Signaling in Markets with Two-Sided Adverse Selection[1]

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September 14, 1999

[1]This paper is an outgrowth of work that has been presented at Carnegie Mellon University; the Studia della Economia/ *Workshop on Economic Theory*, Venice, Italy; the *Conference on Endogenous Incompleteness* at the Federal Reserve Bank of Minneapolis; *SITE* at Stanford University. I would like to thank the participants for their comments, Nicholas Yannelis and an anonymous referee for numerous helpful suggestions, and Piero Gottardi and Alberto Bisin for stimulating conversations over a number of years. The financial support of the National Science Foundation and the C.V. Starr Center for Applied Economics is gratefully acknowledged.

Abstract

The paper analyzes an economy with two-sided adverse selection, focusing on equilibria that satisfy a refinement based on the notion of strategic stability. In the familiar case of one-sided adverse selection, agents reveal all of their private information as long as the contract space is rich enough. However, with two-sided adverse selection, the sufficient conditions for separation are much stronger.
J.E.L. Classification Nos. D5, D82

1 Introduction

One of the classical problems in the analysis of markets with incomplete information is to discover conditions under which agents reveal their private information. There are several reasons why this problem is important. The market has long been recognized as a mechanism for facilitating the transfer of information. If agents have an incentive to withhold information, the resulting equilibrium may be informationally inefficient. The vast empirical and theoretical literature on the informational efficiency of financial markets attests to the interest of this issue. At the same time, it is well known that the attempt to signal private information can cause distortions that make the market allocation inefficient. For example, in markets with adverse selection, an agent's choice of education, insurance, or borrowing may reveal his private information about his productivity, probability of having an accident, or the riskiness of his project. The early papers of Spence (1973) on market signaling and Rothschild and Stiglitz (1976) on screening provide sufficient conditions for the existence of equilibria in which agents reveal their private information. These models are further refined by Wilson (1977) and Riley (1979). In order to signal their private information, agents have to incur a private cost (otherwise their signal could be imitated). The cost of signaling is a deadweight loss from society's point of view and for this reason equilibria in which agents can signal their private information are typically inefficient. An analysis of the efficiency properties of signaling equilibria is contained in Gale (1996).

This paper focuses on the conditions that ensure agents signal their private information. The theoretical results in the literature are mixed. While screening models sometimes have no equilibrium, signaling models have a multiplicity of equilibria in which the amount of information revealed varies from full revelation to no revelation. If we want to reduce the set of equilibria and make tighter predictions about information revelation, some refinement of equilibrium is needed.

Refinements of the Nash equilibrium were developed for games of incomplete information in the nineteen-eighties (Kohlberg and Mertens (1986), Banks and Sobel (1987), Grossman and Perry (1986) and Mailath, Okuno-Fujiwara, and Postlewaite (1993)). Applications to signaling games by Cho and Kreps (1984) and Cho and Sobel (1990) select a separating equilibrium, in which agents' actions reveal their type, as the only outcome that satisfies a (strong) refinement. There are exceptions, however. Hellwig (1987) adds a third stage to the canonical signaling game, to capture the flavor of the Wilson (1977) equilibrium, and finds that for some parameter values the pooling equilibrium is the unique equilibrium satisfying the Kohlberg-Mertens refinement.

The classical signaling game is special in a number of respects.

- There are only two agents, an informed agent and an uninformed agent.
- The informed agent has private information (his type) and moves first.
- The uninformed agent observes the informed agent's action before choosing his own action.

The signaling game can also be interpreted as a market game. Instead of a single informed agent with a probability distribution of types, the market-game interpretation assumes a continuum of informed agents with a known cross-sectional distribution of types. Under certain conditions (e.g., the uninformed agents are risk-netural firms) the reactions of a continuum of uninformed agent mimics the response of a single uninformed agent to a single informed agent with a random type.

The signaling game provides a simple and tractable framework in which to study the informational properties of equilibrium, but it has limitations. There are many ways in which the signaling game could be extended. One alternative framework, which accommodates a richer set of environments and has proved to be very tractable, is developed in Gale (1991, 1992, 1996). A complete set of contracts represents all the possible forms of interactions between agents on the two sides of the markets. Agents choose the contracts they most prefer, taking into account the information revealed by the contract choices of the other agents. Thus, information is symmetrically and simultaneously revealed by equilibrium contract choices and used by agents in making those choices.

In contrast to the signaling game, this framework allows for adverse selection on one or both sides of the market.

- There is a continuum of 'buyers' and 'sellers', each of whom may have private information.
- All agents simultaneously choose the contracts they would like to exchange.

- The ‘buyers’ and ‘sellers’ who have chosen a given contract are randomly matched.

Because agents move simultaneously, they do not observe any information prior to their choice of contract. However, in equilibrium they know the strategies of the other agents and correctly assess the probability of being matched with a given type of ‘buyer’ or ‘seller’ conditional on the contract they choose. Because there is a large number (non-atomic continuum) of agents, the market is competitive. This is reflected in the fact that agents take as given the matching probabilities conditional on each choice of contract. The matching probabilities play the role of prices in the classical theory of competitive equilibrium, that is, they determine which contracts can be traded and which cannot.

The refinements of Nash equilibrium developed in the game theory literature have natural counterparts here. Gale (1992) uses a refinement derived from the Kohlberg-Mertens concept of strategic stability to select a unique separating equilibrium allocation in which each agent chooses to reveal his private information. The selection theorem in Gale (1992) is restrictive in one important respect, however. While it allows for heterogeneous types on each side of the market, there is only adverse selection on one side. More precisely, in a market consisting of buyers and sellers, there are several types of buyers but the sellers do not care which type of buyer they trade with. This model allows for assortative matching, which cannot occur in the standard signaling game, but it avoids the difficulties of analysing two-sided adverse selection.

The present paper has two objectives. First, it is argued that when there is one-sided uncertainty (OSU), separation will always occur if the space of contracts traded in the economy is sufficiently rich relative to the type space. Secondly, it is argued that the case of two-sided uncertainty (TSU) is much more difficult to analyze and requires stronger conditions to ensure full revelation of information.

The reason for the greater difficulty of analyzing TSU is that, roughly speaking, with OSU an informed agent has preferences over contracts but not over types of agents on the other side of the market. For example, if the sellers are workers with different productivities and the buyers are identical, risk neutral firms, then the sellers have preferences over different contracts (specifications of wages, hours of work, and education) but do not care what kind of firm they work for. Thus, beliefs (about the buyers’ types) do not affect the sellers' behavior in a significant way. By contrast, in the case of TSU, an informed agent has preferences over contracts *and* the types of agents on the other side of the market. For example, if there are different types of firms and workers care about the type of firm they work for, then workers have to take into account both the nature of the contract they choose and the type of firm that is likely to offer that particular contract.

Refinements of equilibrium work by restricting 'plausible' out-of-equilibrium beliefs. With TSU it becomes much more difficult to say whether an out-of-equilibrium belief is 'plausible' or not. An agent's choice of contract depends both on the physical characteristics of the contract and the probability distribution of types with which an agent will be matched. This means that in order to discuss what is a reasonable or plausible belief, one has to look at both sides of the market simultaneously. For example, a seller may have the belief that a particular contract is traded by a bad type of buyer. This belief may discourage the seller from choosing that contract. In order to determine whether the buyer's belief is plausible, we have to ask whether it is plausible to expect a bad buyer to choose that contract. That in turn will depend not just on the bad buyer's preferences over contracts, but also on his beliefs about the type of seller he will be matched with if he chooses that contract. But then we have to ask whether the buyer's belief is plausible and that in turn depends on the sellers' behavior, that is, on the preferences and beliefs of the sellers.

