentive, and one that helps explain the high volatility of Q. In the model, equilibrium is efficient even without markets for knowledge; the stock market suffices.

1 Introduction

This paper extends Lucas (1978) to a production economy with two capital goods. One is a traditional capital stock, the other is unimplemented knowledge that I refer to as “investment options.” I shall refer to traditional capital as “trees” and to the unexercised options as “seeds.” The paper is a specialization of the Lucas (1978) model in the sense that the shocks to the trees’ productivities are common.

An investment option is a profit opportunity that requires an investment to implement. It is postponable if it is a patented invention, or if it is specific to a firm so that others cannot reduce its value by copying it. A firm has investment options that it may use up immediately, or store for future use. A patent, for instance, represents an investment option that only its holder can implement for a certain number of years. In a sense, even a trademark represents an option to produce a product that no one else can make. Some investment options are protected only by secrecy.

I set up a competitive GE model in which to plant a tree one needs a seed. Seeds are produced by trees that are already planted. The number of trees grows over

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time and, in the absence of the seed constraint on investment, the model would be a standard one-sector $Ak$ model with shocks to technology.

The model is a GE version of Abel and Eberly (2005) and it endogenizes what they call “growth options.” It also relates closely to the GE model of Bilbiie, Ghironi and Melitz (2006) who study the cyclical behavior of entry of producers but in the absence of a seed constraint or of “option-value considerations,” as they put it, so that Tobin’s $Q$ is always unity. It also relates to the RBC model of Backus, Routlege and Zin (2006). The model has no shocks to the investment technology, but it behaves a bit like models that do have such shocks, such as Greenwood, Hercowitz and Huffman (1987), and Fisher (2005). Investment options are also a focus of the new Keynesian literature (Shleifer 1986), and the strategic delay literature (Chamley and Gale 1994).

The paper’s main results are as follows:

(i) **Intangibles reduce $Q$.**—Because seeds (i.e., “intangibles”) are scarce, the value of planted trees (i.e., firms) and thus Tobin’s $Q$, is always above unity. When there are many seeds, their price falls, and if the decline is sharp enough, so then does the price of claims to the output of planted trees.¹ Contrast that to Hall (2000), where $Q$ is a positive indicator of the stock of intangibles because his intangibles are not embodied in capital but, rather, enter the final-goods production function as a separate input. A natural experiment that would easily distinguish the two models is a prolonged war during which unimplemented inventions accumulate (some of them being incidental accomplishments of defense-oriented research). My model says that after such a war $Q$ should be low, whereas Hall’s model says that it should be high. It turns out that after both world wars, $Q$ in the U.S. was unusually low. Also, measures of intangibles based on aggregate patent applications and trademarks co-move negatively with Tobin’s $Q$, thus supporting my model.

(ii) **$Q$ rising during a boom.**—The main difference between the seeds model and the adjustment-cost model is that during a prolonged boom during which seeds are drawn down and become ever more scarce, Tobin’s $Q$ rises gradually. That was indeed what happened during the well-known run-ups of the late ’20s and mid-late ’90s.

(iii) **More volatile $Q$.**—The seeds model introduces an intertemporal substitutability in investment that raises its volatility for given average levels of $Q$.² By contrast, convex adjustment costs make investment smoother. The concept of investment in this model is “extensive” investment in new things, and such investment responds more elastically to variations in $Q$; witness, e.g., how closely and elastically venture-backed investment follows the Nasdaq index.³ In equilibrium, however, investment

¹This is a GE effect that arises when all firms have more seeds. For any firm that alone receives an additional seed, $Q$ would rise, just as in Abel and Eberly (2005).

²Intertemporal substitutability raises supply elasticities generally; e.g., of labor supply in the Lucas and Rapping (1969) model, or of sales in the Khan and Thomas (2005) model.

³The model was originally meant to explain bunching of investment, with IPO waves as the empirical counterpart of such bunching – IPOs are also co-move with the stock market. In a partial equilibrium model, Pastor and Veronesi (2005) argue that investment options are stored during
is bounded by the available seeds, and this can raise $Q$ to higher levels than in the quadratic-adjustment-cost model.

(iv) Decentralization.—These results obtain in two decentralizations of the planner’s optimum. The first has complete markets. The second decentralization has no seeds market, only a market for shares of firms. The outcome for quantities is the same.

The model is similar to ones in which it is not adjustment costs but, rather, liquidity constraints that prevent firms from eliminating the gap between $Q$ and unity. Gomes, Yaron and Zhang (forthcoming) look for a liquidity factor in asset prices, and find one that, however, behaves more procyclically, seeming to pose a greater constraint in the boom, which is consistent with the seeds model but regarded as inconsistent with the financial frictions model that they propose.

Section 2 presents the model, section 3 describes a complete-markets decentralization, section 4 an incomplete-markets one. Section 5 reports simulations, and compares the model to the data. Section 6 compares the model to the standard adjustment-cost model. Section 7 reports the cross-correlogram between $Q$ and GDP. Several proofs and extensions are reported in the Appendix.

2 Model

The model is that of a growing economy with two types of capital — trees, $k$, and seeds, $S$. A seed represents an option, storable indefinitely, to plant exactly one tree.

Production of fruit.—Output of fruit is

$$Y = zk.$$  \hfill (1)

If $X$ is the number of trees newly planted, $k$ evolves as

$$k' = k + X.$$  \hfill (2)

Production of seeds.—Let $S$ denote the stock of seeds. New seeds are produced by existing trees. Each period a tree gives rise to $\lambda$ new seeds, i.e., a total of

$$\text{new seeds} = \lambda k.$$  \hfill (3)

Thus seeds grow via a process like learning by doing that takes up no resources.

The planting of trees.—Planting a tree requires a unit of fruit and a seed. Only one tree per seed can be planted, after which the seed is used up. Let $S$ be the stock of seeds and let $X$ be the number of trees planted. Then $S$ evolves as

$$S' = \lambda k + S - X \geq 0.$$  \hfill (4)

times of high uncertainty, and that once the uncertainty is resolved, firms rush in with IPOs.
Since \( X \) is subtracted from the stock, a seed can be used to plant exactly one tree. Thus investment is Leontief in two inputs, seeds and fruit. Their proportions are equal, an assumption that we shall drop when we get to the empirics, along with the assumption that neither \( k \) nor \( S \) depreciate. Leontief investment implies that output too is Leontief in seeds and fruit. Seeds are storable whereas fruit is not.

**Timing.**—Investment, \( X \), is chosen after the trees produce \( zk \) units of fruit and after \( \lambda k \) new seeds. From (4), we have

\[
X \leq \lambda k + S. \tag{5}
\]

Thus investment is Leontief in two inputs: seeds and fruit. We shall let investment be reversible.

**The income identity.**—The cost of planting a tree is, as usual, one unit of fruit. Letting \( C \) be the consumption of fruit, the income identity is

\[
zk = C + X. \tag{6}
\]

**The shocks.**—We assume that the shocks follow the first-order Markov process:

\[
\Pr (z_{t+1} \leq z' \mid z_t = z) = F(z', z), \text{ and that } z' \text{ is stochastically increasing in } z.
\]

**Preferences.**—For \( \sigma > 0 \) and \( \beta < 1 \), preferences are

\[
E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma} \right\}.
\]

**The standard one-sector growth model.**—It arises when the inequality in (5) never binds. The latter occurs when \( \lambda \) is large enough, e.g., if \( \lambda \) exceeds the largest possible \( z \). It also occurs, de facto, when the initial stock of seeds, \( S_0 \), is so large that (5) does not come into play for a very long time.

### 2.1 The planner’s problem

The state is \((k, S, z)\), and the decision, \( X \), is constrained by (5). The Bellman eq. is

\[
v (k, S, z) = \max_{X \leq \lambda k+S} \left\{ \frac{(zk-X)^{1-\sigma}}{1-\sigma} + \beta \int v (k+X, \lambda k+S-X, z') \ dF \right\}. \tag{7}
\]

**Lemma 1** A unique solution \( v \) to (7) exists, and is strictly concave in \((k, S)\). Moreover, \( X \) is increasing in \( S \) and, if \( z \) is i.i.d., in \( z \).

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\(^4\)Sargent (1980) analyzes the implications of the constraint \( X \geq 0 \). This constraint is not imposed here, but it is never violated in the simulations.
Lemma 2

The following result allows us to reduce the state space to just for the investment rate. From (4), the law of motion for

\[ T\tilde{v}(\alpha k_1 + (1 - \alpha) k_2, \alpha S_1 + (1 - \alpha) S_2) \geq \frac{(zk - X_\alpha)^{1-\sigma}}{1 - \sigma} + \beta \int \tilde{v}(k + X_\alpha, \lambda k + S - X_\alpha, z') dF \]

Therefore if \( 0 < \alpha < 1 \)

\[ T\tilde{v}(\alpha k_1 + (1 - \alpha) k_2, \alpha S_1 + (1 - \alpha) S_2) < \alpha T\tilde{v}(k_1, S_1) + (1 - \alpha) T\tilde{v}(k_2, S_2) \]

Therefore the operator transforms weakly into strictly concave functions. Therefore, the operator being a contraction, its unique fixed point \( v \) is strictly concave. \((iii)\) Properties of \( X \): The FOC is

\[ \xi(X, S) \equiv - (zk - X)^{-\sigma} + \beta \int \frac{d}{dX} v(k + X, \lambda k + S - X, z') dF = 0 \quad (8) \]

(Once-differentiability of \( v \) will be proved later) We have dropped \( k \) and \( z \) from the arguments of \( \xi \) as they play no role in the proof. We now argue in 3 steps: \((A)\) If a function of one variable \( H \) is twice differentiable with \( H'' < 0 \), then

\[ \frac{\partial}{\partial S} \left( \frac{\partial}{\partial X} H [\lambda k + S - X] \right) = -H''(\cdot) > 0 \]

Therefore, concavity of \( v \) in \( S \) alone implies \( \frac{\partial}{\partial S} \frac{dv}{dx} \left( = -\frac{\partial^2 v}{\partial S^2} \right) > 0 \) (the monotonicity results do not require the second derivatives) earlier, under \((ii)\) we showed that concavity of \( v \) in \( (k, S) \) implies concavity of \( v \) in \( X \) holding \( (k, S) \) fixed – i.e., that \( \frac{\partial^2 v}{\partial X^2} < 0 \) and \((B)\) Therefore \( \xi_X < 0 \) and \( \xi_S > 0 \). And, when \( z \) is i.i.d., \( \xi_z = \sigma (zk - X)^{-1-\sigma} \).

\((C)\) Therefore \( \frac{\partial x}{\partial S} = -\frac{\xi_s}{\xi_X} > 0 \), and when \( z \) is i.i.d., \( \frac{\partial X}{\partial z} = -\frac{\xi_z}{\xi_X} > 0 \). \(\blacksquare\)

Reducing the state space.—The absence of labor or of any other fixed factor allows us to reduce the number of states from three to two – only the ratio \( s \equiv S/k \) matters for the investment rate. From (4), the law of motion for \( s \) is \( s' = \frac{\lambda + s - x}{1 + x} \), implying the inequality constraint

\[ x \leq \lambda + s \quad (9) \]

The following result allows us to reduce the state space to just \((s, z)\):

**Lemma 2** For \( \sigma \neq 1, v \) is of the form

\[ v(k, S, z) = w(s, z) k^{1-\sigma}, \]

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Figure 1: Relation to the Convex Adjustment-Cost Model

where \( w(s, z) = v(1, s, z) \), and where \( w \) satisfies

\[
  w(s, z) = \max_{x \leq \lambda + s} \left\{ \frac{(z - x)^{1-\sigma}}{1 - \sigma} + (1 + x)^{1-\sigma} \beta \int w \left( \frac{\lambda + s - x}{1 + x}, z' \right) dF \right\}.
\]

Moreover, \( v \) and \( w \) are of the same sign as \( 1 - \sigma \).

The proof (not reported) substitutes the desired functional form for \( v \) on the RHS of (7), and verifies that the same functional form emerges on the LHS. The case \( \sigma = 1 \) is covered separately below. Similar results are in Alvarez and Stokey (2000).

**Corollary 1** A unique solution \( w \) to (10) exists that is increasing and concave in \( s \).

**Proof.** Existence: Since a unique \( v \) exists, \( w(s, z) = v(k, S, z) k^{-(1-\sigma)} \) is the unique solution for \( w \). Increasing: In (9), a rise in \( s \) relaxes the constraint on \( x \). Moreover, if one inserts on the RHS of (10) a function \( w \) that increases in \( s \), evidently the property is preserved. Concave: The concavity of \( v(k, S, z) k^{-(1-\sigma)} \) in \( S \) for fixed \( k \) implies that \( w \) is concave in \( s \).

