Liquidity, Interest, and Asset Prices*

Douglas Gale
Department of Economics
New York University
269 Mercer Street
New York, NY 10003
USA
douglas.gale@nyu.edu

January 1, 2005

Abstract

A stylized theory of money and central banking is added to a model of competitive equilibrium in asset markets to explain the determination of the general level of asset prices and interest rates. The cash-in-advance constraint provides a transactions demand for money, but this is not sufficient to guarantee the determinacy of the price level if liquidity is costless or the price level is uncertain. The central bank plays a crucial role in determining interest rates and asset prices both as a supplier of liquidity and through its operation in the goods and asset markets. An extension of the model to allow for segregated markets can be used to show the impact of monetary policy on *real* asset prices and on asset-price volatility. The application of these ideas to financial intermediation, financial crises and bubbles is briefly discussed.

1 Introduction

Recent events, including the financial crises in South Asia and elsewhere and the expansion (and later collapse) of the stock market bubble in the United States have focused attention on the relationship between monetary policy and the financial system. Because of concerns

*This paper was presented as the Fisher Schultz Lecture at the European Meetings of the Econometric Society held on San Giorgio Maggiore in Venice in August 2002. Later versions were presented at conferences at SITE and the LSE. I am indebted to Franklin Allen for his encouragement and extensive discussions throughout the period this paper was being written and to conference participants for useful comments. The financial support of the C.V. Starr Center for Applied Economics and NYU and the National Science Foundation is gratefully acknowledged.

about financial fragility, any turbulence in the financial system inevitably raises questions about what can and should be done by policy makers. Should the Federal Reserve System target stock prices as well as the prices of goods and services in formulating its monetary policy? Can interest rates control the stock market? What impact does the wealth effect have on real consumption? What is the optimal response to financial crises? Can timely provision of liquidity prevent crises? Should crises be prevented at all costs or is a cost-benefit calculus required before intervention is warranted?

Among many other attempts to provide answers to these questions, Allen and Gale (1994, 1998, 2000a,b,c, 2003a,b) investigate the interrelationships between liquidity, asset markets, and financial crises. The models in these papers, like the rest of the literature, are limited in two respects. First, they are essentially real (non-monetary) models. Secondly, they focus on banks and banking, to the exclusion of other parts of the financial system. The present paper contains some strategies for incorporating money and monetary policy in models of the financial system that contain financial markets as well as financial intermediaries. Introducing money into a model of asset markets is a first step towards understanding the role of money and monetary policies in financial crises. The essential idea is that the monetary authorities, by providing liquidity to the traders in asset markets, have a crucial effect on the determination of the level and volatility of asset prices.

The term liquidity is used in various ways throughout the economics literature. Here it refers to an agent's ability to obtain the means of payment ("cash") needed to carry out trades in the asset market. In an Arrow-Debreu model, all assets are traded simultaneously on a centralized exchange and all assets are accepted as means of payment. When markets are incomplete, only a subset of assets can be traded in a particular market at a particular time and relatively few assets are acceptable as a means of payment. A trader's ability to purchase assets is limited not just by his wealth but also by its liquidity. The fact that his some of his wealth is in an illiquid form which cannot immediately be converted into the medium of exchange is an additional constraint on his ability to trade and this fact may reduce the market's ability to absorb unexpected shocks.

Allen and Gale (1994) show that when markets are incomplete the amount of liquidity ("cash") in the market has a crucial role to play in determining the volume of transactions and the prices at which transactions occur. Since theirs is a "real" model, liquidity is represented by the stock of goods available for immediate delivery. The alternative to holding goods as a stock of liquid assets is to invest them in profitable, long-term projects earning a higher rate of return. Because of this opportunity cost of holding goods in liquid form, the amount of liquidity in the system is too small to eliminate fluctuations in asset prices when there is a sudden surge of selling. In a monetary model, by contrast, liquidity consists of access to fiat money. The fact that the supply of liquidity is measured in terms of fiat money makes a significant difference in terms of how the supply of liquidity is determined and how it impacts asset prices. Fiat money is created by the central bank (CB) at virtually zero cost. In other words, it is more or less a free good. Moreover, since the price level is determined endogenously, the real money supply is also endogenous. A sufficiently steep fall in the price level will increase the real quantity of money to arbitrarily large levels.[1] So, in a monetary economy, the question arises of how can there ever be a shortage of liquidity. How can liquidity shortages prevent the existence of stable and efficient asset prices? In the sequel, we show that it is the cost of liquidity in the form of the interest rate on short-term borrowing that matters for the determination of asset prices. If liquidity is costless, fluctuations in nominal asset prices are innocuous (in the absence of debts and other non-state-contingent commitments denominated in terms of money). Even if the money supply is fixed and exogenous, the price level can adjust to provide whatever level of "real" liquidity is needed.