The theory developed in this paper focuses on decentralized and uncoordinated decision making under incomplete information. Other theories focus on the optimality properties of competitive markets (Harris and Townsend (1982), Prescott and Townsend (1984), Yannelis (1991), Myerson (1993), Koutsougeras and Yannelis (1993), Krasa and Yannelis (1994), Jerez (1999)). The two approaches should be thought of as complementary. There is no single paradigm of competitive markets with incomplete information.

The rest of the paper is organized as follows. The basic model of an economy with incomplete information is presented in Section 2, where equilibrium is defined and a refinement of equilibrium is introduced. This refinement is shown to impose restrictions on beliefs about matching probabilities. In Section 3 these restrictions are applied to an economy with OSU. Under conditions that are weaker than the standard Spence-Mirrlees monotonicity conditions, the only kind of equilibrium that satisfies the refinement is a separating equilibrium. An economy with TSU is analyzed in Section 4, where two sets of sufficient conditions for separation are discussed. Proofs are gathered in Section 5.

2 An Economy with Incomplete Information

The economy consists of two mutually exclusive classes of agents who are conventionally referred to as buyers and sellers. There is a finite number of different types of agents. Let $K = S \cup T$ denote the set of types, where S denotes the set of seller types and T denotes the set of buyer types. Each type $k \in K$ consists of a non-atomic continuum of identical agents whose measure is $N(k) > 0$.

The objects traded in this economy are *contracts*. The set of all possible contracts is denoted by Θ. Initially, Θ is assumed to be finite. Later the theory is extended to an infinite set of contracts.

Contracts are exchanged between buyers and sellers, with one buyer and one seller for each contract. If a seller of type s and a buyer of type t exchange a contract of type θ, the seller's payoff is $u(\theta, s, t)$ and the buyer's payoff is $v(\theta, s, t)$. Each agent is assumed to have an outside option that determines his utility if he chooses not to trade a contract. The payoff functions are normalized so that the value of each agent's outside option is equal to 0.

Agents are allowed to trade at most one contract. Under this assumption, the equilibrium choices of all the agents can be described by an *allocation* $f : \Theta \times K \to \mathbf{R}_+$, where $f(\theta, k)$ is the measure of type-k agents who choose a contract θ. An allocation is *attainable* if it satisfies the adding-up condition

$$\sum_{\theta} f(\theta, k) \leq N(k),$$

for every k. The number of agents of type k who choose not to trade is $N(k) - \sum_{\theta} f(\theta, k) \geq 0$. Let F denote the set of attainable allocations.

To avoid some pathological cases, it is assumed that there is a small disutility of participating in a market. Let $c(k) > 0$ denote the participation cost for agents of type k. Note that an agent has to pay the participation cost $c(k)$ even if he is rationed and unable to trade. Only agents who choose not to participate avoid this cost.

Let $\mathcal{E} = \{S, T, N, u, v, c\}$ denote the economy with incomplete information.

2.1 Equilibrium

The buyers and sellers who want to trade contract θ are randomly matched. An agent does not know the type of agent he will be matched with, but he does know the probability of being matched with any particular type of agent. The equilibrium matching probabilities are described by a *probability assessment* $\mu : \Theta \times K \to \mathbf{R}_+$, where $\mu(\theta, t)$ denotes the probability that a seller choosing contract θ will exchange the contract with a type-t buyer and $\mu(\theta, s)$ denotes the probability that a buyer choosing contract θ will exchange the contract with a type-s seller.

The number of buyers and sellers who want to trade θ may not be equal. In that case, the market clears through rationing. A buyer's probability of trading θ is $\sum_s \mu(\theta, s)$. Likewise, a seller's probability of trading θ is $\sum_t \mu(\theta, t)$. Thus, the set of admissible probability assessments is

$$M = \left\{ \mu : \Theta \times K \to \mathbf{R}_+ | \sum_s \mu(\theta, s) \leq 1, \sum_t \mu(\theta, t) \leq 1 \right\}.$$

The probability of being rationed (unable to trade) is $1 - \sum_s \mu(\theta, s) \geq 0$ for buyers and $1 - \sum_t \mu(\theta, t) \geq 0$ for sellers. Note that the equilibrium probability assessment μ is common for all agents. Thus, all sellers have identical beliefs about the trading possibilities open to them and all buyers have identical beliefs about trading possibilities open to them.

It is important to distinguish contracts which are demanded by a positive measure of agents from those which are not demanded by anyone. Matching probabilities are well defined in markets for contracts that are traded in equilibrium, but for non-traded contracts the matching probabilities can be more or less arbitrary. Since Θ is supposed to represent the set of all possible contracts, most of the contracts in Θ will not be actively traded in equilibrium. This means that the equilibrium probability assessment μ is to a large extent arbitrary and this arbitrariness can give rise to a large set of equilibria, as we shall see.

For any attainable alloction f, let $\lambda_f(\theta)$ measure the long side of the market for contract θ, that is,

$$\lambda_f(\theta) = \max \left\{ \sum_s f(\theta, s), \sum_t f(\theta, t) \right\}.$$

The market for contract θ is called *active* if $\lambda_f(\theta) > 0$. Otherwise, the market for θ is called *inactive*. In active markets, beliefs are determined by rational expectations and the random matching process. Since the random matching process treats all buyers and all sellers symmetrically, the probability assessment $\mu(\theta, \cdot)$ must be the same for all agents if the market for θ is active. If the market is inactive, however, the agents' beliefs are more or less arbitrary (i.e., not determined by the matching technology) and here the assumption of common beliefs represents a mild symmetry condition.

The matching rules treat all agents on the same side of the market identically and maximize the probability of trade. A probability assessment is consistent with an allocation if it reflects the actual matching probabilities determined by the allocation in each active market. Formally, the probability assessment μ is *consistent* with the allocation f if

$$\lambda_f(\theta)\mu(\theta, k) = f(\theta, k),$$

for any θ and any k. If the market for θ is active, then $\lambda_f(\theta) > 0$ and the probability $\mu(\theta, k)$ is uniquely determined by the allocation f. If $\lambda_f(\theta) = 0$, the consistency condition is automatically satisfied and does not place any restrictions on the equilibrium probability assessment.

Now we are ready to define an equilibrium. Intuitively, an equilibrium requires each agent to choose the contract that maximizes his expected utility, taking as given the probability assessment, that is, the probability of trading each contract and the distribution of types with whom he may be matched. The probability assessment is determined jointly by the choices of all the agents. Formally, an *equilibrium* consists of an attainable allocation f and a consistent probability assessment μ such that, for every type s and any contract θ, a positive measure $f(\theta, s) > 0$ of agents choose θ only if it is optimal

$$\sum_t u(\theta, s, t)\mu(\theta, t) = u^*(s) = \max_\theta \left\{ \sum_t u(\theta, s, t)\mu(\theta, t) \right\} \geq c(s),$$

and for every type t and any contract θ, a positive measure $f(\theta, t) > 0$ of agents choose θ only if it is optimal

$$\sum_s v(\theta, s, t)\mu(\theta, s) = v^*(t) = \max_\theta \left\{ \sum_s v(\theta, s, t)\mu(\theta, s) \right\} \geq c(t).$$

The significance of the participation cost is simply to ensure that the equilibrium payoff from trading is positive if agents choose to participate. That is, $u^*(s) > 0$ if $f(\theta, s) > 0$ for some θ and $v^*(t) > 0$ if $f(\theta, t) > 0$ for some θ.