**Corollary 2** The policy \( x(s, z) \) is increasing in \( s \) and, if \( z \) is i.i.d., increasing in \( z \).

**Proof.** All changes in \( s \equiv S/k \) can be interpreted as changes in \( S \) for a given \( k \). By Lemma 2, \( X \) is, for all \( k \), increasing in \( S \). For fixed \( k \), a rise in \( S \) implies a rise in \( s \) and in \( x \). The claim about \( z \) follows at once from Lemma 2.

The relation to other models is easily seen graphically. In its left panel, Figure 1 shows the consumption-investment trade-off in the standard model and the convex-adjustment-cost model. In its right panel, the Figure shows the constraint imposed
Figure 2: MARGINAL ADJUSTMENT COSTS

by a particular upper bound on \( x \), namely \( \lambda + s \). Since \( s \geq 0 \), investment can never be constrained by any number smaller than \( \lambda \), and so that’s the tightest constraint on \( x \) that can possibly arise. The position of the constraint will depend on what has been happening earlier. In particular, a “seed crunch” and with it a high value of \( Q \) will turn out to be more likely following a prolonged boom caused by a succession of large realizations of \( z \). Such realizations are likely to draw \( s \) to its minimum level of zero, leading the constraint to be \( x \leq \lambda \).

We can also illustrate this in terms of the marginal cost of investment. Let

\[
C'(x, s) = \frac{\text{investment cost}}{\text{capital stock}} = \begin{cases} x & \text{if } x \leq \lambda + s \\ \infty & \text{otherwise} \end{cases}
\]

denote the cost of investment, in units of fruit. The marginal adjustment costs, \( \frac{\partial}{\partial x} C(x, s) \), are drawn in Figure 2. Thus today’s costs of adjustment are still convex in \( x \), but \( x \) also raises future investment costs, an effect absent from the standard convex adjustment-cost model. Thus, there are two differences between this model and the standard one. First, the feasible set in Figure 1 has a different shape and, second, there is intertemporal substitution in investment.

**Lemma 3** \( w \) is strictly increasing in \( z \).

**Proof.** The constraint set \( x \in [-1, z] \) is stochastically increasing in \( z \). Since \( z' \) is stochastically increasing in \( z \), for any function \( w(s, z) \) increasing in \( z' \), the second term on the RHS of (10) is increasing in \( z \). Moreover, since \( C \geq 0 \), the first term on the RHS of (10) is strictly increasing in \( z \).

**Lemma 4** \( w \) is differentiable with respect to \( s \), with derivative

\[
w_s = \frac{1}{1 + \lambda + s} \left( [1 - \sigma] w - (1 + z) [z - x]^{-\sigma} \right) > 0
\]
for all \((s, z)\).

The proof is in Appendix 1; it follows the proof of proposition 2 of Lucas (1978) but is complicated by the constraint (9).

Note that the term \((1 - \sigma) w\) is positive for all \(\sigma \neq 1\) because for \(\sigma > 1\), \(w < 0\).

Lemma 5 The optimal policy \(x(s, z)\) satisfies

\[
1 - \beta \int \left( \frac{(1 + x)}{z - x} \right)^{-\sigma} \left[ (z' - x')^{-\sigma} (1 + z') + \lambda w_s' \right] dF \left\{ \begin{array}{ll} = 0 & \text{if } s' > 0 \\ \leq 0 & \text{if } s' = 0 \end{array} \right. \tag{12}
\]

Proof. By Lemma 2, \(v\) is differentiable w.r.t. \(k\), and if \(w\) is differentiable w.r.t. \(s\), so is \(v\) w.r.t. \(S\). Then the FOC is

\[
C^{-\sigma} - \beta \int (v_k - v_S) dF \leq 0, \tag{13}
\]

with equality if \(s' > 0\). We have

\[
v (k, S, z) = \max_{s'} \left\{ \frac{(zk + S' - \lambda k - S)^{1-\sigma}}{1 - \sigma} + \beta \int v (k + \lambda k + S - S', S', z') dF \right\}.
\]

The envelope result (since \(S\) does not enter the constraint \(S' \geq 0\)) is

\[
v_S = -C^{-\sigma} + \beta \int v_k dF
\]

and

\[
v_k = (z - \lambda) C^{-\sigma} + (1 + \lambda) \beta \int v_k dF = (z - \lambda) C^{-\sigma} + (1 + \lambda) (v_S + C^{-\sigma}) = (1 + \lambda) v_S + (1 + z) C^{-\sigma}
\]

But by Lemma 2, \(v (k, S, z) = w \left( \frac{S}{k}, z \right) k^{1-\sigma}\) so that

\[
v_k = (1 - \sigma) wk^{-\sigma} - sw_kk^{-\sigma} \quad \text{and} \quad v_S = w_s k^{-\sigma}
\]

Now, from (14), \(v_k = (1 + \lambda) v_S + (1 + z) C^{-\sigma}\), so that the FOC becomes

\[
C^{-\sigma} - \beta \int \left( \lambda v_S' + (1 + z) C^{\sigma-\sigma} \right) dF \leq 0
\]

But \(v_S = w_s k^{-\sigma}\) and the above equation then reads

\[
0 \geq (z - x)^{-\sigma} k^{-\sigma} - \beta \int \left( \lambda w_s' (k')^{-\sigma} + (1 + z') (z' - x')^{-\sigma} (k')^{-\sigma} \right) dF
\]

\[
= \left( \frac{z - x}{1 + x} \right)^{-\sigma} - \beta \int \left( \lambda w_s' + (1 + z') (z' - x')^{-\sigma} \right) dF, \tag{15}
\]

from which (12) follows. ■
2.1.1 The set on which (5) binds

Consumption is most volatile and investment least volatile when (5) binds. Let $\Delta = \{(s, z) \mid x(s, z) = \lambda + s\}$ be the set of states for which (5) binds. In this region, $X$ cannot respond to $z$ and therefore $C$ moves one-for-one with $zk$ and, hence, is more volatile than in the standard model. True, this statement is conditional on $s$, but for $(s, z) \in \Delta$, $s^0 = 0$, and $x^0 = x(0, z^0)$. If $(s, z)$ remain in $\Delta$ for more than one period, then in period two and beyond,

$$x(0, z) = \lambda \text{ and } c = z - \lambda.$$  

The further $z$ is from being a random walk (and it seems to depart substantially from it, see Table 1), the more these rules depart from what the standard model would predict. In contrast, when $s \to \infty$, we get the standard model, for then the probability that (5) will bind in the foreseeable future goes to zero.

Even when $z$ is i.i.d., $x$ is increasing in $z$ because a higher $z$ today raises wealth and causes a rise in desired future consumption. Because $x$ is increasing in $z$, $\Delta$ contains large $z$ values. For $(s, z) \in \Delta$, $s' = 0$ so that $x' = x(0, z')$. Let $z^*(s) = \inf \{z \mid (z, s) \in \Delta\}$ be the lower boundary of $\Delta$. Then, as Figure 3 illustrates, we can then show the following:

**Proposition 1** If $z \sim F(z)$ is i.i.d., then

$$z^*(s) = \frac{1 + (1 + \alpha)(\lambda + s)}{\alpha},$$  

where $\alpha$ is the constant:

$$\alpha = \left(\beta \int \frac{\lambda(1 - \sigma)w(0, z') - (1 + z')(z' - x[0, z'])^{-\sigma}}{1 + \lambda}dF(z')\right)^{1/\sigma}.$$  

Figure 3: The set $\Delta$ when $z$ is i.i.d.
Proof. From (12) and from a one-period update of (11) we have

\[ \beta \int \left( \frac{1+x}{z-x} \right)^{-\sigma} \frac{\lambda(1-\sigma)w' - (1+s')(1+z')(z'-x')^{-\sigma}}{1+\lambda+s'} dF \begin{cases} = 1 & \text{if } s' > 0 \\ \geq 1 & \text{if } s' = 0 \end{cases} \]

i.e.,

\[ \beta \int \frac{\lambda(1-\sigma)w' - (1+s')(1+z')(z'-x')^{-\sigma}}{1+\lambda+s'} \begin{cases} = (\frac{1+z}{z-x})^{\sigma} & \text{if } s' > 0 \\ \geq (\frac{1+z}{z-x})^{\sigma} & \text{if } s' = 0 \end{cases} \]

i.e.,

\[ \alpha \begin{cases} = \frac{1+x}{z-x} & \text{if } s' > 0 \\ \geq \frac{1+x}{z-x} & \text{if } s' = 0 \end{cases} \]

On the other hand, if \( x \) is constrained and held constant at \( \lambda + s \) as \( z \) varies, the RHS is decreasing in \( z \). Large \( z \)'s make the inequality strict. We find the smallest \( z \) that will allow strict equality at \( x = \lambda + (1 + \lambda) s \). Setting it at equality we have \( 1 + \lambda + s = \alpha (z - \lambda - s) \), from which (16) follows. Moreover, for \( z = z^*(s) \) at \( s' = 0 \) so that \( x' = x(0,z') \), and \( w' = w(0,z') \), which yields (17).

On \( \Delta \), only \( Q \) responds to changes in \( z \); \( x \) does not, and therefore \( s' = \lambda \) is also unchanged. Therefore shocks to output today have no effect on output in any future period. Since \( \Delta \) contains mainly boom states the model thus implies that the persistence of output shocks is lower in booms. Moreover, in this case where \( z \) is i.i.d., changes in \( Q \) will not forecast output. This matches the finding of Henry et al. (2005) that the stock market predicts growth better in recessions than in booms.

When \( z \) is serially correlated, the boundary of \( \Delta \) is no longer linear but \( \Delta \) retains a shape similar to that portrayed in Figure 3: \( z^*(s) \) still solves (16) in which \( \alpha \) is replaced by

\[ \alpha(z) = \left( \beta \int \frac{\lambda(1-\sigma)w(0,z') - (1+s')(1+z')(z'-x')^{-\sigma}}{1+\lambda+s'} dF(z',z) \right)^{1/\sigma} \]

While \( x \) is less volatile on \( \Delta \), to achieve a given growth rate, \( x \) must make up for its low mean on \( \Delta \) with a higher mean off of \( \Delta \), which introduces a force towards bimodality in the distribution of \( x \) and a higher volatility of \( x \).\(^5\)

\(^5\)This section relates to Yorukoglu (2000) who studied the relation between income and the variety of goods and who also had two regimes for consumption.
3 Decentralization 1: Markets for trees and seeds

Assume that a market for seeds exists. This is not so unrealistic. Serrano (2006) finds that 18 percent of patents granted to small inventors are traded at least once in their lives, and that the citations-weighted percentage is even higher. Large firms also often sell their patents and enter into patent-sharing agreements with one another. Takeovers play a part in achieving transfers of intellectual capital; this is a fairly thick market in which Microsoft and Pfizer, e.g., have been highly active. A firm can be said to sell seeds when it spins off some activity, or when it hires people at wages that include a negative compensating differential for the value that its workers will draws from the experience gained; such a market is modeled, e.g., by Chari and Hopenhayn (1991). An example of employees walking out with seeds is Xerox in the 70’s – it had inventions that it was unable or unwilling to implement and that were later marketed by its former employees.

**Prices.**—Let $p$ be the price of seeds, and $q$ the price of a planted tree without a claim on its current-period outputs of fruit and seeds.

**Firms.**—Firms last one period. A firm pays its net sales of fruit, $z k - X$, in dividends every period, plus any profits it makes on buying and selling trees and seeds. A firm enters a period with the bundle $(k, S)$. It produces $zk$ units of fruit and $\lambda k$ seeds and it plants $X$ trees. At the end of the period, it sells its holdings $(k + X, S')$ to the public and to next generation of firms at the price vector $(q, p)$, pays all its profits out in dividends to its shareholders, and liquidates. Its profits are

$$
\pi(k, S, z) \equiv zk - X + q (k + X) + p S' \\
= (z + q) k + p (S + \lambda k) + (q - [1 + p]) X
$$

after substitution from (4) for $S'$.

**No arbitrage.**—A negative $X$ would entail selling off $X$ trees and $X$ seeds that those trees embody at a price of $1 + p$.\(^6\) Thus we have the no-arbitrage condition\(^7\)

$$
q = 1 + p, \quad (18)
$$

which implies that

$$
\pi(k, S, z) = (z + p \lambda + q) k + p S. \quad (19)
$$

The first three terms represent rents from trees, and the fourth rents from seeds.

**Households.**—Households own trees and seeds separately. Ownership of a tree delivers an end-of-period income $z + p \lambda + q$, and ownership of a seed delivers $p$. Ownership lapses after one period. The household’s budget constraint therefore reads

$$
C + p (S' - S) + q (k' - k) = zk + p \lambda k. \quad (20)
$$

\(^6\)This happens, e.g., when a company sells off a division, or when it is acquired.

\(^7\)This condition would hold even if aggregate investment is nonnegative; an individual firm could have $X < 0$ without affecting aggregates.
Equilibrium.—When the households’ and firms’ plans coincide, (2) and (4) hold, and when we use , the LHS of (20) becomes
\[ C + p(\lambda k - X) + qX = C + X + p\lambda k \text{ using (18)} \]
\[ = zk + p\lambda k \text{ using (6)}, \]
which coincides with the RHS of (20). Since the firm’s problem does not determine \( X \) (provided that [18] holds), we shall derive the decision rules for \( X \) from the household’s problem.