In Section 2, we introduce a simple model of exchange in which a cash-in-advance constraint (Clower, 1968) provides a transactions demand for money. Before we can answer questions about how the CB's control of money supply and interest rates affects asset prices, we first need to understand how the general level of asset prices is determined in a monetary economy. This question is of some independent interest because, as Allen and Gale (1998) have shown, the control of the price level can exercise an importance influence on the level of real debt and the incidence of financial crises. At first glance, the model introduced in Section 2 is reminiscent of the classical monetary theory in which relative prices are determined by a Walrasian system of market-clearing conditions and the general price level is determined by the excess demand for money. If p is a vector of nominal prices of goods and assets and $Z(p)$ is the vector of aggregate excess demands for goods and assets, the market-clearing condition for goods and assets is

$$Z(p) = 0. \tag{1}$$

The excess demand function is homogeneous of degree zero, so these equations only determine the price vector p up to a scalar multiple. To determine the absolute prices, we need to appeal to the condition that money demand equals money supply,

$$M^d(p) = \bar{M}. \tag{2}$$

In this special case, the "classical dichotomy" holds: the relative prices of goods and (real) assets are determined by real demands and supplies independently of the price level and the price level is determined by the excess demand for money. In general, equilibrium prices are determined by the entire system of simultaneous equations of supply of and demand for goods, assets, and money. In what follows it is often convenient to adopt the language of the classical dichotomy and refer to the analogue of equation (2) as determining the price level, with the unspoken qualification "together with the other equilibrium conditions." In any case, we shall see that even in a monetary economy with a complete set of demand and supply functions, the determinateness of the price level is not guaranteed.

To see how the cash-in-advance constraint leads to an equilibrium condition analogous to (2), suppose that for every every trader in the economy, the cash-in-advance constraint is strictly binding. Then we can interpret the cash-in-advance constraint as a demand-for-money equation, since it gives the required money balances as a function of prices and excess

[1] Note that we are interested here in real asset prices, that is, the price of assets in terms of goods, so a change in the absolute prices of goods and assets together does not imply real asset-price volatility.

demands. Adding up the individual demands for money and equating the aggregate demand to the aggregate supply provides an equilibrium condition that looks very much like condition for monetary equilibrium (2) from classical monetary theory. The demand for an individual agent i can be written

$$m_i^d(p) = p \cdot z_i(p)^+$$

where $m_i^d(p)$ is i's demand for money, $z_i(p)$ is his excess demand, and $z_i(p)^+$ is the vector of positive excess demands. The market-clearing condition says that the money supply $\bar{M}$ is equal to the aggregate demand for money $M^d(p) = \sum_i m_i^d(p)$ so we get the analogue of (2):

$$\sum_{i=1}^{I} p \cdot z_i(p)^+ = M^d(p) = \bar{M} \tag{3}$$

This equation, together with the conditions (1), allows us to determine the price level. However, the equilibrium conditions are sufficient to determine the price level only if we make an implicit assumption that (a) the cash-in-advance constraints are binding and (b) the price level is known with certainty at the moment individuals choose how much money to hold. If either of these conditions is violated, the price level is once again indeterminate.

If it is costless to obtain liquidity, individuals will be willing temporarily to hold cash balances in excess of their needs. If $m_i^d(p) > p \cdot z_i(p)^+$ for one or more agents i, the money market condition (3) will be replaced by a strict inequality

$$\sum_{i=1}^{I} p \cdot z_i(p)^+ < M^d(p) = \bar{M}.$$

The market-clearing conditions (1) by themselves are not enough to determine the price level.

To ensure that the cash-in-advance constraint is always binding, we introduce a cost of liquidity: individuals are assumed to obtain money balances from the CB, for which they must pay the CB a positive rate of interest. If liquidity is costly, individuals will economize on money balances held for transactions purposes and hold the minimum amount consistent with their planned transactions. Once again, (3) holds as an equation.