2.2 Perfection and Stability

As has already been noted, the probability assessment $\mu(\theta, \cdot)$ is more or less arbitrary when the market for θ is inactive. The problem this poses for the theory is that there may be many different equilibrium allocations supported by different beliefs about trading probabilities in inactive markets. Some of these equilibria are of little interest because they depend on implausible beliefs about the trading possibilities in inactive markets. For example, if we assume that $\mu(\theta_0, k) = 0$ for some fixed but arbitrary θ_0 and any k, then it is easy to see that no one will attempt to trade the contract θ_0 in equilibrium. Then $f(\theta_0, k) = 0$ for every k, and the probability assessment $\mu(\theta_0, \cdot)$ will be consistent with the allocation $f(\theta_0, \cdot)$. In this way we can "close" the market for any contract θ_0 without violating the equilibrium conditions. By closing markets, we can generate a large number of different equilibrium allocations; but these equilibria are not very interesting.

To rule out such trivial equilibria, it is usually assumed that markets must be orderly, which means that at most one side of the market can be rationed in equilibrium (Hahn and Negishi (1962), Dreze (1975), Hahn (1978)). The probability assessment μ is *orderly* if, for every θ,

$$\max \left\{ \sum_t \mu(\theta, t), \sum_s \mu(\theta, s) \right\} = 1.$$

This restriction rules out some equilibria, but it does not eliminate the multiplicity of equilibria caused by arbitrary beliefs in inactive markets. To rule out these equilibria, a further restriction of beliefs in inactive markets is required. There are various strategies for refining an equilibrium concept. One is based on the idea of the 'trembling hand' introduced by Selten (1975). Here it is the 'invisible hand' of the market that 'trembles'. The essential idea is to perturb the economy so that all markets are active, find an equilibrium for the perturbed economy, and then let the perturbation become vanishingly small. The limit of this sequence of perturbed equilibria will be an equilibrium of the unperturbed economy.

Formally, a *perturbation* is an attainable allocation g such that $\lambda_g(\theta) > 0$ for all θ. For any perturbation g, denote the set of attainable allocations of the perturbed economy by $F(g)$, where

$$F(g) = \{f \in F | f \geq g\}.$$

The perturbed economy is denoted by $\mathcal{E}(g) = \{S, T, N, u, v, c, g\}$. Note that the parameters are the same as in the original economy $\mathcal{E}$; only the set of attainable allocations $F(g)$ has changed.

Define an equilibrium for the perturbed economy $\mathcal{E}(g)$ to be an attainable allocation $f \in F(g)$ and a consistent probability assessment μ such that, for each type s,

$$f(\cdot, s) \in \arg\max_{F(g)} \sum_{\theta} f(\theta, s) \left\{ \sum_{t} u(\theta, s, t)\mu(\theta, t) - c(s) \right\}$$

and, for each type t,

$$f(\cdot, t) \in \arg\max_{F(g)} \sum_{\theta} f(\theta, t) \left\{ \sum_{s} v(\theta, s, t)\mu(\theta, s) - c(t) \right\}.$$

A *perfect equilibrium* is the limit (f^0, μ^0) of a sequence of equilibria $\{(f^n, \mu^n)\}$ where, for each n, (f^n, μ^n) is an equilibrium of the perturbed economy $\mathcal{E}(n^{-1} \cdot g)$. Standard arguments suffice to show that there exists a perfect equilibrium for any economy $\mathcal{E}$.

In the perturbed economy, all markets are active, so the equilibrium probability μ assessment is uniquely determined by the equilibrium allocation $f \in F(g)$. A perfect equilibrium, being the limit of perturbed equilibria, has a probability assessment that is the limit of probability assessments generated by matching probabilities attainable allocations.

Note that an equilibrium of a perturbed economy is orderly by construction. For any θ, $\lambda_f(\theta) > 0$ so the consistency condition implies that $\mu(\theta, k) = f(\theta, k)/\lambda_f(\theta)$ and

$$\begin{aligned}
\max\left\{ \sum_{t} \mu(\theta, t), \sum_{s} \mu(\theta, s) \right\} &= \max\left\{ \frac{\sum_t f(\theta, t)}{\lambda_f(\theta)}, \frac{\sum_s f(\theta, s)}{\lambda_f(\theta)} \right\} \\
&= \frac{\max\left\{\sum_t f(\theta, t), \sum_s f(\theta, s)\right\}}{\lambda_f(\theta)} \\
&= \frac{\lambda_f(\theta)}{\lambda_f(\theta)} = 1.
\end{aligned}$$

A perfect equilibrium, being the limit of a sequence of orderly equilibria, is also orderly.

Although perfection imposes some restrictions on the equilibrium beliefs, they are pretty mild. In order to restrict beliefs further we have to use a stronger refinement. This refinement is related to the notion of strategic stability introduced by Kohlberg and Mertens (1986). A perfect equilibrium (f, μ) is robust to a single perturbation g in the sense that, for any $\varepsilon > 0$ and each n sufficiently large, there is an equilibrium of the perturbed economy $\mathcal{E}(n^{-1} \cdot g)$ which is ε-close to (f, μ). The Kohlberg-Mertens notion of stability requires this kind of robustness in the face of all possible perturbations. It may not be possible to find a single equilibrium that has such a property, so we consider sets of equilibria. Call S a *stable set* if it is a minimal set of equilibria with the property that:

> for any perturbation g and any $\varepsilon > 0$ there exists a number $n_0 > 0$ such that for any $n > n_0$ there exists an equilibrium of the perturbed economy $\mathcal{E}(n^{-1} \cdot g)$ that is ε-close to S.

One reason why we need to consider sets of equilibria is that by choosing different perturbations we generate different probability assessments in the inactive markets. So sequences of equilibria corresponding to different perturbations may have different limiting probability assessments. However, this may be the only difference between the equilibria belonging to the stable set. In that case, there is a unique allocation that satisfies the stability criterion. If all the equilibria in S have the same allocation f then f is called a *unique stable allocation*. When there is no risk of ambiguity we refer to f as a *stable allocation* for short.

The existence of a unique stable allocation is established for generic economies with incomplete information in Gale (1992).

2.3 Stable Beliefs

When an economy is perturbed, the probability assessments are forced to change. If the probability assessments change in a way that makes some unused contract θ more attractive, it may cause a large number of agents to deviate to θ, in which case the original equilibrium is deemed to have been non-robust or unstable. On the other hand, if the number of agents deviating to θ is small and if the probability assessment changes in such a way that no one strictly prefers θ to the equilibrium payoff, then a small perturbation has led to a small change in the equilibrium and the equilibrium is considered robust. In other words, if an allocation is stable then any perturbation can be stabilized by the endogenous re-allocation of a small number of agents. This principle plays a crucial role in what follows. The next step is to characterize exactly what this means for beliefs in a stable equilibrium.

Let f be a stable allocation and let g be a fixed but arbitrary perturbation. There is a sequence of equilibria $\{(f^n, \mu^n)\}$, where (f^n, μ^n) is an equilibrium of the perturbed economy $\mathcal{E}(n^{-1} \cdot g)$ for each n and

$$\lim_{n \to \infty} (f^n, \mu^n) = (f, \mu) \in S.$$

As was noted above, the probability assessment μ may depend on the particular sequence $\{(f^n, \mu^n)\}$. Let $u^*(k)$ denote the equilibrium payoff of type k in the limiting equilibrium (f, μ). Suppose that for some fixed but arbitrary contract θ_1, $u^*(s) > \sum_t \mu(\theta_1, t) u(\theta_1, s, t)$. Since the payoff functions are continuous, $\max_\theta \sum_t \mu^n(\theta, t) u(\theta, s, t) > \sum_t \mu^n(\theta_1, t) u(\theta_1, s, t)$, for all n sufficiently large and this implies that no agent of type s voluntarily chooses contract θ_1, i.e., $f^n(\theta_1, s) = n^{-1} \cdot g(\theta_1, s)$, for all n sufficiently large. A symmetric conclusion holds for buyers. Thus, we have established that, for any contract θ,

$$\left[u^*(s) > \sum_t \mu(\theta, t) u(\theta, s, t) \right] \implies \left[\mu^n(\theta, s) = \frac{n^{-1} g(\theta, s)}{\lambda^n(\theta)} \right], \forall s$$

$$\left[v^*(t) > \sum_s \mu(\theta, s) v(\theta, s, t) \right] \implies \left[\mu^n(\theta, t) = \frac{n^{-1} g(\theta, t)}{\lambda^n(\theta)} \right], \forall t,$$

for all n sufficiently large. Taking limits as $n \to \infty$ immediately proves the following result.