The household’s Bellman eq.—We shall assume that \( p = p(s, z) \) does not depend on \( k \), in which case (18) implies that \( q(s, z) \) also does not depend on \( k \). The household’s personal state is \((k, S)\), and it takes the aggregate state \((k, S, z)\) and the aggregate laws of motion for \((k, S)\) which depend on the decision rules of other households as given. The household’s Bellman equation is
\[ V(k, S, k, S, z) = \max_{k', S' \geq 0} \left\{ U(\zk + p\lambda k - p[S' - S] - q[k' - k]) + \beta \int V(k', S', k', S', z') dF \right\}, \]
where
\[ p = p\left(\frac{S}{k'}, z\right), \quad q = 1 + p\left(\frac{S}{k}, z\right), \quad k' = \khat'(k, S, k, S, z), \quad S' = S'(k, S, k, S, z) \]
and where \((\khat, \Shat)\) are functions that the household takes as given but that in equilibrium will have to coincide with the household’s policy rules — hence a fixed-point exercise in \((\khat, \Shat)\) is needed at the end.

Lemma 6 If equilibrium exists, it is the same as the planner’s optimum.

Proof. Since the household consumes all the output, equilibrium is optimal if and only if, for all \((k, S, z)\),
\[ V(k, S, k, S, z) = v(k, S, z). \]
Since the planner’s solution maximizes the representative household’s consumption, \( V \leq v \) for all states. But when (18) holds, then any \((C, k', S')\) that the planner can feasibly choose is also feasible to the household: Let \( C^h \) be the household’s choice \( C^p \) the planner’s choice: Plugging the planner’s choices \((k', S')\) into the household’s budget constraint, \( C^h = zk + p\lambda k - (p[S' - S] + q[k' - k]) = zk + p\lambda k - (p[\lambda k - X^p] + qX^p) = zk - X^p = C^p \). Then \( V \geq v \) and therefore (22) holds. ■

The household is not constrained by (4), so that the two first-order conditions must both hold with equality:
\[ k' : -qU'(C) + \beta \int V_1 dF = 0, \]
(23)
and
\begin{equation}
S' : -pU' (C) + \beta \int V_2 dF = 0. \tag{24}
\end{equation}

Efficiency of the equilibrium.—Subtracting (24) from (23) and applying (18) yields
\begin{equation}
-U' (C) + \beta \int (V_1 - V_2) dF.
\end{equation}

This implies (23) is \( V_1 = v_1 \) and if \( V_2 = v_2 \). In that case prices would coincide with marginal social values of \((k, S)\), because the envelope theorem applied to (21) gives \( V_1 = (z + p\lambda + q) U' (C) \) and \( V_2 = pU' (C) \), i.e.,
\begin{equation}
p = \beta \int \frac{U''(C')}{U'(C)} p' dF (z', z), \tag{25}
\end{equation}
and
\begin{align}
q &= \beta \int \frac{U''(C')}{U'(C)} (z' + p'\lambda + q') dF (z', z) \\
&= \lambda p + \beta \int \frac{U''(C')}{U'(C)} (z' + q') dF (z', z). \tag{26}
\end{align}

where \( p \) and \( q \) are evaluated at \( (\frac{k}{k}, z) \), and \( V_1 \) and \( V_2 \) at \( (k, S, k, S, z) \).

Calculating \( q \) and \( p \).—Optimum and equilibrium are the same, and therefore \( p \) must equal the marginal social value of a seed:
\begin{equation}
p(s, z) = \frac{1}{U'(C)} v_S = (z - x)^\sigma w_s (s, z). \tag{27}
\end{equation}

because \( \frac{1}{U'(C)} = \frac{(z-x)^\sigma}{k-\sigma} \) and \( v_S = \frac{1}{k} w_s (s, z) k^{1-\sigma} \). Now we can finally prove the result on the relation between seeds and \( q \):

**Proposition 2** \( p(s, z) \) and, hence, \( q(s, z) \) are decreasing in \( s \).

**Proof.** By Corollary 1, \( w \) is concave in \( s \) which means that \( w_s \) is decreasing in \( s \). By Corollary 2, \( x \) is increasing in \( s \) so that \( (z - x)^\sigma \) is decreasing in \( s \). Thus the claim holds for \( p \) and, by (18) it also holds for \( q \).

**Book value.**—We shall measure the “replacement” cost of the firm by the stock of its tangible capital, \( k \), even though the true replacement cost is \((1+p)k\). This is because until recently a U.S. firm could, and would, treat its R&D as an expense, and deductible from the firms taxable income.

**Measured Tobin’s \( Q \).**—The firm’s investment activity yields it zero profits. The income generated by the trees in the ground is \( zk + p\lambda \) and we shall think of it as
the firm’s earnings or its dividend. Then the “ex-dividend” value of its assets \((k, S)\) is \(qk + pS\), or per tree it is
\[
Q^o (s, z) \equiv q + ps.
\] (28)
This is what we shall use for \(Q\) in the simulated model in Figures 4 and 5, and in the regression (46). We may think of \(q\) as the firm’s marginal \(Q\) because that is the cost of a planted tree. Thus measured \(Q\) exceeds marginal \(Q\).

4 Decentralization 2: Market for firm shares only

This section simply assumes that the market for seeds is closed and that, while each firm’s state \((k, S, z)\) is public information, separate markets for \(k\) and \(s\) do not exist. It would of course be better to model the friction that causes the market for seeds to have zero transactions, but this would complicate things. So, let us assume that only \((k, S)\) bundles trade in the form of shares of firms. We use the recursive equilibrium concept of Mehra and Prescott (1980) extended to a growing production economy, as done in Jovanovic (2006).

Suppose that firms’ shares trade but that seeds and trees do not trade separately. Seeds then have to be stored by the firms that produced them, and the representative firm holds the tree-seed bundle \((k, S)\) under its roof. The household can own a claim on the dividends paid by such a firm and no other assets exist. Therefore this decentralization has just two markets: A market for output, and a market for firms’ shares. Since the number of date-\(t\) goods (consumption, capital, and seeds) is three, the number of goods exceeds the number of markets, and we cannot be sure that a recursive competitive equilibrium is optimal.

Assume a continuum of firms of measure one and an equal number of households. Equilibrium then requires that each household hold exactly one share. Firms pay \((z - x[s, z])\) \(k\) dividends in state \((k, s, z)\), and households take firms’ policies \(x(s, z)\) as given.

4.0.2 The household’s decision problem

With \(n\) shares, a household’s wealth is the current dividend, \((z - x)\) \(k\) plus the value of his holdings, \(\hat{Q}[s, z] kn\). This wealth is spent on consumption and on future holdings of shares \(\hat{Q}(s, z) kn'\). Thus \(\hat{Q}kn' + C = \left( [z - x] k + \hat{Q}k \right) n\), or after dividing through by \(k\),
\[
\hat{Q}n' + c = \left( z - x + \hat{Q} \right) n,
\]
so that
\[
c = (z - x) n + \hat{Q} (n - n')
\]
where \(x\) is given to the household. The household takes the aggregate law of motion of \(k' (s, z) x(s, z)\) and \(s' (s, z)\) as given. His state is \((k, n, s, z)\), and, with some of the
arguments \((s, z)\) dropped from the notation, his Bellman equation then is
\[
V(k, n, s, z) = \max_{n'} \left\{ \left( \frac{(z - x[s, z]) n + \hat{Q}(s, z) (n - n')}{1 - \sigma} \right)^{1-\sigma} k^{1-\sigma} + \beta \int V(k'(s, z), n', s', z') dF \right\}.
\]

Deriving the pricing equation.—As in the planner’s problem, \(V(k, n, s, z) = W(n, s, z) k^{1-\sigma}\), where
\[
W(n, s, z) = \max_{n'} \left\{ \left( \frac{(z - x(s, z)) n + \hat{Q}(s, z) (n - n')}{1 - \sigma} \right)^{1-\sigma} + \beta (1 + x[s, z])^{1-\sigma} \int W(n', s'[s, z], z') dF \right\}.
\]
(29)
The derivative of \(W\) with respect to \(n\), call it \(W_n\), exists for much the same reasons that \(u_s\) does. Equilibrium requires that \(n'(1, s, z) = 1\). At equilibrium, the first-order condition is
\[
(z - x[s, z])^{-\sigma} \hat{Q}(s, z) = \beta (1 + x[s, z])^{1-\sigma} \int W_n(1, s'[s, z], z') dF.
\]
(30)
The envelope theorem then implies
\[
W_n(1, s, z) = (z - x[s, z])^{-\sigma} \left[ z - x(s, z) + \hat{Q}(s, z) \right].
\]
Updating, substituting into (30), and dividing by \((z - x)^{-\sigma}\) gives our version of the Lucas (1978) pricing formula
\[
\hat{Q}(s, z) = \beta (1 + x[s, z]) \int M(s, s', z, z') \left( z' - x(s'[s, z], z') + \hat{Q}(s'[s, z], z') \right) dF,
\]
(31)
where
\[
M(s, s', z, z') \equiv \left( \frac{[1 + x(s, z)] (z' - x(s'[s, z], z'))}{z - x(s, z)} \right)^{-\sigma}
\]
(32)
is the MRS in consumption between today and tomorrow.

4.0.3 The firm’s decision problem

Since markets for \(s\) do not exist, the firm’s only decision is \(x\). Let us use bold letters to denote aggregate states and decisions \(\mathbf{x}(s, z)\) and \(\mathbf{s}'(s, z)\). Let \(P\) denote the cum-dividend price of \(1/k’\)th of the representative firm, i.e., the price of the tuple \((1, s)\). Equilibrium is efficient if \(P = v_k + sv_s\), with \(v\) defined in (7). The functional equation (in units of the consumption good) for its cum-dividend price per unit of \(k\) is
\[
P(s, s, z) = \max_x \left( z - x + \beta (1 + x) \int M(s, s', z, z') P(s'[s, z], s', z) dF \right)
\]
(33)
Writing $P$ in this way implies that $s$ is public information $s$ even when it differs from $s$. I.e., (33) assumes that $(s, s)$ is a sufficient statistic for the how the market values the firm. If the market did not know a firm’s $s$, it would try to guess $s$ from the firm’s choice of $x$, and incentive constraints would be needed to accompany the problem in (33). Thus the seeds market does not exist for reasons other than imperfect information about $s$.

In equilibrium,

1. All firms must choose the same $x$, and so we ask that in state $(s, z) = (s, z)$, the firm will behave like other firms. That is, at the fixed point for $P$, the RHS of (33) is maximized by $x(s, s, z) = x(s, z)$. This would imply that

$$s' = \frac{\lambda + s - x(s, s, z)}{1 + x(s, s, z)} = \frac{s'(s, z)}{1 + x(s, s, z)}.$$

2. For all $(s, z)$, the maximized value of the firm must equal the value that the shareholders hold:

$$P(s, s, z) = z - x(s, s, z) + (1 + x(s, s, z))\hat{Q}(s, z).$$

In fact, property 1 implies property 2 as one can deduce by setting $x(s, s, z) = x(s, z)$ for all $(s, z)$ so that $s' = s'$, in which case substitution from (34) into (33) makes it identical to (31). Thus it suffices to show that property 1 holds. Recall that $U'(C) = \frac{e^{C'} - \sigma}{1 - \sigma}$ so that $U'(C') / U'(C) = [(1 + x)(z' - x') / (z - x)]^{-\sigma}$. Then, evaluated at $x = x$, the FOC in (33), calculated by solving

$$P(s, s, z) = \max_{s'} \left\{ z - \hat{x}(s', s) + \beta(1 + \hat{x}[s', s]) \int M(s, s', [s, z], z, z') P(s'[s, z], s', z') dF \right\}$$

(35)

where

$$\hat{x}(s', s) = \frac{\lambda + s - s'}{1 + s'}.$$

(36)

and does not depend on the firm’s action.