As long as the price level is certain, the conditions (1) and (3) are sufficient to determine the price level. But what if prices are uncertain? Suppose that the equilibrium price vector $p(s)$ is a function of the state of nature s and money balances are chosen before s is realized. The cash-in-advance constraint for agent i takes the form $m_i \geq p(s) \cdot z_i(p(s))^+$ in every state s and, if liquidity is costly, it will hold with equality in at least one state s. But there is no reason to believe that it will hold for all agents i in any state s, let alone for all i in all states s. Thus, even costly liquidity will not ensure that the counterpart of (3) holds in every state. This leaves us with the problem that adding a single equation to the system is not sufficient to determine the price level if the price level is a random variable. The determination of the price level under uncertainty is discussed in Section 3, where we show that the CB has another instrument it can use to control the price level. As long as the interest rate is positive (i.e., liquidity is costly), the CB earns seigniorage which it must spend in equilibrium. The CB's budget constraints in each state provide us with additional equilibrium conditions, one for each possible price level. If the CB can commit to its demand for assets and supply of fiat money in each state, we have enough equations to determine the remaining equilibrium variables, including the value of money (the inverse of the price level).

The objective of Sections 2 and 3 is to clear away a number of purely technical or theoretical issues before we can get down to the more practical issue of how liquidity affects asset prices. Section 4 is concerned with the determinants of asset prices relative to goods prices when the central bank targets the level of goods prices. Fluctuations in the general level of asset prices clearly have important implications for the real economy whenever contracts or securities are denominated in terms of money. Inflation can reduce the real value of debts, and thus reduce the chance of default or a financial crisis, and conversely deflation will have opposite effects. Here we focus on one aspect of this question, how the CB's control of liquidity influences the determination of asset prices. In Section 4 we extend the model to allow for segregated markets for goods and assets and proceed to analyze the determinants of the relative prices of assets and goods. In particular, we analyze the impact of costly liquidity on the pricing and trading of assets.

Open market operations that involve the purchase and sale of assets can obviously influence relative prices if the CB is a large trader, so we rule out open market operations and assume that seigniorage is collected in the form of goods, not assets. The impact of CB policy is felt indirectly, if at all. Nonetheless, we can show that asset prices are influenced by the supply of liquidity, even if the CB does not participate directly in asset markets. The reason for the non-neutrality of money is that additional liquidity is injected into the system through the goods market, rather than the asset markets. It is crucial for this effect that the trading of assets and the trading of goods are segregated on different markets. This structure implies distinct cash-in-advance constraints and budget constraints for the asset market and goods market. Liquidity shocks (changes in the demand for liquidity) in the asset market, in the presence of a fixed amount of liquidity, will change (nominal) asset prices relative to (nominal) goods prices. In fact, the CB's attempt to maintain a constant (goods) price level will exacerbate the real volatility of asset prices. Thus, costly liquidity by itself may lead to asset-price volatility, even while the CB targets the (goods) price level.

In Section 5 we return to some of the issues with which we began and discuss the way in which these ideas can be used to understand the influence of monetary policy on financial markets, financial intermediation, crises and bubbles.

Some simple proofs are gathered in Section 6.

2 Money and the price level

In modern payments systems, access to liquidity is (virtually) free within the trading day, whereas obtaining liquidity overnight involves significant costs. Within the trading day, banks are able to run very large "daylight overdrafts" that enable them to bridge the gap between payments and receipts at minimal cost, whereas payments that are not covered at the end of the day must be balanced by overnight borrowing on the interbank market (Coleman, 2002; Martin, 2002; Zhou, 2000). In general, the liquidity costs will depend on the type of transactions involved. For example, spot transactions generate a very short-term demand for liquidity. A typical example would be rebalancing a portfolio, selling some securities in order to re-invest the proceeds in another class of securities. This involves a series of spot transactions in which payment and delivery are closely matched. The cost of liquidity for these kinds of transactions may be very low or negligible. On the other hand, the precautionary or speculative demand for liquidity has a longer duration. A typical example would be a "carry trade" in which the trader takes advantage of the difference between short- and long-term interest rates by financing the purchase of long-term bonds with short-term debt. Arbitrage transactions of this sort may involve holding or shorting assets over a long and perhaps uncertain period of time. The carrying costs of this kind of transaction may be significant. In this section, we focus on the short-term, transactions demand for liquidity. Although the liquidity cost is assumed to be small, it still plays an important role in the determination of the price level. Later, in Section 4, we will consider arbitrage transactions in asset markets, where the costs of financing a complete series of transactions may be higher.