Theorem 1 *Suppose that f is a unique stable allocation and g is a fixed but arbitrary perturbation. Then there exists a probability assessment μ such that (f, μ) is an equilibrium and, for any contract θ and for some constant $\gamma(\theta) \geq 0$,*

$$\left[u^*(s) > \sum_t \mu(\theta, t) u(\theta, s, t) \right] \implies [\mu(\theta, s) = \gamma(\theta) g(\theta, s)],$$

for any seller type s and

$$\left[v^*(t) > \sum_s \mu(\theta, s) v(\theta, s, t) \right] \implies [\mu(\theta, t) = \gamma(\theta) g(\theta, t)],$$

for any buyer type t.

To sum up, a perturbation changes the equilibrium probability assessment, but exactly how it changes depends on the equilibrium responses of the agents. So the relationship between the equilibrium probability assessment and the perturbation in the market for a contract θ depends on whether agents find it optimal to choose that contract in equilibrium. In particular, if no one finds it optimal to choose θ in equilibrium, then $f^n(\theta, \cdot) = n^{-1}g(\theta, \cdot)$ and the probability assessment $\mu^n(\theta, \cdot)$ is determined by the perturbation $n^{-1}g(\theta, \cdot)$. On the other hand, if the perturbation $g(\theta, \cdot)$ by itself would have made θ attractive to some types (would give them a payoff higher than their equilibrium payoff) then in equilibrium a small number of agents must be endogenously re-allocated to θ in order to make θ less attractive and prevent a large deviation by other agents. So one way to show that an allocation is not stable is to find a perturbation g that cannot be stabilized by a small re-allocation of agents.

2.4 General Contract Spaces

Even in finite games, the definition of a perturbation requires some care (see Kohlberg and Mertens (1986)). There is no comparable development of the theory for infinite games or economies with an infinite number of contracts. Similarly, the existence of equilibrium in an economy with a finite number of contracts is straightforward matter (it uses a standard fixed point argument) but the existence of a unique stable allocation is more difficult (see Gale (1992)). Again, the theory has not been developed to deal with an infinite number of contracts.

The assumption of a finite number of contracts is a convenient simplification, but for some purposes it is more convenient to have a continuum of contracts. In particular, when it comes to characterizing the degree of separation in equilibrium it is nice to be able to consider neighboring contracts. The theory can be extended from a finite to an infinite set of contracts by taking limits (this was the approach taken in Gale (1992)), but for simplicity an alternative approach is adopted here. I take as a definition of stability the characterization of stable beliefs derived in Theorem 1. From now on it is assumed that Θ is an open subset of some finite-dimensional Euclidean space endowed with the usual topology. The payoff functions $u(\cdot, s, t)$ and $v(\cdot, s, t)$

are assumed to be continuously differentiable functions of θ on Θ, for every pair (s,t).

When the contract space is infinite, an allocation is represented by a measure. To keep things simple (and it really does not make much difference to the analysis to follow), we assume that the number of contracts traded in equilibrium is finite. In that case, we can continue to define an allocation as a function $f : \Theta \times K \to \mathbf{R}_+$ with finite support, where $f(\theta,k)$ is the number of agents of type k that choose $\theta \in \Theta$. The allocation f is attainable if

$$\sum_\theta f(\theta,k) \leq N(k), \forall k.$$

An orderly probability assessment is a measurable function $\mu : \Theta \times K \to \mathbf{R}_+$ such that

$$\max\left\{\sum_t \mu(\theta,t), \sum_s \mu(\theta,s)\right\} = 1, \forall\theta.$$

The probability assessment μ is consistent with an attainable allocation f if

$$\lambda_f(\theta)\mu(\theta,k) = f(\theta,k)$$

for every θ, where $\lambda_f(\theta) = 0$ for all but a finite number of contracts θ.

An attainable allocation f is said to be *stable* if it satisfies the condition that

> for any attainable allocation g, there exists an orderly probability assessment μ consistent with f such that for any contract θ and some constant $\gamma(\theta) \geq 0$,
>
> $$\left[u^*(s) > \sum_t \mu(\theta,t)u(\theta,s,t)\right] \Longrightarrow [\mu(\theta,s) = \gamma(\theta)g(\theta,s)] \quad (1)$$
>
> for any seller type s and
>
> $$\left[v^*(t) > \sum_s \mu(\theta,s)v(\theta,s,t)\right] \Longrightarrow [\mu(\theta,t) = \gamma(\theta)g(\theta,t)] \quad (2)$$
>
> for any buyer type t, where $u^*(s)$ and $v^*(t)$ are the payoffs of types s and t respectively in the allocation f.

3 Separation with One-Sided Uncertainty

Paralleling the familiar signaling models in the literature, we first consider the special case of an economy in which there is a single type of buyer $T = \{t_0\}$. This is a special case of one-sided uncertainty (OSU).[1] We can suppress the reference to the buyers' type and write the payoff functions as $u(\theta, s)$ and $v(\theta, s)$. Normalize the number of buyers to 1. Then the economy $\mathcal{E} = \{S, N, \Theta, u(\cdot), v(\cdot), c(\cdot)\}$ is defined by the set of seller types S, the distribution of seller types $N(s)$, the set of contracts Θ, the payoff functions $u(\theta, s)$ and $v(\theta, s)$, and the participation cost function $c(k)$.

The result presented in this section gives conditions that are sufficient to rule out pooling of different types of sellers at a single contract if the allocation is stable. We are only interested in contracts that are actually traded in equilibrium, so there is no loss of generality in focusing attention on a contract θ_0 such that

$$\sum_s f(\theta_0, s) > 0 \text{ and } f(\theta_0, t_0) > 0.$$

Let S_0 denote the set of seller types that find θ_0 optimal, that is,

$$S_0 = \{s \in S | \mu(\theta_0, t_0) u(\theta_0, s) = u^*(s)\}.$$

Then it is assumed that there is a single type s_0 in S_0 that is most preferred by the buyers, that is,

$$v(\theta_0, s_0) > v(\theta_0, s), \forall s \in S_0, s \neq s_0.$$

[1] An economy is characterized by OSU (one-sided adverse selection) if the agents on one side of the market, say the sellers, are indifferent about being matched with different types of buyers. If there is only one type of buyer, this condition is automatically satisfied, but it would also be satisfied when there are heterogeneous types of buyers, as long as sellers do not care which type of buyer they are matched with. Formally, OSU requires $u(\theta, s, t) = u(\theta, s, t')$ for every θ, s, t and t'. Since sellers do not care about the buyers' types, they do not face an adverse selection problem and this simplifies the analysis considerably. The existence of heterogeneous types of buyers may still be important in equilibrium, however. Because different buyer types have different preferences over contracts and types of sellers, they will typically choose different contracts in equilibrium. Gale (1992) shows how this can lead to positive assortative matching in equilibrium. This phenomenon, which cannot arise when all buyers are identical, illustrates one way in which the model encompasses a richer set of equilibrium possibilities than the classical signaling game, even when they are superficially similar.