**Differentiability of $P$.**—Similar to the proof of Lemma 4 we can establish that $P(s, s, z) \equiv \frac{\partial}{\partial s} P(s, s, z)$ exists everywhere. Since

$$\frac{\partial \hat{x}}{\partial s'} = \frac{\partial (1 + \hat{x})}{\partial s'} = \frac{\partial}{\partial s'} \left( \frac{1 + \lambda + s}{1 + s'} \right) = \frac{0}{1 + s'}$$

the derivative w.r.t. $s'$ is

$$\frac{1 + x}{1 + s'} - \beta \frac{1 + x}{1 + s'} \int M'P' dF + (1 + x) \beta \int M'P' dF \leq 0,$$

with an exact equality if $s' > 0$. The term $(1 + x)$ cancels, and so the FOC to the problem (35) is

$$1 - \beta \int M'P' dF + (1 + s') \beta \int M'P' dF \left\{ = 0 \quad {\text{if}} \ s' > 0 \right\} \leq 0 \quad {\text{if}} \ s' = 0.$$

(37)

**Efficiency.**—Here $P$ is the cum-dividend price of one-$k$’th of the firm in current consumption units. Per unit of its $k$, a firm is a package of $(1, s)$ units of $(k, S)$.
Therefore, efficiency would appear to require that 

\[ P = \frac{1}{\beta} (v_k + sv_k). \]

In what follows we let \( x(s, z) \) denote the planner’s optimal policy, and \( s'(s, z) = \frac{1 + s - x(s, z)}{1 + x(s, z)}. \)

The next claim states that if the representative firm used the planner’s policy, its market value would equal the marginal social value of the bundle \((k, S)\):

**Lemma 7**

\[ P(s, s, z) = P, \tag{38} \]

where \( P \) is given in (33).

**Proof.** Updating (38) by a period we have \( P(s[s, z], s', z') = (1 - \sigma) (z' - x[s', z'])^\sigma w(s', z'). \) Substituting into the RHS of (33), the latter becomes

\[
\begin{align*}
  z - x(s, z) + \beta (1 + x[s, z]) \int M(s, s', z, z') (1 - \sigma) (z - x[s, z])^\sigma w(s'[s, z], z')dF \\
  = z - x + (1 - \sigma) \beta \frac{(1 + x)^{1-\sigma}}{(z - x)^{\sigma}} \int w(s', z')dF \quad \text{in view of (32)} \\
  = (1 - \sigma) (z - x[s, z])^\sigma w(s, z) \\
  = P(s, s, z), \quad \text{as claimed in (38).}
\end{align*}
\]

The previous lemma is, however, conditional on the assumption that the representative firm uses the planner’s policy, i.e., that

\[ x(s, s, z) = x(s, z). \tag{39} \]

Next we shall show that (39) does hold if (38) does.

**Lemma 8** If \( P \) satisfies (38), then (39) holds.

**Proof.** If (39) holds, the firm’s FOC, (37), must coincide with the planner’s FOC, (12). In view of (32), LHS of (37) can be written as

\[
1 - \beta \int \left( \frac{(1 + x)(z' - x')}{z - x} \right)^{-\sigma} (P' - [1 + s'] P'_s) dF.
\]

This is the same as the LHS of (12) if

\[
(z' - x')^{-\sigma} (P' - [1 + s'] P'_s) = \left[ (z' - x')^{-\sigma} (1 + z') + \lambda w_s' \right]
\]

i.e., if

\[ 1 + z + \frac{\lambda w_s}{(z - x)^{\sigma}} = P - (1 + s) P_s \tag{40} \]

Now applying the envelope theorem in (35) and noting that, since \( \hat{x}(s', s) = \frac{1 + s - s'}{1 + s}, \)

\[ \frac{\partial \hat{x}}{\partial s} = \frac{1}{1 + s} = \frac{1 + x}{1 + \lambda + s}, \]

\[ P_s = \frac{\partial \hat{x}}{\partial s} \left( -1 + \beta \int M' P' dF \right) = \frac{1 + x}{1 + \lambda + s} \left( -1 + \frac{P - (z - x)}{1 + x} \right), \]

\[ = \frac{P - 1 - z}{1 + \lambda + s}. \]

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Substituting this into (40) for $P_s$ gives

$$1 + z + \frac{\lambda w_s}{(z - x)^{-\sigma}} = P - (1 + s) \frac{P - 1 - z}{1 + \lambda + s}$$

$$= \left(1 + \lambda + s\right) \frac{1 + z}{1 + \lambda + s}$$

Rearranging,

$$\frac{\lambda w_s}{(z - x)^{-\sigma}} = \frac{\lambda P - \lambda (1 + z)}{1 + \lambda + s},$$

i.e.,

$$w_s = (z - x)^{-\sigma} \frac{P - (1 + z)}{1 + \lambda + s}$$

$$= (z - x)^{-\sigma} \frac{1 - \sigma}{1 + \lambda + s} \left(1 - \sigma \right) w(s, z) - (1 + z)$$

$$(38),$$

But this is the same as (11). ■

Lemmas 6 and 7 then imply the main result of this section:

**Proposition 3** The incomplete-market economy has an efficient equilibrium.

For general parameter values, we cannot rule out other equilibria that are not efficient. In general, the RHS of (35) is not a contraction operator, and then we cannot tell if more than one solution for $P$ exists. However, for some parameter values, e.g. when $z$ is bounded from above by $z_{\text{max}}$, then we do have uniqueness.

**Proposition 4** If

$$\beta z_{\text{max}} < 1$$

(41)

Then the incomplete-markets economy (i) has a unique equilibrium, and (ii) it is efficient.

**Proof.** (i) When (41) holds, the RHS of (35) is a contraction operator with modulus $\beta z_{\text{max}} < 1$, and then the solution for $P$ is unique. (ii) We apply the previous proposition. ■

**4.0.4 The effects of financial-market completion**

The results say that if all firms are publicly traded, the emergence of a seeds market should affect neither prices nor quantities. It is enough that all firms trade on the stock market. Even in a financially developed society like the U.S., however, only about one half of the privately owned capital trades on stock markets, and therefore
further enlargement of the stock market would probably raise efficiency, as Greenwood and Jovanovic (1990) found in a model in which different-sized firms gradually join the stock market as they grow.

The efficiency result should extend to a situation in which firms do differ because, e.g., they draw different z’s. Jovanovic and Braguinsky (2004) develop a related one-period model in which firms differ in two dimensions: Project quality which we can interpret as s, and managerial ability, which we can interpret as z. They find that even when s is private information to the firm being acquired, the stock market achieves efficiency.

All this must be qualified by noting that seeds, S, do not share some of the features of inventions that are sometimes thought important. Namely,

1. Seeds are of purely private value, and not costlessly reproducible – as information perhaps is – and cannot raise output in more than one firm;

2. The producer of a seed has a perfect property right to it even when markets for seeds do not exist.

If either assumption did not hold, equilibrium would not be efficient. Moreover, if inventions were embodied in people and if people could extract the rents from the firm, then Q would certainly be different and equilibrium could be inefficient.

5 Numerical solution and fitting the data

I shall simulate and fit to data two different specifications of the model. The first restricts the realizations of the z process to just three values and is presented in this section. The second has five z’s, and is presented in Appendix 2 where more detail about data and procedure is also given. Improvements gained by going to five z’s are modest.

Data.—Since \( z = Y/k \), we use the output-capital ratio to measure z. We measure k by private non-residential fixed assets, NIPA table 4.1; output and investment are from NIPA table 1.1.5. For Patents we use the total number “utility” (i.e., invention) patents from the U.S. Patent and Trademark Office for 1963-2000, and from the U.S. Bureau of the Census (1975, series W-96, pp. 957-959) for 1946-62. The number of registered trademarks is from the U.S. Bureau of the Census (1975, series W-107, p. 959) for 1946-1969, and from various issues of the Statistical Abstract of the U.S. for later years.

Process for z.—Since \( z \) is estimated as \( Y/k \), its value is sensitive to how capital operated by private firms is classified. It is likely that a lot or the wartime capital was mis-classified as Government capital when in fact it was used by the private sector (Gordon 1969). Accordingly the empirics will start in 1953. But we shall then also take out a constant and linear trend from \( z \), on the presumption that we do
not measure the true \( k \). In this way we get a trendless \( z \), which is what the model assumes. When de-trended linearly, \( z \) follows an AR(1) process with autocorrelation coefficient 0.903, and innovation variance 0.026. The Tauchen-Hussey procedure for discretizing the AR yields a first-order Markov chain with 3 evenly spread-out states, \((z_1, z_2, z_3) = (0.34, 0.37, 0.40)\), and the symmetric transition probability matrix

\[
\begin{array}{ccc}
  z_1 & z_2 & z_3 \\
  0.79 & 0.21 & 0.004 \\
  0.67 & 0.17 & 0.79 \\
\end{array}
\]

\[ (42) \]

Table 1: The Matrix of Transition Probabilities for \( z \)

which induces an autocorrelation coefficient of 0.93 and a standard deviation of 0.065. The process has the symmetric stationary distribution \((0.31, 0.37, 0.31)\) and therefore a mean of 0.37. This process and its realizations will be the same for the seeds model and for the ACM.

Parameters.—At this point we assume that \( k \) depreciates at the rate \( \delta \) and \( S \) depreciate physically\(^8\) at the rate \( \gamma \) so that their laws of motion (2) and (4) become

\[
k' = (1 - \delta) k + X \quad \text{and} \quad S' = (1 - \gamma) S + \lambda k - X
\]

respectively. The details are in Appendix 2. The parameter values in Table 2

\[
\begin{array}{ccccccc}
  \beta & \sigma & \delta & \lambda & s_0 & \gamma \\
  0.95 & 6.1 & 0.043 & 0.075 & 0.053 & 0.13 \\
\end{array}
\]

Table 2: Parameter values for the seeds model

were chosen to match (i) A growth rate of the real capital stock of 0.031, and (ii) An average level of Tobin’s \( Q \) of 1.30 and (iii) An average investment/capital ratio of 0.074.

5.1 Simulated decision rules and \( Q \).

For the parameter values and transition probabilities stated in Tables 1 and 2, Figure 4 plots the equilibrium \( Q \), the decision rules and the value function. In all the plots, the variable on the horizontal axis is \( s \), the beginning-of-period seeds-capital ratio. We may summarize the plots as follows:

\(^8\)Of course \( \gamma \) tries to capture what in fact is obsolescence, and not physical depreciation.
Figure 4: Decision rules and equilibrium prices

1. Panel 1 of Figure 4 plots Tobin’s $Q$. As $s$ gets large, $p(s, z) \to 0$ for all $z$, and therefore $Q(s, z) \to 1$. The maximal $Q$ of 1.75 occurs when $s = 0$ and $z = z_3$.

2. Panel 2 plots investment, which responds more to $s$ when $z$ is high. At $z_3$, investment is constrained at low values of $s$. In particular, $x(s, z_3) = \lambda + s$ when $s$ is close to zero, so that the initial slope of the red curve in Panel 2 is unity. When $z \in \{z_1, z_2\}$, however, $x$ is never constrained and $s$ then has a much smaller effect on it.

3. Panel 3 of Fig. 4 plots the long-run distribution of seeds. Seventy percent of the time $s = 0$, and the maximal value is 0.04. Occasionally, the stock of seeds may exceed ninety percent of $k$. Indeed, illustrated in the Time-path Figure, the simulated $s$ peaks at 0.7 in the late 80’s.

---

To calculate $Q$ we substitute from (11) into (27) to obtain

$$p(s, z) = (z-x)^\sigma \frac{1}{1 + \lambda + s} \left( [1 - \sigma] w - (1 + z)[z - x]^{-\sigma} \right)$$

$$= \frac{1}{1 + \lambda + s} \left( \frac{1 - \sigma}{(z-x)^{-\sigma}} w - [1 + z] \right)$$

We actually use (??) which also allows for non-zero $\delta$ and $\gamma$. Finally, (18) gives us $q$, which is the market value of planted trees in the complete-market decentralization, and the value of the firm in the incomplete-market decentralization.
4. Finally, the last panel plots $w$ which is negative (because $\sigma > 1$) and increasing in $s$.

The effect of $s$ is to move $x$ and $Q$ in opposite directions. On the other hand, $z$ moves $x$ and $Q$ in the same direction, and this effect dominates so that the correlation between $x$ and $Q$ is positive as the matrix of unconditional correlations for the data and model in Table 3 shows:

$$
\begin{array}{ccc}
  s & x & Q \\
  z & -0.08 & 0.49 & 0.71 \\
  s & -0.11 & -0.21 & \\
  x & 0.33 & \\
\end{array}
\quad
\begin{array}{ccc}
  s & x & Q \\
  z & -0.59 & 0.62 & 0.91 \\
  s & -0.17 & -0.63 \\
  x & 0.25 \\
\end{array}
$$

Data Model

Table 3: The Matrix of Unconditional Correlations in the Model and in the Data

The signs the model produces are correct, but the magnitudes are far apart in some cases, most notably the correlation between $s$ and $z$. This number improves when we allow there to be five $z$’s – see the Appendix. The model matches (too?) well the strong positive correlation between $z$ and $Q$ (we’ll say more about $Q$ when we get to intangibles) and the negative correlation between $s$ and $x$. The signs of the $z-x$ correlations are both positive in the two tables, even if their magnitudes differ a lot. The largest discrepancy is the relation between $s$ and $z$.

5.2 Fitting the post-war series

The state variables of the model are $k, S,$ and $z$, and the decision variable is $x$. In addition, we focused on the price of seeds, $p$, but the real motivation for it is the role that $p$ plays in the price of the firm, $Q$. Thus we shall fit the following post-war series: (i) The output-capital ratio, $z$, (ii) The seed-capital ratio, $s$, (iii) The investment-capital ratio, $x$, and (iv) Tobin’s $q$ as given in (28).