Pigou (1943) observed that a change in the price level has real effects if securities and contracts are denominated in terms of money. While the effect of money and the price level has been thoroughly studied in macroeconomics, less attention has been paid to the impact of money in financial economics, which tends to rely on "real" models (cf., Duffie, 2001). A prominent exception, which illustrates the importance of nominal prices in a financial context, is the model of general equilibrium with financial securities introduced by Cass (1984) and Werner (1985). Financial securities are claims that promise payment in an abstract unit of account contingent on the state of nature. Any change in the future price level changes the "real" returns of these securities. The Cass-Werner model in its original form does not explain how the price level is determined. The price level is, in fact, indeterminate. Moreover, the indeterminacy is not of the innocent kind found in classical models where only relative prices matter. Here, the indeterminacy of the price level may cause real indeterminacy, that is, a change in the price level is associated with a change in the equilibrium allocation of commodities.

The indeterminacy of the price level in the Cass-Werner model results from the fact that financial securities are defined as claims on an abstract unit of account rather than as claims on a real commodity, as in Genakoplos and Polemarchakis (1986), or on fiat money that also serves as a medium of exchange, as in Magill and Quinzii (1992). To resolve the indeterminacy of the Cass-Werner model, one could introduce financial securities that are claims on an object that actually exists and is priced in markets. We do this by introducing fiat money that is held for transactions purposes. Then the corresponding market-clearing condition (demand for money equals supply of money) provides an additional equation which together with the other equilibrium conditions determines the equilibrium price level. This is the approach adopted by Magill and Quinzi (1992), and Geanakoplos and Dubey (1992), who introduce a transactions demand for money based on the cash-in-advance constraint (Clower, 1968; Lucas, 1990; Fuerst, 1992).

Unfortunately, introducing fiat money with a well-defined transactions demand is not sufficient to guarantee a determinate (locally unique) price level. If there is no cost of holding excess money balances, the quantity of money no longer determines the price level, since any quantity of money can be absorbed by the system without affecting the demand for and supply of goods. In a related vein, Magill and Quinzii point out that the price level in their model is determinate if money is held as medium of exchange in equilibrium, but not if it is held as a store of value. A central contribution of the present paper is the observation that, in order to use the demand for and supply of money to determine the price level, liquidity must be costly.

In the next section we present a simple model of **costly** liquidity. Agents can obtain as much liquidity as they want from the central bank, but they must pay interest on their borrowing. Because liquidity is costly, the cash-in-advance constraint is always binding. In the first model we consider, the binding cash-in-advance constraint is sufficient to determine the price level. Subsequently, we shall see that even this may not be enough.

2.1 Transactions demand for money

Clower (1968) enunciated the axiom that in actual markets, unlike classical general equilibrium theory, "money buys goods and goods buy money". To capture this realistic feature of exchange, he added the now-famous cash-in-advance constraint to a model of pure exchange. We plan to follow Clower and then go further by adding a central bank (CB) and a cost of liquidity.

A *pure exchange economy* consists of I agents, indexed by $i = 1, ..., I$. The agents trade ℓ commodities and fiat money. Each agent i is characterized by a consumption set $\mathbf{R}^\ell_+$, an endowment $e_i \in \mathbf{R}^\ell_+$, and a utility function $u_i : \mathbf{R}^\ell_+ \to \mathbf{R}$. One interpretation of the pure exchange economy is that the ℓ commodities are a mixture of goods and assets. For example, suppose there are two dates and a single consumption good at each date. Each agent i has a von Neumann-Morgenstern utility function $U_i(c_0, c_1)$, where c_t is consumption of the good at date $t = 0, 1$. There are K assets $k = 1, ..., K$. The return to one unit of asset k is represented by the random variable $\tilde{R}_k$, where $\tilde{R}_k$ is the number of units of the consumption good at date 1 produced by one unit of the asset. Then the indirect utility function of agent i is defined by

$$u_i(x_i) = E\left[U_i\left(x_{i0}, \sum_{k=1}^{K} x_{ik}\tilde{R}_k\right)\right],$$

where x_{i0} is demand for consumption and x_{ik} is the demand for asset k at date 0. This economy is isomorphic to a pure exchange economy with $\ell = K + 1$ commodities, namely, the single consumption good at date 0 plus the K assets.