This condition merely rules out ties and avoids some complications that do not seem to be important in the general analysis. The main condition in the theorem is that there exists a contract θ_1 that is arbitrarily close to θ_0 and that is preferred by type s_0 and not by any types $s \in S_0, s \neq s_0$:

$$\begin{array}{ll}(S1) & u(\theta_1, s_0) > u(\theta_0, s_0) \\ (S2) & u(\theta_1, s) < u(\theta_0, s), \forall s \in S_0, s \neq s_0.\end{array}$$

This condition is much weaker than the famous single-crossing property, although it is implied by the latter. It requires only that we can find some dimension of the contract and some direction in that dimension that is preferred by type s_0 and only by that type. Beyond this, the argument is much the same as in the standard analysis, except that it works through perturbations of the game and has to take account of the possiblity of rationing, which does not appear in models with market-clearing prices.

Under these conditions, it is impossible for two types of sellers to pool at a contract θ_0. The formal proof is left until Section 5. A heuristic proof follows. First, consider a perturbation of the game that assigns only type s_0 to contracts that are not used in the stable allocation f. If no type finds θ_1 weakly optimal, then buyers must believe that only type s_0 will exchange contract θ_1. By assumption, sellers of type s_0 prefer θ_1 to θ_0 and, for θ_1 close to θ_0, buyers will always prefer to trade θ_1 with s_0 rather than trade θ_0 with a mixture of s_0 and less desirable types. Since markets are orderly, some agents can trade θ_1 with probability one and this contradicts the definition of stability. So at least one type of seller other than s_0 must find θ_1 weakly optimal, but then condition $(S1)$ implies that s_0 strictly prefers θ_1 to θ_0, again contradicting the stability condition.

This proves the following result.

Theorem 2 *Let f be a stable allocation and let θ_0 be a contract in Θ that is traded in equilibrium: $f(\theta_0, s) > 0$ for some s and $f(\theta_0, t_0) > 0$. Suppose (i) that there is a unique best type $s_0 \in S_0 = \{s : \mu(\theta_0, t_0)u(\theta_0, s) = u^*(s) \geq c(s)\}$, and (ii) that for any $\varepsilon > 0$ there is a contract θ_1 that is ε-close to θ_0 and satisfies conditions $(S1)$ and $(S2)$. Then $f(\theta_0, s) = 0$ for any $s \neq s_0$.*

While Theorem 2 provides conditions under which at most one type chooses θ_0, it will often be the case that θ_0 will be optimal for more than one type. In other words, the self-selection constraint is binding in equilibrium.

3.1 Discussion

The model analyzed in this section is a special case of the model described in Gale (1992). In particular, Gale (1992) allows for private information on one side of the market and heterogeneous agents on both sides. However, the assumptions used in Gale (1992) to prove separation are much stronger than the assumptions of Theorem 2. In particular, in order to prove that all private information is revealed in a stable equilibrium, Gale (1992) assumes that preferences satisfy the Spence-Mirrlees or single-crossing condition.

The separation conditions ($S1$-$S2$) say that, for any contract θ_0, we can find a nearby contract θ_1 that is better for s_0 and worse for every type $s \in S_0, s \neq s_0$. Since S_0 is determined endogenously, conditions ($S1$-$S2$) are not an assumption about primitives. However, it is easy to find a condition on primitives that is sufficient for conditions ($S1$-$S2$). Say that $s < s_0$ if $v(\theta_0, s) < v(\theta_0, s_0)$. Then assume that for any s_0 and all $s < s_0$ there exists a contract θ_1 arbitrarily close to θ_0 such that

$$u(\theta_1, s_0) > u(\theta_0, s_0)$$

and

$$u(\theta_1, s) < u(\theta_0, s), \forall s < s_0.$$

This condition can be interpreted as a condition on the 'richness' of the contract set Θ. More precisely, it is a joint condition on the rank of the Jacobian matrix

$$\left[\frac{\partial u(\theta_0, s)}{\partial \theta}\right]$$

and the dimension of the contract space Θ. If the dimension of Θ is greater than $|S|$ and the Jacobian matrix has full rank, then the conditions ($S1$-$S2$) will be satisfied. We can thus see the conditions ($S1$-$S2$) as a combination of a regularity assumption and an assumption on the relative dimensions of Θ and S. This discussion is formalized in the following result.

Corollary 3 *Let f be a stable allocation and let θ_0 be a contract in Θ that is traded in equilibrium: $f(\theta_0, s) > 0$ for some s and $f(\theta_0, t_0) > 0$. Suppose (i) that there is a unique best type $s_0 \in S_0 = \{s : \mu(\theta_0, t_0)u(\theta_0, s) = u^*(s) \geq c(s)\}$, and (ii) the dimension of Θ is greater than $|S|$ and the Jacobian matrix has full rank. Then $f(\theta_0, s) = 0$ for any $s \neq s_0$.*

Note that it is not enough to assume that Θ is 'big' because some dimensions of Θ may not be payoff relevant. That is why the regularity (full rank) condition has to be added.

4 Separation with Two-Sided Uncertainty

Returning to the 'general' model of an economy with two-sided uncertainty (TSU), it is interesting to see how the conditions for separation change. Theorem 2 shows that conditions on preferences over contracts are sufficient for equilibrium separation of types. More precisely, conditions on preferences over contracts ensure that s_0 is more likely to choose θ_1 than any other type of seller and this ensures that the pooling equilibrium is destabilized when a small measure of type s_0 are assigned to θ_1.

In an economy with TSU, things are more complicated. For all agents, the payoff to a contract $\theta_1 \neq \theta_0$ depends on both the contract θ_1 and the distribution of types $\mu(\theta_1, \cdot)$ associated with θ_1. Looking at it from the point of view of sellers, one cannot say which type of seller will be attracted to θ_1 until one knows the distribution of buyers that are expected to trade θ_1. Similarly, one cannot say which type of buyer will be attracted to θ_1 until one knows the distribution of sellers that are expected to trade θ_1. Even if sellers of type s_0 prefer θ_1 to θ_0, other things being equal, the probability assessment $\mu(\theta_1, t)$ can make θ_0 more attractive than θ_1. Furthermore, Theorem 1 tells us that a reasonable probability assessment $\mu(\theta_1, t)$ depends on which types of buyers find it optimal to choose θ_1. For this reason, one cannot determine what is a reasonable probability assessment $\mu(\theta, \cdot)$ by looking at one side of the market only. To determine what is a reasonable belief one must look at both sides of the market simultaneously.

4.1 The Inclusion Principle

To generalize the separation condition used in the case of OSU, we need to be able to make the following kind of statement: if s_0 chooses a contract θ_0 then there exists a nearby contract θ_1 such that whenever a less desirable type $s < s_0$ finds θ_1 weakly optimal, s_0 will find θ_1 strictly optimal. In other words,

$$\left[\sum_t \mu(\theta_1,t)u(\theta_1,s,t) \geq \sum_t \mu(\theta_0,t)u(\theta_0,s,t)\right]$$
$$\Longrightarrow \left[\sum_t \mu(\theta_1,t)u(\theta_0,s_0,t) > \sum_t \mu(\theta_0,t)u(\theta_0,s_0,t)\right].$$

If this is true for all contracts θ_1 arbitrarily close to θ_0 then, given the continuity of the payoff functions, we can take the limit as $\theta_1 \to \theta_0$ to get

$$\left[\sum_t [m(t)-\mu(\theta_0,t)]u(\theta_0,s,t) \geq 0\right] \tag{3}$$
$$\Longrightarrow \left[\sum_t [m(t)-\mu(\theta_0,t)]u(\theta_0,s_0,t) \geq 0\right],$$

where $m(t)$ represents the limit of the probability assessments $\mu(\theta_1,t)$ as $\theta_1 \to \theta_0$. Since we have little information about the probability assessments that might obtain in equilibrium, if we want the implication (3) to hold in equilibrium we will have to assume that it holds for all possible probability assessments. That is, for any vector of weights $\lambda = (\lambda(t))$ we must assume that

$$\left[\sum_t \lambda(t)u(\theta_0,s,t) \geq 0\right] \Longrightarrow \left[\sum_t \lambda(t)u(\theta_0,s_0,t) \geq 0\right]. \tag{4}$$

The implication (4) is generally valid if and only if $u(\theta_0,s,\cdot)$ is a positive scalar multiple of $u(\theta_0,s_0,\cdot)$, that is,

$$u(\theta_0,s,t) = \alpha(\theta_0,s)u(\theta_0,s_0,t), \forall t,$$

for any $s < s_0$ and for some constant $\alpha(\theta_0,s) > 0$.