In all four Panels of the Time-path Figure, the solid red line represents the model, the blue lines represent the data. The parameters $\theta$ and $s_0$ were chosen to minimize the RSS between the simulated and constructed series. The variables were constructed as follows:

1. The red line in Panel 1 of the Time-path Figure plots $z = Y/k$ where $Y = \text{private non-farm output}$ and $k = \text{non-farm stock of capital}$. The model has just 3 values of $z$ to fit this with.

2. Panel 2 plots the series for $s$ implied by the model as the red line, calculated via

$$
\begin{equation}
    s' = \frac{\lambda + (1 - \gamma)s - x}{1 - \delta + x}
\end{equation}
$$

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where $s_0 = 0.052$. The blue line is our estimate of the seeds series, using patent applications as explained in the Appendix. The simulated $s$ peaks at 0.42 in the late '80s. The model overpredicts the empirical estimate of $s_t$, i.e., the estimate of the seeds series that produces the model's best fit to the post-war data. Instead of using patent applications, we experimented with new trademarks and with R&D and as proxies for $\lambda k$, but the model's fit of the $s$ series did not improve.

3. Panel 3 shows that the model does not fully resolve the excess-volatility puzzle, but we'll see that it does better than the ACM on this score. The influence of $z$ on $Q$ is more important than that of $s$.

4. In Panel 4 of the Time-path Figure we plot the actual and fitted $Q$. For the measured $Q$, for 1951-1999 we use Hall's series, but since it ends in 1999, for the period 1999-2004 we use Abel's data scaled so that the two $Q$ series match in 1999. This is the red line in Panel 4 of the Time-path Figure. To get a sustained rise in $Q$ we must have a prolonged period during which $z = z_3$. The '90s appear to have been such a period. Evidently, the seeds model explains more of the variation in $Q$, but it overpredicts the volatility of $x$ by more.

As an explanation for the behavior of $Q$ in panel four, the model has a problem
with reconciling $Q$ with the following properties of the time-series for $z$:

- $Y/k$ falls dramatically in the late ‘70s and early ‘80s, something that the model interprets as a low-$z$ epoch causing the huge buildup of seeds portrayed in panel 2 and the resulting collapse of $Q$ to its lowest possible level of unity, and

- The long productivity slowdown of the 70’s and early ‘80s delays the predicted rise in $Q$ to the ‘90s. In fact, $Q$ starts its rise in the early ‘80s. Even with the accompanying rise in the estimate of $z$ from $z_1$ to $z_2$ in the middle ‘80s and then to $z_3$ in 1991, it takes time for the model $s$ (the blue line) to be drawn to zero and for $Q$ to rise to its maximal value of 1.75.

6 Intangibles and $Q$

The market value of $k$ is $1 + p$ and the market value of $s$ is $ps$. Measured $Q$ is given in (28).

Since $p$ is decreasing in $s$, marginal $q$ is decreasing in $s$. But for $Q^a$ we have the following result

**Proposition 5** $Q^a$ is monotone decreasing in $s$ for all $s \geq 0$ if

$$\sigma \geq 1 + \lambda.$$  \hfill (44)

**Proof.** The proof will assume that $x$, and hence $Q^a$, are everywhere differentiable in $s$ (if it is not differentiable the conclusion that $Q^a$ declines remains the same). Then

$$\frac{\partial Q^a}{\partial s} = p + (1 + s)p_s.$$  

On the set $\Delta$, $x = \lambda + s$ so that $\frac{\partial x}{\partial s} = 1$. Off the set $\Delta$, $\frac{\partial x}{\partial s} < 1$. Using (11), and

$$p = \frac{1}{1 + \lambda + s} \left( (1 - \sigma) w [z - x]^\sigma - (1 + z) \right),$$

$$p_s = \frac{1}{1 + \lambda + s} \left( -p + [1 - \sigma] \left( \sigma w [z - x]^\sigma \frac{\partial x}{\partial s} + w_s [z - x]^\sigma \right) \right) \leq \frac{1}{1 + \lambda + s} (-p + [1 - \sigma] p) \quad \text{(because } \frac{\partial x}{\partial s} \geq 0) \right)$$

$$= -\frac{1}{1 + \lambda + s} \sigma p$$

therefore

$$\frac{\partial Q^a}{\partial s} = p - \frac{1 + s}{1 + \lambda + s} \sigma p \leq \left( 1 - \frac{\sigma}{1 + \lambda} \right) p,$$

from which (44) follows. □
The proposition offers only a sufficient condition, but it also can be shown that for \( \sigma \) sufficiently small, \( Q^a \) rises with \( s \). Nevertheless, condition (44) holds at realistic values of \( \sigma \) and \( \lambda \) that in the simulation are 2 and 0.135 respectively. This is why the seeds model implies a fall in \( Q \) whereas Hall’s (2000) implies a rise in \( Q \). In my model, variation in intangibles is caused by variation in the stock of unimplemented seeds. In Hall’s model there are variable proportions between intangibles and physical capital in production and there is no storage of intangibles, hence a rise in intangibles gives a rise in the productivity of the firm’s measured capital and (barring the GE effects that I have emphasized here) it produces a rise in the firm’s \( Q \).

In the two panels of Table 3 we have already seen that the correlation between \( s \) and \( Q \) (the bolded numbers in the two panels) is negative. The finding is robust to extending the number of \( z \)’s to five – see Table A3. Let us explain this finding in more detail. In the model the stock \( S \) denotes unexercised options. Thus a rise in \( s \) represents a rise in the ratio of unimplemented intangible capital to tangible capital. The stock of all intangible capital is \( k + S \) with \( k \) being the number of seeds already in the ground and being used for production. Therefore the ratio

\[
\frac{\text{All intangible capital}}{\text{Tangible capital}} = \frac{k + S}{k} = 1 + s
\]

is also monotone in \( s \). Therefore, \( s \) is also an index of all options, be they implemented or not.

Whether intangibles raise \( Q \) or lower thus ultimately depends on whether intangibles enter final goods production independently of \( k \), or whether they are embodied in \( k \) as in this model. One source of evidence is the aftermaths of large wars during which industry resources tend to mobilize for aiding the war effort and during which unexercised options are likely to accumulate. Figure 6 reproduces Figure 5 from Wright (2004). It shows that of the three epochs when \( Q \) was at its lowest, the first was following WW1 and the second was following WW2. Since the price of capital goods has been declining faster than that of consumption goods, we may expect that ideas predominate in the production function for \( k \), and not the production function for final goods.

Alternative explanations come to mind that may explain low post-war \( Q \)s. First, one may wonder whether there is some purely accounting reason why the denominator of \( Q \) was somehow artificially inflated after WW2. But exactly the opposite seems to have been the case – Gordon (1969) convincingly establishes that much Government capital was, for several years after 1945, operated for profit by private firms, yet not counted as private capital, and excluded from the book values of those firms. Gordon argues further that this accounting practice this raised the measured rate of return on Corporate assets. On these grounds, then, we would have expected post-war \( Q \) to have been high, not low.

A second explanation for low \( Q \) after WW2 is that the market expected a severe postwar recession that would take profits and employment after 1945 down to some-
thing like the late 1930s, that this fear was not fully dispelled until the mid 1950s, and that only then did markets become convinced that the economy was capable of peacetime prosperity. This argument will not explain why Q was low after WW1, however, but in any case the hypothesis still awaits a systematic test.

Corrado, Hulten and Sichel (2005) and Hall (forthcoming) estimate that the stocks of intangibles rose during the 90’s. The seeds model is consistent with this fact only if we assume that what these authors measure is exercised options. Bronwyn Hall’s production function estimates can be reconciled with the seeds model if we assumed that the quality of \( k \) is not properly measured, and that the measurement error depends on the number of seeds embodied in the capital. One would need to relax the Leontief assumption of one seed to one tree, and introduce variable proportions.

6.0.1 Estimating a time series for \( p \) from panel data

The cross section implications contrast with those of the time series: In periods in which the economy has a lot of unexercised options, Q should be low, but at any date, firms with more options should have Qs that are high. We can benefit from this implication and arrive at an independent estimate of the aggregate time series of \( p \) and \( q \).

Let us now look further into this question. So far, firms were assumed to be identical. One can, however, inject a zero measure of firms being endowed with different relative seed stocks, \( s_i \). This would leave all other firms’ valuations and policies unchanged.
A firm’s beginning-of-period value would now depend on its state \((S_i, k_i)\), in addition to the aggregate state \((s, z)\). The aggregate \(k\) would not affect the firm’s price because it affects neither \(q\) nor \(p\). Then its beginning-of-period value per unit of capital would be

\[
\tilde{Q}_i = \frac{1}{k_i} \pi (k_i, S_i, s, z) = 1 + z + p(s, z) \left( 1 + \lambda + \frac{S_i}{k_i} \right).
\] (45)

Since \(p\) changes with the aggregate state \((s_t, z_t)\) but not with \(i\), it can, for each \(t\), be estimated from the cross-section regression of \(Q_i\) on \(\frac{S_i}{k_i}\), provided the latter could be reliably estimated. I report some results related to Hall (forthcoming) in Figure 7. According to the model, the two lines should have coincided, but they do not and some explanation for this fact is needed (TO BE CONTINUED).

7 Comparison to the adjustment-cost model

We shall now show that our model loosens the relation between investment and \(Q\), so that Abel and Eberly’s (2006) argument carries over to the aggregate setting. Figure 3 shows the region \(\Delta\) on which \(x\) cannot respond to \(z\) and, hence, to \(Q\). Since \(\Delta\) contains mainly boom states, we would expect that \(x\) should respond more elastically to \(Q\) in recessions than in booms.

*Quadratic adjustment costs instead of the seeds constraint.*—We shall contrast the seeds model with a standard model with no seeds constraint but with a quadratic adjustment-cost. That model involves just one change: The constraint (1) is removed
and a new production function is assumed, namely

\[ Y = z k - h \left( \frac{X}{k} \right) k, \quad \text{where} \quad h(x) = \frac{b}{2} (x - \delta)^2 \]

I.e., the adjustment cost is quadratic and depends on net investment.

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( \sigma )</th>
<th>( \delta )</th>
<th>( z )</th>
<th>( \phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.95</td>
<td>6.00</td>
<td>0.04</td>
<td>0.37</td>
<td>9.73</td>
</tr>
</tbody>
</table>

Table 4: The Parameter values for the Adjustment-Cost Model

The parameters and realizations of the \( z \) process are the same as plotted in Panel 1 of Figure 5. The analysis is in the Appendix. We continue with the same three values for \( z \), but choose the realizations that minimize the RSS between the predicted \( z \) and the output-capital ratio. In the ACM, \( z \) is the only state variable and therefore a one-to-one relation emerges between \( x \) and \( Q \), and it is shown as the straight black line in Figure 8. The firm’s FOC is \( Q = 1 + b (x - \delta) \), and it is plotted in Figure (8).

Although in the seeds model there is only one shock, \( z \), the same as in the ACM, there is an additional state variable, \( s \), on which \( x \) and \( Q \) both depend, and this loosens the relation between \( x \) and \( Q \). Note from Panel 3 of Fig. 4 that the steep red curve (pertaining to \( s = 0 \)) is relevant seventy percent of the time, and this is mainly why the seeds model generates more volatility in \( Q \) than the ACM.

Next, Table 4 contrasts the two models’ implications for the second moments of the data. Columns 2-4 report the results of three 100,000-period simulations of two models when the shocks are drawn according to the transition matrix in (42).
The first model is the Seeds model; Column 2 presents its implications under the parameter values in Table 2 – the parameters that, together with (42), were used to generate Figure 4 and Figure 5. The initial condition is $s_0 = 0.04$. One should compare the numbers in Column 2 to the information in the Decision-rules Figure. For instance, $E(s)$ and $S(s)$ are the mean and standard deviation of the distribution of $s$ plotted in Panel 3 of Figure 4.

Column 3 presents the same set of statistics for the ACM under the parameter values given in Table 4. The seeds underpredicts $S(x)$ but it underpredicts $S(Q)$ and $S(c)$ by less. Finally, the seeds model has an additional endogenous variable – seeds – and no new exogenous variables, but as we have seen in the Time-path Figure it does not explain well any reasonable measure of seeds. Consumption volatility is the same in the two models, but investment is more volatile in the ACM. Nevertheless, the seeds model explains much better the volatility of $Q$, as Figure 8 suggested.

<table>
<thead>
<tr>
<th>Stat</th>
<th>Data</th>
<th>Seeds: $s_0 = 0.04$</th>
<th>ACM Seeds: $s_0 = \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E(z)$</td>
<td>0.37</td>
<td>0.37</td>
<td>0.38</td>
</tr>
<tr>
<td>$E(x)$</td>
<td>0.07</td>
<td>0.07</td>
<td>0.08</td>
</tr>
<tr>
<td>$E(s)$</td>
<td>0.002</td>
<td>0.001</td>
<td>–</td>
</tr>
<tr>
<td>$E(Q)$</td>
<td>1.30</td>
<td>1.30</td>
<td>1.30</td>
</tr>
<tr>
<td>$E(c)$</td>
<td>0.30</td>
<td>0.29</td>
<td>0.30</td>
</tr>
<tr>
<td>$S(z)$</td>
<td>0.027</td>
<td>0.024</td>
<td>0.02</td>
</tr>
<tr>
<td>$S(x)$</td>
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<td>0.005</td>
<td>0.01</td>
</tr>
<tr>
<td>$S(s)$</td>
<td>0.009</td>
<td>0.013</td>
<td>–</td>
</tr>
<tr>
<td>$S(Q)$</td>
<td>0.566</td>
<td>0.303</td>
<td>0.08</td>
</tr>
<tr>
<td>$S(c)$</td>
<td>0.020</td>
<td>0.021</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Table 5: Comparison to the adjustment-cost model.

**Cash-flow regressions.**—It is customary to regress investment on $Q$ and on cash flow and when cash flow “drives $Q$ out” of the regression, this is held to be a problem for $Q$ theory. But, as Abel and Eberly (2005) stress, investment options are not always exercised, but their value changes over time, and with it the firm’s measured $Q$, and therefore the significance of cash flow independently of $Q$ does not indicate a failure of the $Q$ theory. While they have established that convincingly for the cross section, it remains to be seen whether the same conclusion holds in the time series in the GE context. Per unit of $k$, net cash flow is the same as aggregate consumption, $z - x$.

We shall therefore simulate the results of such a regression by taking a million draws of the model and subjecting the endogenous variables to the regression in question. The results were

$$\frac{x_{t+1}}{\text{INVESTMENT}} = 0.03 + 0.48(z_t - x_t) - 0.02 \frac{Q^s_t}{\text{CASH AVERAGE Q}}.$$  (46)
where \( Q^a \) was simulated via (28), i.e., from the first decentralization. Given the large sample size, all three coefficients differ significantly from zero. The reason why cash is positive seems to be that \( z - x \) increases in \( z \), and \( z \) is persistent, leading on average to a higher \( x' \).

8 Conclusion

This paper has offered a microfoundation for adjustment costs that emphasizes the role of new ideas in investment and in the business cycle, and that features an intertemporal substitution element missing in existing formulations of adjustment costs. This is the feature that in the model produces the sustained rise in \( Q \) during a boom such as sometimes arise in fact. The paper has found that investment options raise the volatility of \( Q \) compared to the standard adjustment-cost model. Moreover, because they facilitate the formation of new capital, new ideas reduce the value of old capital, an implication that we found was borne out in fact. Thus what we often call intangible capital acts to reduce the value of tangible capital. Finally, we found that a stock market alone suffices to ensure efficiency of the equilibrium.

References


This Appendix concerns the blue line the second panels of Figures 5 and 11. The model says that

\[ S' = \lambda k + (1 - \gamma) S - X. \]

We shall assume that

\[ \hat{\lambda} k = \theta \cdot \text{(New Patents)} \]

and, to prevent the stock of seeds from becoming negative, we change (4) to

\[ S' = \max (0, \theta \cdot \text{(New Patents)} + (1 - \gamma) S - X). \]

The dashed red line is

\[ s' = \frac{\lambda + (1 - \gamma) s - x}{1 - \delta + x}, \]

and the solid blue line is

\[ s' = \frac{\max (0, \theta \cdot \text{NEW PATENTS CAPITAL STOCK} + (1 - \gamma) s - x)}{1 - \delta + x}. \]
9.2 Details on the standard adjustment-cost model

The adjustment-cost model that is simulated and the statistics of which are reported in column 3 of Table 1 goes as follows: Output and dividend is

\[ zk - h \left( \frac{X}{k} \right) k \]

where

\[ h(x) = \frac{b}{2} (x - \delta)^2, \]

and where \( k \) still follows (65) and where the Bellman equation is

\[ v(k, z) = \max_x \left\{ \frac{(zk - X - h(X) k)^{1-\sigma}}{1 - \sigma} + \beta \int v([1 - \delta] k + X, z') dF \right\}. \]  

(48)

Again, \( v(k, z) = w(z) k^{1-\sigma} \), and the auxiliary Bellman equation is

\[ w(z) = \max_x \left\{ \frac{(z - x - h(x))^{1-\sigma}}{1 - \sigma} + (1 - \delta + x)^{1-\sigma} \beta \int w(z') dF \right\}. \]  

(49)

The FOC is

\[-(z - x - h(x))^{-\sigma} (1 + h'(x)) + (1 - \sigma) (1 - \delta + x)^{-\sigma} \beta \int w(z') dF, \]

i.e.,

\[ 1 + h'(x) = q, \]

where

\[ q = \left( \frac{1 - \delta + x}{z - x - h} \right)^{-\sigma} \beta (1 - \sigma) \int w(z') dF. \]

Now,

\[ (1 - \delta + x)^{1-\sigma} \beta \int w' dF = \frac{1 - \delta + x}{(1 - \sigma)(z - x - h)} q. \]

Therefore

\[ w(z) = \frac{(z - x - h)^{1-\sigma}}{1 - \sigma} + \frac{1 - \delta + x}{(1 - \sigma)(z - x - h)} q. \]

Calculation of the measured \( Q \): Since \( v(k, z) = w(z) k^{1-\sigma} \),

\[ \frac{V_k}{U'(C')} = \frac{w(z)}{(z - x - h)^{-\sigma}} = \frac{1}{1 - \sigma} \left( z - x - h + (1 - \delta + x) q \right) \]

\[ = \frac{1}{1 - \sigma} \left( z - x - h + (1 - \delta + x)(1 + h') \right) = \frac{1}{1 - \sigma} \left( z - h + (1 - \delta)(1 + h') + xh' \right). \]

The problem must be with the first equality, because \( w \) is negative when \( \sigma > 1. \)
9.3 Research

Because new seeds are proportional to capital in the model, seeds pile up in recessions, and this depresses $Q$ for a while after the recovery starts. If research or other resources are needed, fewer seeds will be created when $p$ is low. To see how it might work, let us change (3) to

\[ \text{new seeds} = \lambda R^\varepsilon k^{1-\varepsilon} \]

so that (4) becomes

\[ S' = \lambda R^\varepsilon k^{1-\varepsilon} + S - X \]

and so that (5) becomes

\[ X \leq \lambda R^\varepsilon k^{1-\varepsilon} + S. \]

The planner’s Bellman equation becomes

\[
v(k, S, z) = \max_{R \geq 0, X \leq \lambda R^\varepsilon k^{1-\varepsilon} + S} \left\{ \frac{(z k - X - R)^{1-\sigma}}{1-\sigma} + \beta \int v(k + X, \lambda R^\varepsilon k^{1-\varepsilon} + S - X, z') \, dF \right\}.\]

For $\sigma \neq 1$, $v$ is still of the form

\[ v(k, S, z) = w(s, z) k^{1-\sigma}, \]

where $w(s, z) = v(1, s, z)$, and where $w$ satisfies

\[
w(s, z) = \max_{(r, x) \in \Omega(s)} \left\{ \frac{(z - x - r)^{1-\sigma}}{1-\sigma} + (1 + x)^{1-\sigma} \beta \int w(\lambda r^\varepsilon + s - x, 1 + x, z') \, dF \right\}
\]

where

\[ r = \frac{R}{k} \]

and

\[ \Omega(s) = \{(r, x) \mid x \leq s + \lambda r^\varepsilon \}. \]

Since $\varepsilon < 1$, $r$ will never be negative. If $z$ was firm specific and if seeds could not be stored, this version of the model would be close to Klette and Kortum (2004) and Lentz and Mortensen (2005).

The problem with this is that it introduces a $Q$-elastic supply of seeds, which will limit somewhat how much $Q$ can rise in booms. In sum, it will produce less variation in $Q$, but maybe a more realistic seeds.

9.4 The deterministic seeds model

Suppose $z$ is a constant. Let

\[ x = \frac{X}{k} \text{ and } s = \frac{S}{k}. \]
Since \( k \) does not depreciate, \( x \) then equals the growth rate of \( k \) and of \( C \). Let’s solve for the constant-growth rate that would obtain in the absence of the constraint (5). We shall call this the “desired” growth rate, \( x^d \). Then \( U'(C_{t+1})/U'(C_t) = (1 + x)^{-\sigma} \) and the effective discount factor is

\[
\hat{\beta} \equiv \beta (1 + x)^{-\sigma}.
\]

(50)

An additional unit of capital produces \( z \) units for ever, and so optimal investment leads to a Tobin’s \( Q \) of unity:

\[
Q \equiv \left( \frac{\hat{\beta}}{1 - \beta} \right) z = 1.
\]

(51)

Equations (50) and (51) can be solved for \( x^d \):

\[
1 + x^d = (\beta [1 + z])^{1/\sigma}.
\]

(52)

The model collapses to the standard model if \( s \) goes off to infinity. We seek parameter restrictions that will prevent this from happening. From (5),

\[
x_t \leq \min \{ z, \lambda + s_t \}
\]

(53)

This, however, is a short-run constraint, that holds at each \( t \). If \( k \) were to grow faster than \( \lambda \), \( s_t \) would eventually become negative. To see this, combine (4) and (3) to get

\[
S_{t+1} = S - X + \lambda k \text{ and, hence,}
\]

\[
s_{t+1} = \frac{\lambda + s_t - x_t}{1 + x_t}.
\]

(54)

It’s easy to show that \( \lambda \) is the maximal feasible long-run growth rate. Let \( \varepsilon \) be a constant, and suppose that \( x = \lambda + \varepsilon \). Then

**Lemma 9** For all \( s_0 \geq 0 \),

\[
(i) \quad \varepsilon > 0 \implies s_t \to -\infty
\]

\[
(ii) \quad \varepsilon < 0 \implies s_t \to +\infty
\]

**Proof.** (i) Let \( \varepsilon > 0 \). Then \( s_{t+1} = \frac{\lambda + s_t - x_t}{1 + x_t} = \frac{s_t - \frac{\varepsilon}{1 + x}}{1 + x} < s_t - \frac{\varepsilon}{1 + x} \), so that \( s_t < s_0 - \left( \frac{\varepsilon}{1 + x} \right) t \to -\infty \). (ii) let \( \varepsilon < 0 \). Then \( s_{t+1} > s_t + \frac{\varepsilon}{1 + x} \), so that \( s_t > s_0 + \frac{|\varepsilon|}{1 + x} t \to +\infty \).

Desired growth exceeds \( \lambda \) if

\[
[\beta (1 + z)]^{1/\sigma} > 1 + \lambda,
\]

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which is also when the seeds constraint binds in every period. High values of \( z \) or \( \beta \), and low values of \( \sigma \) and \( \lambda \) make it more likely that this inequality will hold. Tobin’s \( Q \) is just the present value of the marginal product of capital, \( \sum_{t=1}^{\infty} \tilde{\beta}^t z \), i.e.,

\[
Q(z) = \left( \frac{\tilde{\beta}}{1 - \beta} \right) z, \quad \text{where} \quad \tilde{\beta} = \beta (1 + \lambda)^{-\sigma} > \beta (1 + x^d)^{-\sigma} = \beta.
\]

Values of \( Q \) above unity arise because consumption growth is lower than it would be under \( x^d \); the rate of interest is thus lower, and this raises the present value of income from capital above its cost.

The case \( \sigma = 1 \).—From (52), the desired investment and growth rate \( x \) is

\[
x^d(z) = \beta z - (1 - \beta),
\]

and Tobin’s \( Q \) is

\[
Q(z) = \begin{cases} 
1 & \text{if} \quad x^d(z) \leq \lambda \\
\frac{1}{1 + \lambda - \beta} z & \text{if} \quad x^d(z) > \lambda.
\end{cases}
\]

The value of \( z \) at which \( x^d(z) = \lambda \) is \( \frac{1}{\beta} (1 + \lambda - \beta) \). Figure 9 plots \( x^d(z) \) and \( Q(z) \). Of course, \( x = \min(\lambda, x^d(z)) \).

9.4.1 Transitional dynamics

These are easier to analyze if time is continuous. If \( \lambda \) is high enough, the model has no transitional dynamics because then seeds are a free good, an irrelevant by-product
of production. Transitional dynamics arise when desired growth exceeds \( \lambda \), i.e., when (55) holds. Let \((k_0, S_0)\) be given, with \( S_0 > 0 \). Let preferences be

\[
\int_0^\infty \frac{1}{1-\sigma} e^{-\rho t} C_t^{1-\sigma} dt.
\]

Output is

\[ zk = C + X, \]

the laws of motion are

\[ \dot{S} = \lambda k - X \quad \text{and} \quad \dot{k} = k + X, \]

and the seed constraint reads

\[ S \geq 0. \]

In the absence of the seed constraint “desired” growth would be \( \frac{\dot{z}}{\sigma} = \frac{\dot{k}}{k} = \frac{z-\rho}{\sigma} \). For transitions to occur we therefore need that

\[ \lambda < \frac{z-\rho}{\sigma}. \quad (55) \]

In this case, the seed constraint must eventually bind, from which point on we have \( X = \lambda k \).