If this relationship holds for every contract θ and every type s (how else can we guarantee that it will hold in equilibrium as required?), then the payoff functions can be written in the form

$$u(\theta,s,t) = a(\theta,s)b(\theta,t), \forall(\theta,s,t) \in \Theta \times S \times T.$$

In other words, the preferences are separable and each type of seller has identical marginal preferences over different types of buyers. This is a strong but natural condition that is sufficient for extending the argument used in Section 3 to the case of TSU. Later, we consider the extent to which this condition might be necessary as well.

4.2 Separation with Separable Preferences

Assume that all the agents on one side of the market have similar preferences over the types of agents on the other side and vice versa, that is,

$$\begin{aligned} u(\theta, s, t) &= a(\theta, s)b(\theta, t) \\ v(\theta, s, t) &= c(\theta, t)d(\theta, s). \end{aligned}$$

Let f be a stable allocation and θ_0 a contract that is traded in f. Let $S_0 = \{s | \sum_t \mu(\theta_0, t)u(\theta_0, s, t) = u^*(s) \geq c(s)\}$ be the set of seller types for which θ_0 is an optimal choice. We assume that there is a unique best type $s_0 \in S_0$. Then $s \in S_0$ and $s \neq s_0$ imply that $d(\theta_0, s) < d(\theta_0, s_0)$. We shall also assume that $d(\theta_0, s_0) \neq d(\theta_0, s)$ for any $s \notin S_0$. The separation condition is the following: for any $\varepsilon > 0$ there is a contract θ_1 that is ε-close to θ_0 and satisfies

$$(S3) \quad \frac{a(\theta_1, s_0)}{a(\theta_0, s_0)} > \frac{a(\theta_1, s)}{a(\theta_0, s)}, \forall t, \forall s \ni d(\theta_0, s) < d(\theta_0, s_0).$$

Note that the separation condition $(S3)$ restricts the preferences of all the types s that are less attractive than s_0 to buyers, and not just the types $s \in S_0, s \neq s_0$. A similar condition is required for the other side of the market:

$$(S4) \quad \frac{c(\theta_1, s_0)}{c(\theta_0, s_0)} > \frac{c(\theta_1, s)}{c(\theta_0, s)}, \forall s, \forall t \ni b(\theta_0, s) < b(\theta_0, s_0).$$

Note that ($S3$-$S4$) does not require that θ_1 is preferred to θ_0 by types s_0 and t_0 other things being equal. This would be an implausible condition given the nature of trade. Rather, it requires that types s_0 and t_0 dislike θ_1 relatively less than types $s < s_0$ and $t < t_0$, respectively.

With these assumptions we can show that a stable allocation must be separating.

Theorem 4 *Suppose that the payoff functions have the additively separable form*

$$\begin{aligned} u(\theta, s, t) &= a(\theta, s)b(\theta, t) \\ v(\theta, s, t) &= c(\theta, t)d(\theta, s). \end{aligned}$$

Let f be a stable allocation and θ_0 a contract belonging to the interior of Θ that is traded in equilibrium. Let S_0 (resp. T_0) denote the set of seller types (resp. buyer types) that find θ_0 optimal in equilibrium. Let s_0 (resp.

t_0) denote the best type in S_0 (resp. T_0). Assume that $d(\theta_0, s) \neq d(\theta_0, s_0)$ if $s \neq s_0$ (resp. $b(\theta_0, t) \neq b(\theta_0, t_0)$ if $t \neq t_0$) and that for any $\varepsilon > 0$ we can find a contract θ_1 that is ε-close to θ_0 and satisfies ($S3$-$S4$). Then $f(\theta_0, s) = 0$ for $s \in S_0, s \neq s_0$ and $f(\theta_0, t) = 0$ for $t \in T_0, t \neq t_0$.

Proof. See Section 5. ■

Again, we can see condition ($S3$-$S4$) is implied by the joint assumption that the space of contracts Θ is sufficiently rich (has a high dimension) and that the payoff functions are regular (the Jacobian matrix has full rank).

Corollary 5 *Suppose that the payoff functions have the additively separable form and let f be a stable allocation and θ_0 a contract belonging to the interior of Θ that is traded in equilibrium. Let s_0 (resp. t_0) denote the best type in S_0 (resp. T_0), the types of sellers (resp. buyers) that find θ_0 optimal in equilibrium. Assume that $d(\theta_0, s) \neq d(\theta_0, s_0)$ if $s \neq s_0$ (resp. $b(\theta_0, t) \neq b(\theta_0, t_0)$ if $t \neq t_0$). If Θ has a sufficiently high dimension and that the payoff functions are regular (the Jacobian matrix has full rank) then $f(\theta_0, s) = 0$ for $s \in S_0, s \neq s_0$ and $f(\theta_0, t) = 0$ for $t \in T_0, t \neq t_0$.*

Separability is a natural assumption to make in this context in order to extend the argument used in the OSU case. Nonetheless, separability is restrictive so it is important to ask how far one can relax this assumption and still guarantee separation of types in a stable allocation.

4.3 Separation without Separability

Without separability, there is no hope of applying the general line of argument used above, but there is additional structure that might be used to argue that private information will be fully revealed even if preferences are not separable. To illustrate the problems and possibilities of this approach, consider the case where there are two types on each side of the market, $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$.

To simplify, the types are ranked in order of attractiveness to the other side of the market. Types s_2 and t_2 are the 'good' types and s_1 and t_1 are the 'bad' types. This means that, for any contract θ, and for any seller type s

$$u(\theta, s, t_1) < u(\theta, s, t_2),$$

and for any buyer type t

$$v(\theta, s_1, t) < v(\theta, s_2, t).$$

Suppose that f is a stable allocation and let θ_0 be a traded contract. Let S_0 and T_0 denote the types of sellers and buyers, respectively, for whom θ_0 is an optimal choice. If S_0 and T_0 are singletons, there is nothing to prove; if one of them is a singleton, the argument used in Section 3 will suffice to establish separation. So, without loss of generality, we can assume that $S_0 = \{s_1, s_2\}$ and $T_0 = \{t_1, t_2\}$.

Suppose then, contrary to what we want to prove, that $f(\theta_0, s_1) > 0$ and $f(\theta_0, t_1) > 0$. Let θ_1 be an arbitrary contract that is very close to θ_0. Consider a perturbation g such that $g(\theta_1, s_2) = g(\theta_1, t_2) > 0$ and $g(\theta_1, s_1) = g(\theta_1, t_1) = 0$. Let (f, μ) be the equilibrium satisfying (1-2) relative to the perturbation g.

Markets are orderly, so at most one side of the market is rationed. Suppose that the buyers are constrained in the market for the contract θ_1 (the other case is exactly symmetrical). Then the sellers are unconstrained and their probability of trade is $\sum_t \mu(\theta_1, t) = 1$.