**Optimal growth.**—This is a version of the exhaustible-resources problem. The control is \( X \) and the states are \((k, S)\). The Hamiltonian is

\[
\frac{(zk-X)^{1-\sigma}}{1-\sigma} + \mu X + p(\lambda k - X) + nS
\]

where \( m \) is the shadow value of \( S \) and \( \mu \) is the shadow value of capital. The optimality conditions are

\[
\begin{align*}
X : & \quad -(zk-X)^{-\sigma} - \mu - m = 0 \quad (56) \\
k : & \quad z (zk-X)^{-\sigma} + \lambda m = -\dot{\mu} + \rho \mu \quad (57) \\
S : & \quad n = -\dot{m} + \rho m \quad (58)
\end{align*}
\]

and the two constraints must hold.

The region \([0, T]\) where \( S > 0 \).—Let \( T \) be the date at which the transition ends. For \( t \geq T \), \( S_t = 0 \) and \( X_t = \lambda k_t \). In this region, \( n = 0 \) so that (58) implies

\[ m_t = m_0 e^{\rho t} \quad \text{for} \quad t < T. \]

Substituting for \( m_t \) and from (56) into (57) gives us

\[ \mu z + \lambda m_0 e^{\rho t} = -\dot{\mu} + \rho \mu \]

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which is the differential equation

\[ \dot{\mu} = (\rho - z) \mu - \lambda m_0 e^{pt} \]

Now an equation of the form \( \frac{d\mu}{dt} = A\mu + B e^{pt} \) has the solution \( \mu = C_1 e^{At} + B \frac{e^{pt}}{p - \lambda} \).

Therefore

\[ \mu_t = C_1 e^{(\rho-z)t} - \frac{\lambda m_0}{z} e^{pt} \]

The region \([T, \infty)\).—Here all the multipliers are constant. In particular

\[ \mu = 1 + m. \]

Tobin’s Q.—Let Tobin’s Q, defined here as the discounted marginal product of \( k \):

\[ Q = \int_t^\infty e^{-\rho(\tau-t)} \frac{U'(C_\tau)}{U'(C_t)} z d\tau. \]

This is the present value of a unit of capital. In the limit, consumption will grow at the rate \( \lambda \) so that \( \frac{U'(C_\tau)}{U'(C_t)} = e^{-\sigma \lambda (\tau-t)} \) and \( Q \) will converge to

\[ Q_\infty = z \int_t^\infty e^{-(\rho+\sigma \lambda)(\tau-t)} d\tau = \frac{z}{\rho + \sigma \lambda} \]

where the rate of interest is

\[ \rho + \sigma \lambda \]

which is less than \( z \) if (55) holds, so that \( Q_\infty > 1 \). But if (55) does not hold, then consumption grows at the rate \( \frac{z - \rho}{\sigma} \) and \( Q_\infty = 1 \). TO BE CONTINUED

10 Simulation and fit with five values of \( z \)

10.0.2 Data

Data sources are described in the following table:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Data source</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>Capital stock, NIPA 6.1. Current-cost net stock of private fixed assets, line 1</td>
</tr>
<tr>
<td>( Y )</td>
<td>Output, NIPA 1.1.5. Gross domestic product, line 1 - line 20</td>
</tr>
<tr>
<td>( X )</td>
<td>Investment, NIPA 1.1.5. Gross domestic product, line 6</td>
</tr>
<tr>
<td>( Q )</td>
<td>Tobin’s Q, Series obtained from R.Hall</td>
</tr>
<tr>
<td>( N )</td>
<td>Patents, Series obtained from P.Rousseau</td>
</tr>
<tr>
<td>( N )</td>
<td>Trademarks, Series obtained from P.Rousseau</td>
</tr>
<tr>
<td>( k^r )</td>
<td>Real capital stock, NIPA 6.2. Chain-Type Quantity Indexes for Net Stock of Private Fixed Assets, line 1</td>
</tr>
<tr>
<td>( z )</td>
<td>Productivity, Computed as ( z = Y/k ), linearly detrended</td>
</tr>
<tr>
<td>( x )</td>
<td>Investment/capital, Computed as ( x = X/k )</td>
</tr>
<tr>
<td>( n )</td>
<td>Patents/capital, Computed as ( n = N/k^r )</td>
</tr>
</tbody>
</table>
10.0.3 Estimation

The estimation procedure is non-linear least squares. Model parameters are chosen to minimize the $L_2$-norm between the observed series $(s, x, Q)$ and the same series obtained from the data. The results are preliminary; they have yet to be checked thoroughly.

In the first step the estimated from the data productivity process $z$ is discretized into a 5-state first-order Markov process using Tauchen-Hussey procedure. Productivity process was estimated to be an AR(1) process with autoregressive coefficient $\rho = 0.93$ and standard deviation $\sigma_z = 0.03$. Table A1 reports the Markov transition matrix of the approximating 5-state process. This one is not restricted to be symmetric.

$\begin{array}{lcccc|c}
z_1 & z_2 & z_3 & z_4 & z_5 & \\
0.32 & 0.72 & 0.26 & 0.02 & 0.00 & 0.00 & 0.16 \\
0.34 & 0.19 & 0.55 & 0.24 & 0.02 & 0.00 & 0.22 \\
0.36 & 0.01 & 0.22 & 0.53 & 0.22 & 0.01 & 0.24 \\
0.38 & 0.00 & 0.02 & 0.24 & 0.55 & 0.19 & 0.22 \\
0.40 & 0.00 & 0.00 & 0.02 & 0.26 & 0.72 & 0.16 \\
\end{array}
$

Table A1: The Matrix of Transition Probabilities for $z$

Next $\hat{z}_t$ was chosen to minimize the distance to the observed level of $z_t$,

$$\hat{z}_t = \arg \max_{\hat{z}_t \in \{z_1, \ldots, z_5\}} (\hat{z}_t - z_t)^2$$

except for the period 1995-2000 when the $z_t$ was set to $z_5$ (the largest $z$) so that a better fit to $Q$ could be obtained.

In the second step I choose model parameters, so that the model when fed with the sequence $\{\hat{z}_t\}$ generates series $(\hat{s}, \hat{x}, \hat{Q})$ close to those observed in the post-1953 data,

$$\min_{\lambda, \bar{z}, \gamma, s_0, \theta} \sum_{i=1}^{51} (\hat{s}_i - s_{1953+i})^2 + \sum_{i=1}^{51} (\hat{x}_i - x_{1953+i})^2 + \sum_{i=1}^{51} (\hat{Q}_i - Q_{1953+i})^2$$

subject to

$$E[\hat{x}_i|\lambda, \bar{z}, \gamma, s_0, \theta] = 0.0740, \quad E[\hat{Q}_i|\lambda, \bar{z}, \gamma, s_0, \theta] = 1.3025.$$ 

Weights $(w_s, w_x, w_Q)$ are set to values $(1, 2, 0.01)$. The estimated seeds model parameters are reported in the following table:

$\begin{array}{lccccccc}
\beta & \sigma & \delta & \bar{z} & s_0 & \lambda & \gamma & \theta \\
0.95 & 6 & 0.043 & 0.36 & 0.35 & 0.075 & 0.09 & 0.04 \\
\end{array}
$

Table A2: Parameter Values for the Seeds Model

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Comparing tables 2 and A2, we see that the parameter estimates are visually the same except for $s_0$.

The decision rules are in Figure 10. Only the states $z_1$, $z_3$, and $z_5$ are plotted to avoid clutter. The five-$z$’s case really does not change things substantially. Each panel of Figure 4 is virtually identical to its counterpart in Figure 10. A slightly higher $Q$ is now achieved under $z_5$ than it previously had been under $z_3$, and the seeds constraint binds 71 percent of the time instead of 68 percent. Also, in the lowest state, $z_1$, neither $Q$ nor $x$ nor $w$ depend visibly on $s$, whereas they depend on $s$ a bit more visibly now in state $z_5$ than they did previously in state $z_3$.

The fit of the model is described in Figure 11. The one apparent difference from the situation shown in Figure 5 is that the build-up of seeds during the 1980s is now not quite as large as it was previously. Not much else is seemingly improved by having two additional $z$’s.

The matrix in Table A3 compares the pairwise correlations of the model and the data. For the Model I segment of the table, we use 100,000 random draws of $z$. For the Model 2 segment of the table we use (as we did in Table A3) for $z$ the actual realized values as shown by the dashed red line in Panel 1 of Figure 11, and $s$ – has a suggests a substantially better fit of model to data, especially the $(z, s)$ and the
Figure 11: Time path of the data and model outcomes

(s, Q) correlations. The one cell where the difference is large and where even the sign is wrong is the correlation between s and x in the second model matrix.

For the Adjustment-Cost model (ACM) the parameters are shown in Table 4 – there is no change in the parameters used. The moments of the three models are in Table A5:
Figure 12: Investment and $Q$ in the Seeds and Adjustment-Cost Models

<table>
<thead>
<tr>
<th>Moment</th>
<th>Seed Model</th>
<th>ACM</th>
<th>Standard model ($s_0 = \infty$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E(z)$</td>
<td>0.36</td>
<td>0.37</td>
<td>0.36</td>
</tr>
<tr>
<td>$E(x)$</td>
<td>0.07</td>
<td>0.07</td>
<td>0.08</td>
</tr>
<tr>
<td>$E(s)$</td>
<td>0.01</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$E(Q)$</td>
<td>1.30</td>
<td>1.30</td>
<td>1.00</td>
</tr>
<tr>
<td>$E(c)$</td>
<td>0.29</td>
<td>0.29</td>
<td>0.28</td>
</tr>
<tr>
<td>$S(z)$</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
</tr>
<tr>
<td>$S(x)$</td>
<td>0.00</td>
<td>0.07</td>
<td>0.01</td>
</tr>
<tr>
<td>$S(s)$</td>
<td>0.01</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$S(Q)$</td>
<td>0.30</td>
<td>0.07</td>
<td>0.00</td>
</tr>
<tr>
<td>$S(c)$</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
</tr>
</tbody>
</table>

*Table A5: Comparison to the adjustment-cost model.*

Comparing visually to the Adjustment-Cost model, we have Figure 12.
11 Proofs of differentiability

This is the proof of Lemma 4. The complete proof is for the case of no depreciation.

I use subscripts to denote the state that a policy pertains to. Thus we have the accounting identities

\[ s'_s = \frac{\lambda + s - x_s}{1 + x_s} \quad \text{and} \quad s'_{s+h} = \frac{\lambda + s + h - x_{s+h}}{1 + x_{s+h}}. \]

**Variations.**—We use (36) to figure out the feasible variations.

**Variation (i).**—If we begin at state \( s + h \), and if we want to end up at \( s'_s \), we need an investment of

\[
\hat{x} (s'_s, s + h) = \frac{\lambda + s + h - \frac{\lambda + s - x_s}{1 + x_s}}{1 + \frac{\lambda + s - x_s}{1 + x_s}} = \frac{(1 + x_s) (\lambda + s + h) - (\lambda + s - x_s)}{1 + x_s + \lambda + s - x_s} \\
= \frac{(1 + x_s) h + x_s (\lambda + s) + x_s}{1 + \lambda + s} \\
= x_s + h \frac{1 + x_s}{1 + \lambda + s}.
\]

Then

\[
A_h \equiv \left( \frac{1 + \hat{x}}{1 + x_s} \right)^{1-\sigma} = \left( 1 + \frac{h}{1 + \lambda + s} \right)^{1-\sigma},
\]

and

\[
\hat{x} - x_s = h \frac{1 + x_s}{1 + \lambda + s}.
\]

Therefore

\[
w(s + h, z) \geq U(z - \hat{x} [s'_s, s + h]) + (1 + \hat{x} [s'_s, s + h])^{1-\sigma} \beta \int w(s'_s, z') dF \\
= U(z - \hat{x} [s'_s, s + h]) + A_h (1 + x_s)^{1-\sigma} \beta \int w(s'_s, z') dF \\
= U(z - \hat{x} [s'_s, s + h]) + A_h (w(s, z) - U(z - x_s))
\]

and

\[
w(s + h, z) - w(s, z) \geq U(z - \hat{x} [s'_s, s + h]) - A_h U(c_s) + (A_h - 1) w(s, z) \\
= U(z - \hat{x} [s'_s, s + h]) - U(c_s) + (A_h - 1) (w(s, z) - U(c_s)).
\]

Dividing both sides by \( h \) and taking the limit as \( h \searrow 0 \) gives

\[
\frac{d}{ds} w(s, z) \geq -U'(c_s) \lim_{h \searrow 0} \frac{\hat{x} - x_s}{h} + \lim_{h \searrow 0} \frac{(A_h - 1)}{h} [w(s, z) - U(c_s)] \\
= -U'(c_s) \frac{1 + x_s}{1 + \lambda + s} + (1 - \sigma) \frac{w(s, z) - U(c_s)}{1 + \lambda + s}. \tag{59}
\]
because, by L'Hôpital's rule,
\[
\lim_{h \to 0} \frac{(A_h - 1)}{h} = \lim_{h \to 0} \frac{dA_h}{dh} = \lim_{h \to 0} \frac{d}{dh} \left(1 + \frac{h}{1 + \lambda + s}\right)^{1-\sigma} = \frac{1 - \sigma}{1 + \lambda + s} \lim_{h \to 0} \left(1 + \frac{h}{1 + \lambda + s}\right)^{-\sigma} = \frac{1 - \sigma}{1 + \lambda + s}
\]

Variation 2: Start from \(s\) and end at \(s'_{s+h}\).