For any θ_1 sufficiently close to θ_0, the continuity of the payoff functions and the assumption that type t_2 is better (more desirable) than type t_1 implies that

$$\begin{aligned} u(\theta_1, s_2, t_2) &> \sum_t \mu(\theta_0, t) u(\theta_0, s_2, t) \\ &= u^*(s_2). \end{aligned}$$

So the optimality conditions require that $\mu(\theta_1, t_2) < 1$ and $\mu(\theta_1, t_1) > 0$. Given the construction of the perturbation g and the stability condition (2) this can only be true if

$$v^*(t_1) = \sum_s \mu(\theta_1, s) v(\theta_1, s, t_1). \quad (5)$$

There are two cases to consider.

Case 1. First, suppose that θ_1 is not optimal for sellers of type s_1. Then (1) and the construction of g imply that $\mu(\theta_1, s_1) = 0$. Then the optimality condition (5) reduces to

$$\sum_s \mu(\theta_0, s) v(\theta_0, s, t_1) = \mu(\theta_1, s_2) v(\theta_1, s_2, t_1).$$

and the optimality condition for buyers of type t_2 can be written

$$\sum_s \mu(\theta_0, s)v(\theta_0, s, t_2) \geq \mu(\theta_1, s_2)v(\theta_1, s_2, t_2).$$

Dividing these conditions by $v(\theta_1, s_2, t_1)$ and $v(\theta_1, s_2, t_2)$ respectively and letting θ_1 converge to θ_0 gives

$$\mu(\theta_0, s_1)\frac{v(\theta_0, s_1, t_1)}{v(\theta_0, s_2, t_1)} + \mu(\theta_0, s_2) = m(s_2)$$

and

$$\mu(\theta_0, s_1)\frac{v(\theta_0, s_1, t_2)}{v(\theta_0, s_2, t_2)} + \mu(\theta_0, s_2) \geq m(s_2),$$

where $m(s)$ denotes the limiting value of $\mu(\theta_1, s)$ as $\theta_1 \to \theta_0$. These conditions are mutually inconsistent if the relative variation in payoffs for type t_1 is greater than for type t_2:

$$\frac{v(\theta, s_1, t_2)}{v(\theta, s_2, t_2)} > \frac{v(\theta, s_1, t_1)}{v(\theta, s_2, t_1)}, \forall\theta. \tag{6}$$

Thus, condition (6) is sufficient to rule out pooling in this case.

Case 2. The second case assumes that θ_1 is optimal for sellers of type s_1. Then

$$\sum_t \mu(\theta_1, t)u(\theta_1, s_1, t) = \sum_t \mu(\theta_0, t)u(\theta_0, s_1, t). \tag{7}$$

We need to show that condition (7) implies that sellers of type s_2 will strictly prefer the contract θ_1. The following two conditions will be shown to be sufficient. The first condition is that type s_2 is more sensitive than type s_1 to the type of buyer he is matched with:

$$u(\theta, s_2, t_2) - u(\theta, s_2, t_1) > u(\theta, s_1, t_2) - u(\theta, s_1, t_1) \tag{8}$$

for any contract θ. The second condition is that type s_1 has a stronger relative preference for θ_0 over θ_1 than does type s_2:

$$0 < u(\theta_0, s_1, t) - u(\theta_1, s_1, t) > u(\theta_0, s_2, t) - u(\theta_1, s_2, t), \forall t. \tag{9}$$

The condition (7) can be rewritten as

$$\sum_t \left[\mu(\theta_1, t) - \mu(\theta_0, t)\right] u(\theta_1, s_1, t) = \sum_t \mu(\theta_0, t) \left[u(\theta_0, s_1, t) - u(\theta_1, s_1, t)\right]. \tag{10}$$

The right hand side of (10) is positive (condition (9) says that s_1 prefers θ_0 to θ_1) so the left hand side must be positive and this implies that the probability distribution $\mu(\theta_1, t)$ puts more weight on type t_2 relative to $\mu(\theta_0, t)$:

$$\mu(\theta_1, t_2) - \mu(\theta_0, t_2) > 0 > \mu(\theta_1, t_1) - \mu(\theta_0, t_1).$$

Then condition (8) implies that

$$\sum_t [\mu(\theta_1, t) - \mu(\theta_0, t)]\, u(\theta_1, s_2, t) > \sum_t [\mu(\theta_1, t) - \mu(\theta_0, t)]\, u(\theta_1, s_1, t)$$

and condition (9) implies that

$$\sum_t \mu(\theta_0, t)\, [u(\theta_0, s_1, t) - u(\theta_1, s_1, t)] > \sum_t \mu(\theta_0, t)\, [u(\theta_0, s_2, t) - u(\theta_1, s_2, t)]\,.$$

Combining these conditions with (10) gives

$$\begin{aligned}\sum_t [\mu(\theta_1, t) - \mu(\theta_0, t)]\, u(\theta_1, s_2, t) &> \sum_t [\mu(\theta_1, t) - \mu(\theta_0, t)]\, u(\theta_1, s_1, t) \\ &= \sum_t \mu(\theta_0, t)\, [u(\theta_0, s_1, t) - u(\theta_1, s_1, t)] \\ &> \sum_t \mu(\theta_0, t)\, [u(\theta_0, s_2, t) - u(\theta_1, s_2, t)]\,,\end{aligned}$$

which proves the θ_1 is strictly preferred to θ_0 by type s_2, contradicting the optimality conditions.

A symmetric argument applies for the case in which the sellers are constrained in the market for θ_1, but note that in order to deal with both cases we need to find a single contract θ_1 that satisfies the condition (6) for the buyers and the analogue for sellers, and that satisfies conditions (8) and (9) for sellers and their analogues for the buyers. These are not innocuous conditions but they are not extremely restrictive either. They are, however, much more restrictive than the conditions required in Section 3.

Theorem 6 *Suppose that there are two types of buyers and sellers,* $S = \{s_1, s_2\}$ *and* $T = \{t_1, t_2\}$*, and that the types can be ranked as follows:*

$$u(\theta, s, t_1) < u(\theta, s, t_2), \forall s,$$

$$v(\theta, s_1, t) < v(\theta, s_2, t), \forall t.$$

Let f be a stable allocation and θ_0 a contract belonging to the interior of Θ that is traded in equilibrium. Assume that for any contract θ

$$\begin{aligned} \frac{v(\theta, s_1, t_2)}{v(\theta, s_2, t_2)} &> \frac{v(\theta, s_1, t_1)}{v(\theta, s_2, t_1)}, \\ \frac{u(\theta, s_2, t_1)}{u(\theta, s_2, t_2)} &> \frac{u(\theta, s_1, t_1)}{u(\theta, s_1, t_2)}, \end{aligned}$$

and that

$$\begin{aligned} u(\theta, s_2, t_2) - u(\theta, s_2, t_1) &> u(\theta, s_1, t_2) - u(\theta, s_1, t_1) \\ v(\theta, s_2, t_2) - v(\theta, s_1, t_2) &> v(\theta, s_2, t_1) - v(\theta, s_1, t_1). \end{aligned}$$

For any $\varepsilon > 0$ suppose that there is a contract θ_1 that is ε-close to θ_0 and that

$$\begin{aligned} 0 &< u(\theta_0, s_1, t) - u(\theta_1, s_1, t) > u(\theta_0, s_2, t) - u(\theta_1, s_2, t), \forall t, \\ 0 &< v(\theta_0, s, t_1) - v(\theta_1, s, t_1) > v(\theta_0, s, t_2) - v(\theta_1, s, t_2), \forall s. \end{aligned}$$

Then if θ_0 is optimal for types s_2 and t_2, it is not traded by types s_1 and t_1, that is, $f(\theta_0, s_1) = 0 = f(\theta_0, t_1)$.

4.4 Discussion

For the case of OSU, Theorem 2 shows that conditions on preferences over contracts ($S1$-$S2$) are sufficient for separation of types in a stable allocation. With the assumption of separability, it is possible to extend this result to the case of TSU. Theorem 4 shows that separation of types in a stable allocation is implied by the conditions ($S3$-$S4$), which only refer to agents' preferences over contracts. Furthermore, these conditions on preferences over contracts can be interpreted as requiring that the contract space be "sufficiently rich".

Separability is a natural condition to impose if we want to extend the separation argument from OSU to TSU. At the same time it is restrictive and so it would be nice to relax it. The 2×2 example suggests that sufficient conditions for full revelation of private information will have to be much stronger under TSU without separability than under OSU or under TSU with separability. Essentially, we have seen that with OSU a stable allocation will be fully revealing as long as the contract space is sufficiently rich. With TSU, a lot more structure has to be placed on the preferences of the agents to ensure that a stable allocation is separating. In particular, we need to put restrictions on the *intensity* of agents' preferences over the different types of agents on the other side of the market.

The conditions obtained so far are sufficient but not necessary. However, the restrictiveness of the sufficient conditions does suggest that full revelation may be less likely under TSU without separability than in the other cases.

5 Proofs

Proof of Theorem 2

Let θ_1 be a contract arbitrarily close to θ_0 that satisfies conditions $(S1)$ and $(S2)$. Let g denote a perturbation such that for any choice of θ_1 close to θ_0

$$g(\theta_1, s) = \begin{cases} \delta > 0 & s = s_0 \\ 0 & s \neq s_0 \end{cases}.$$

From the definition of a stable allocation, there exists an equilibrium (f, μ) whose probability assessment satisfies conditions (1-2).

I claim that $\mu(\theta_1, s) = 0$ for any $s \neq s_0$. To prove this, we start by noting that the equilibrium condition for s_0 implies that

$$\mu(\theta_1, t_0)u(\theta_1, s_0) \leq \mu(\theta_0, t_0)u(\theta_0, s_0). \tag{11}$$

From conditions ($S1$-$S2$), we know that $u(\theta_1, s_0) > u(\theta_0, s_0) > 0$, so inequality (11) implies that

$$\mu(\theta_1, t_0) < \mu(\theta_0, t_0). \tag{12}$$

This fact is enough to show that θ_1 is not optimal for any $s \notin S_0$. To see this, we consider two cases. If $\mu(\theta_1, t_0)u(\theta_1, s) < 0$, the conclusion is obvious so suppose that $0 \leq \mu(\theta_1, t_0)u(\theta_1, s) \leq \mu(\theta_0, t_0)u(\theta_1, s)$. If θ_0 is not optimal for s, then either some other contract is strictly preferred or no trade is strictly preferred:

$$\mu(\theta_0, t_0)u(\theta_0, s) - c(s) < \max\{u^*(s) - c(s), 0\}. \tag{13}$$

Then (12) and (13) imply that

$$\mu(\theta_1, t_0)u(\theta_1, s) - c(s) < \max\{u^*(s) - c(s), 0\}$$

because $u(\theta_1, s)$ is approximately equal to $u(\theta_0, s)$ for θ_1 arbitrarily close to θ_0. In other words, θ_1 is not an optimal choice for $s \notin S_0$, as claimed.

Now consider the types $s \in S_0, s \neq s_0$. To show that θ_1 is not an optimal choice for s there are three cases that need to be considered. If $\mu(\theta_1, t_0) = 0$ there is nothing to prove since $u^*(s) > 0$. If $\mu(\theta_1, t_0) > 0$ and $u(\theta_1, s) < 0$ then again there is nothing to prove since $\mu(\theta_1, t_0)u(\theta_1, s) < 0$ violates individual rationality. So we are left with the case where $\mu(\theta_1, t_0) > 0$ and $u(\theta_1, s) \geq 0$. Then $\mu(\theta_1, t_0) < \mu(\theta_0, t_0)$ and $u(\theta_1, s) < u(\theta_0, s)$ imply $\mu(\theta_1, t_0)u(\theta_1, s) < \mu(\theta_0, t_0)u(\theta_0, s)$ as required.

We have shown that θ_1 is not an optimal choice for any $s \neq s_0$ if θ_1 is chosen close enough to θ_0. Then $(S1)$ and the definition of g imply that $\mu(\theta_1, s) = 0$ for every $s \neq s_0$, as claimed.

Orderliness and $\mu(\theta_1, t_0) < 1$ imply that buyers are not rationed at θ_1, that is, $\mu(\theta_1, s_0) = \sum_s \mu(\theta_1, s) = 1$. The optimality condition for type t_0 requires

$$\begin{aligned} \mu(\theta_1, s_0)v(\theta_1, s_0) &= \sum_s \mu(\theta_1, s)v(\theta_1, s) \\ &\leq \sum_s \mu(\theta_0, s)v(\theta_0, s). \end{aligned}$$

For θ_1 sufficiently close to θ_0, we have $v(\theta_1, s_0) \approx v(\theta_0, s_0) > v(\theta_0, s)$ for every $s \in S_0, s \neq s_0$, so the equilibrium condition will only be satisfied if $\mu(\theta_0, s) = 0$ for every $s \in S_0, s \neq s_0$. Since θ_0 is traded in equilibrium, this implies that $f(\theta_0, s) = 0$ for all $s \neq s_0$. ■

Proof of Theorem 4

Let θ_1 be a contract arbitrarily close to θ_0 that satisfies $(S3\text{-}S4)$. We want to show that θ_1 is not optimal for any type s such that $d(\theta_0, s) < d(\theta_0, s_0)$. The optimality condition

$$\sum_t \mu(\theta_1, t)u(\theta_1, s_0, t) \leq \sum_t \mu(\theta_0, t)u(\theta_0, s_0, t)$$

can be rewritten as

$$\frac{a(\theta_{01}, s_0)}{a(\theta_1, s_0)} \sum_t \mu(\theta_0, t)b(\theta_1, t) \leq \sum_t \mu(\theta_1, t)b(\theta_1, t). \tag{14}$$

If $\sum_t \mu(\theta_1, t)b(\theta_1, t) = 0$ then θ_1 is clearly not optimal for any type s so without loss of generality we can assume that $\sum_t \mu(\theta_1, t)b(\theta_1, t) > 0$. Then

(14) and the separation condition imply that

$$\frac{a(\theta_0, s)}{a(\theta_1, s)} \sum_t \mu(\theta_0, t) b(\theta_1, t) < \sum_t \mu(\theta_1, t) b(\theta_1, t),$$

for s such that $d(\theta_0, s) < d(\theta_0, s_0)$ and this last inequality can be reorganized to give

$$\sum_t \mu(\theta_1, t) u(\theta_1, s, t) < \sum_t \mu(\theta_0, t) u(\theta_0, s, t)$$

for s such that $d(\theta_0, s) < d(\theta_0, s_0)$. This proves that θ_1 is not optimal for $s < s_0$.

A similar argument applies to the other side of the market.

To complete the proof, consider a perturbation g such that

$$g(\theta_1, k) = \begin{cases} \delta > 0 & k = s_0, t_0 \\ 0 & k \neq s_0, t_0 \end{cases} .$$

Since f is stable there exists a consistent probability assessment μ satisfying the conditions (1-2), which tells us that $\mu(\theta_1, s) = 0$ if $d(\theta_0, s) < d(\theta_0, s_0)$ and $\mu(\theta_1, t) = 0$ if $b(\theta_0, t) < b(\theta_0, t_0)$. The orderly markets condition requires that either $\sum_t \mu(\theta_1, t) = 1$ or $\sum_s \mu(\theta_1, s) = 1$. Then for some θ_1 sufficiently close to θ_0, some types will strictly prefer θ_1 to θ_0, contradicting the equilibrium conditions, unless $f(\theta_0, s) = 0$ for $s \neq s_0$ and $f(\theta_0, t) = 0$ for $t \neq t_0$. This proves the desired result. ■

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