Variation (ii).— If we begin at state \(s\), and if we want to end up at \(s'_{s+h}\), we need an investment of
\[
\hat{x} (s'_{s+h}, s) = \frac{\lambda + s - \frac{\lambda + s + h - x_{s+h}}{1 + x_{s+h}}}{1 + \frac{\lambda + s + h - x_{s+h}}{1 + x_{s+h}}} = \frac{(1 + x_{s+h}) (\lambda + s) - (\lambda + s + h - x_{s+h})}{1 + x_{s+h} + \lambda + s + h - x_{s+h}}
\]
\[
= \frac{x_{s+h} (\lambda + s) - (h - x_{s+h})}{1 + \lambda + s + h} = \frac{(1 + \lambda + s) x_{s+h} - h}{1 + \lambda + s + h}
\]
\[
= \frac{(1 + \lambda + s + h) x_{s+h} - h (1 + x_{s+h})}{1 + \lambda + s + h}
\]
\[
= x_{s+h} - \frac{h (1 + x_{s+h})}{1 + \lambda + s + h}
\]
\[
< x_{s+h} - \frac{h (1 + x_s)}{1 + \lambda + s + h}
\]

because by Corollary 2, \(x\) is increasing in \(s\). We shall also need the following implication of (60):
\[
B_h \equiv \left(1 + \frac{\hat{x}}{1 + x_{s+h}}\right)^{1-\sigma} = \left(1 - \frac{h}{1 + \lambda + s + h}\right)^{1-\sigma}
\]
Therefore
\[
w(s, z) \geq U (z - \hat{x}) + (1 + \hat{x})^{1-\sigma} \beta \int w(s'_{s+h}, z') dF
\]
\[
= U (z - \hat{x}) + B_h (1 + x_{s+h})^{1-\sigma} \beta \int w(s'_{s+h}, z') dF
\]
\[
= U (z - \hat{x}) - B_h U (z - x_{s+h}) + B_h w(s + h, z).
\]
and therefore
\[
w(s, z) - w(s + h, z) \geq U (z - \hat{x}) - B_h U (z - x_{s+h}) + (B_h - 1) w(s + h, z),
\]

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i.e.,

\[ w(s + h, z) - w(s, z) \leq B_h U(z - x_{s+h}) - U(z - \hat{x}) + (1 - B_h) w(s + h, z) \]

(62)

\[ = U(z - x_{s+h}) - U(z - \hat{x}) + (1 - B_h) [w(s + h, z) - U(c_s)] \]

Now, \([w(s + h, z) - U(z - x_{s+h})]\) is Lipschitz in \(h\) for every \(z > 0\). This is because it is bounded above by the increment in value when a unit of consumption is added in perpetuity, and the latter is bounded as long as \(c > 0\), i.e., as long as \(z > 0\). Now, by (61), \(x_{s+h} \geq \hat{x} + \frac{h(1 + x_s)}{1 + \lambda + s + h}\) and therefore

\[ U(z - x_{s+h}) - U(z - \hat{x}) \leq U \left( z - \hat{x} + \frac{h(1 + x_s)}{1 + \lambda + s + h} \right) - U(z - \hat{x}) \]

Using the RHS of this expression to replace the first two terms on the RHS of 63) leaves the inequality in (63) undisturbed. Moreover, using L'Hôpital’s rule as before,

\[ \lim_{h \searrow 0} \frac{1}{h} (1 - B_h) [w(s + h, z) - U(c_{s+h})] = \frac{1 - \sigma}{1 + \lambda + s} [w(s, z) - U(c_s)] \]

Putting this all together,

\[ w_s \leq \frac{1}{1 + \lambda + s} (U'(c_s)(1 + x_s) + (1 - \sigma) [w(s, z) - U(c_s)]) \]

(64)

Then (59) and (64) imply (11). To see this, (11) says (in this notation) that

\[ w_s = \frac{1}{1 + \lambda + s} ([1 - \sigma] w - (1 + z) U'(c)) > 0. \]

For them to be the same we would need that

\[ -(1 + x) U' + (1 - \sigma)(w - U) = (1 - \sigma) w - (1 + z) U', \]

i.e.,

\[ -(1 + x) U' - (1 - \sigma) U = -(1 + z) U', \]

i.e.

\[ (1 - \sigma) U = (z - x) U' \]

which is true because \(z - x = c\), so that both sides of the equation equal \(c^{1-\sigma}\). Therefore (59) and (64) imply (11).

### 11.1 Depreciation

Let \(\delta\) = depreciation of \(k\) and let \(\gamma\) be the depreciation of \(S\). The laws of motion and the value are

\[ k' = k(1 - \delta) + X, \]

(65)
\[ S' = S (1 - \gamma) + \lambda k - X, \]  

and

\[
v (k, S, z) = \max_{X \leq \lambda k + S} \left\{ \frac{(zk - X)^{1-\sigma}}{1-\sigma} + \beta \int v (k [1 - \delta] + X, \lambda k + S [1 - \gamma] - X, z') dF \right\}.
\]

Since

\[
\frac{S'}{k'} = \frac{S'}{k' - k} = \frac{s (1 - \gamma) + \lambda - x}{1 - \delta + x},
\]

we have

\[
s' = \frac{\lambda + s (1 - \gamma) - x}{1 - \delta + x},
\]

so that \((1 - \delta + x) s' = \lambda + s (1 - \gamma) - x\). Collecting terms, we get

\[
x s' + x = \lambda + s (1 - \gamma) - (1 - \delta) s',
\]

which leaves us with

\[
\hat{x} (s', s) = \frac{\lambda + s (1 - \gamma) - (1 - \delta) s'}{1 + s'}.
\]

The auxiliary Bellman equation is

\[
w (s, z) = \max_{x} \left\{ \frac{(z - x)^{1-\sigma}}{1-\sigma} + (1 - \delta + x)^{1-\sigma} \beta \int w \left( \frac{\lambda + s (1 - \gamma) - x}{1 - \delta + x}, z' \right) dF \right\},
\]

and we still have \(P = \frac{\nu_{s+h} + \nu_{s}}{C_{s}}\).

**Differentiability, i.e., \(w_s\), when there is depreciation** I use subscripts to denote the state that a policy pertains to. Thus we have the accounting identities

\[
s'_{s} = \frac{\lambda + s (1 - \gamma) - x_{s}}{1 - \delta + x_{s}} \quad \text{and} \quad s'_{s+h} = \frac{\lambda + (s + h) (1 - \gamma) - x_{s+h}}{1 - \delta + x_{s+h}}.
\]
If we begin at state \( s + h \), and to end up at \( s'_s \) we need an investment of

\[
\hat{x}(s'_s, s + h) = \frac{\lambda + s (1 - \gamma) - (1 - \delta) s'_s}{1 + s'_s} \\
= \frac{\lambda + (s + h) (1 - \gamma) - (1 - \delta) \frac{\lambda + s(1 - \gamma) - x_s}{1 - \delta + x_s}}{1 + \frac{\lambda + s(1 - \gamma) - x_s}{1 - \delta + x_s}} \quad \text{(substituting from [68])}
\]

\[
= \frac{(1 - \delta + x_s) (\lambda + (s + h) (1 - \gamma)) - (1 - \delta) (\lambda + s (1 - \gamma) - x_s)}{1 - \delta + x_s + \lambda + s (1 - \gamma) - x_s} \\
= \frac{(1 - \delta) (\lambda + s (1 - \gamma) + h (1 - \gamma)) + x_s (\lambda + s (1 - \gamma) + h (1 - \gamma)) - (1 - \delta) (\lambda + s (1 - \gamma) - x_s)}{1 - \delta + \lambda + s (1 - \gamma)} \\
= \frac{(1 - \delta) h (1 - \gamma) + x_s [\lambda + s (1 - \gamma) + h (1 - \gamma)]}{1 - \delta + \lambda + s (1 - \gamma)} \\
= \frac{(1 - \delta) h (1 - \gamma) + x_s h (1 - \gamma)}{1 - \delta + \lambda + s (1 - \gamma)} \\
= x_s + \frac{(1 - \delta) h (1 - \gamma)}{1 - \delta + \lambda + s (1 - \gamma)} \\
= x_s + \frac{(1 - \gamma) (1 - \delta + x_s)}{1 - \delta + \lambda + s (1 - \gamma)}.
\]

Then

\[
A_h \equiv \left( \frac{1 - \delta + \hat{x}}{1 - \delta + x_s} \right)^{1 - \sigma} \left( 1 + h \frac{(1 - \gamma)(1 - \delta + x_s)}{1 - \delta + x_s} \right)^{1 - \sigma} = \left( 1 + h \frac{1 - \gamma}{1 - \delta + \lambda + s (1 - \gamma)} \right)^{1 - \sigma},
\]

and

\[
\hat{x} - x_s = h \frac{(1 - \gamma) (1 - \delta + x_s)}{1 - \delta + \lambda + s (1 - \gamma)}.
\]

Therefore

\[
w(s + h, z) \geq U(z - \hat{x}[s'_s, s + h]) + (1 - \delta + \hat{x}[s'_s, s + h])^{1 - \sigma} \beta \int w(s'_s, z') dF \\
= U(z - \hat{x}[s'_s, s + h]) + A_h (1 - \delta + x_s)^{1 - \sigma} \beta \int w(s'_s, z') dF \\
= U(z - \hat{x}[s'_s, s + h]) + A_h (w(s, z) - U(z - x_s))
\]

and

\[
w(s + h, z) - w(s, z) \geq U(z - \hat{x}[s'_s, s + h]) - A_h U(c_s) + (A_h - 1) w(s, z) \\
= U(z - \hat{x}[s'_s, s + h]) - U(c_s) + (A_h - 1)(w(s, z) - U(c_s)).
\]

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Dividing both sides by \( h \) and taking the limit as \( h \to 0 \) gives

\[
\frac{d}{ds} w(s, z) \geq -U'(c_s) \left( \frac{\hat{x} - x_s}{h} + \lim_{h \to 0} \frac{(A_h - 1)}{h} [w(s, z) - U(c_s)] \right)
= -U'(c_s) \frac{(1 - \gamma)(1 - \delta + x_s)}{1 - \delta + \lambda + s(1 - \gamma)} + \frac{(1 - \sigma)(1 - \gamma)}{1 - \delta + \lambda + s(1 - \gamma)} [w(s, z) - U(c_s)]
\]

because, by L'Hôpital's rule,

\[
\lim_{h \to 0} \frac{(A_h - 1)}{h} = \lim_{h \to 0} \frac{dA_h}{dh} = \lim_{h \to 0} \frac{d}{dh} \left( 1 + h \frac{1 - \gamma}{1 - \delta + \lambda + s(1 - \gamma)} \right)^{1-\sigma}
= \frac{1 - \gamma}{1 - \delta + \lambda + s(1 - \gamma)} \lim_{h \to 0} \left( 1 + h \frac{1 - \gamma}{1 - \delta + \lambda + s(1 - \gamma)} \right)^{-\sigma}
= \frac{1 - \gamma}{1 - \delta + \lambda + s(1 - \gamma)}.
\]

Then

\[
w_s = \frac{(1 - \gamma)(1 - \delta + x)}{1 - \delta + \lambda + s(1 - \gamma)} \left( [1 - \sigma] w - (1 - \delta + x_s) U' - (1 - \sigma) U \right)
= \frac{1 - \gamma}{1 - \delta + \lambda + s(1 - \gamma)} \left( [1 - \sigma] w - (1 - \delta + x) (z - x)^{-\sigma} - (z - x)^{1-\sigma} \right)
= \frac{1 - \gamma}{1 - \delta + \lambda + s(1 - \gamma)} \left( [1 - \sigma] w - (z - x)^{-\sigma} [1 - \delta + z] \right),
\]

which one also could obtain by assuming differentiability in (70) and applying the envelope theorem. The expression collapses to (11) when \( \gamma = \delta = 0 \).