Exchange, in an Arrow-Debreu economy, is assumed to take place at a single instant or date, but in order to model the process of monetary exchange more precisely, time is here divided into three sub-periods. This intertemporal structure takes on added significance when we allow for uncertainty in Section 3. Each sub-period represents a different stage in the trading process.

- In the first sub-period, agents borrow money from the CB.
- In the second sub-period, agents exchange goods (and assets) for money and money for goods (and assets).
- In the final sub-period, agents repay their loans to the CB with interest.

The CB provides liquidity to the market so agents can carry out transactions. The CB earns interest on the loans it makes in the first sub-period and wants to spend this income on goods and assets in the second sub-period. Because the interest payments are not received until the third sub-period, after the goods and assets are traded, the CB injects additional money into the economy in the second period to pay for its purchases. These injections allow the agents to repay their loans with interest in the final period.

The interest charged by the CB for supplying liquidity generates income or seigniorage, which the CB spends on commodities (goods and assets). In some emerging markets, seigniorage may be an important source of income for the government and the impact of government demand on general equilibrium prices will be significant. In developed economies, on the other hand, seigniorage is relatively insignificant. Inflation and interest rates are low and the monetary base (M0) is a very small fraction of the broader measures of the money supply (M1, M2, etc.). The results presented in the sequel do not depend on the assumption of large liquidity costs—in fact, many of them refer to the limiting case in which the interest rate and seigniorage become vanishingly small—but they do depend on the existence of small but positive costs.

The liquidity costs represented by the CB's interest rate should be interpreted as a proxy for the residual frictions that exist in a sophisticated financial system. Participants in financial markets do not literally obtain cash balances from the CB, but there is a positive slope to the yield curve: more liquid assets yield lower returns than less liquid assets and the difference in returns is the opportunity cost of liquidity. A more realistic model would explicitly model the opportunity of investing in alternative real assets. Then the possibility of investing funds borrowed from the CB in these productive assets would imply a positive opportunity cost of holding idle balances. Similarly, the seigniorage that is generated by supplying liquidity does not necessarily accrue to the CB. It may accrue to other institutions in the financial system. For example, banks and other financial institutions can make money on the float that is provided by their ability to provide liquidity services to their customers. Modeling the complexities of a modern financial system is beyond the scope of this paper. The simplifying assumption that the CB provides liquidity services directly to the market participants will serve as a reduced form for present purposes.

Let $M > 0$ denote the amount of money supplied by the CB in the first sub-period and let $g \in \mathbf{R}^{\ell}_{+}$ and $\Delta M \geq 0$ denote, respectively, the bundle of goods and assets demanded and the injection of money supplied in the second sub-period. We assume that the CB's choice of M, g and ΔM is exogenous but has to satisfy a budget constraint. Alternative formulations of the CB's policy are discussed in Section 2.4.

Agents require money in order to buy goods and assets and so, in the first sub-period, each agent borrows an amount of money that will allow him to carry out the planned transactions in the second sub-period. The money he receives in exchange for selling goods and assets must exceed the amount he originally borrowed so that he can repay his loan with interest in the final period. Let $m_i \geq 0$ denote the money balance agent i borrows from the CB in the first sub-period and let $x_i \in \mathbf{R}^\ell_+$ denote the bundle of goods and assets demanded in the second sub-period. Let $p \in \mathbf{R}^\ell_+$ and $r \geq 0$ denote, respectively, the vector of money prices of goods and assets and the interest rate on loans from the CB. The cash-in-advance constraint requires that the value of purchases be less than or equal to the agent's cash balance, that is,

$$p \cdot (x_i - e_i)^+ \leq m_i, \tag{4}$$

where for any vector $z = (z_1, ..., z_\ell)$, the notation z^+ denotes the vector of non-negative excess demands defined by

$$z^+ = (\max\{z_1, 0\}, ..., \max\{z_\ell, 0\}).$$

After trade in the second sub-period, the amount of money agent i holds is equal to his initial balance m_i minus the value of his net trades $p \cdot (x_i - e_i)$. In order to repay his loan with interest he needs $(1+r)m_i$ units of money in the final sub-period. So agent i can repay his loan with interest if and only if his budget constraint

$$p \cdot (x_i - e_i) + rm_i \leq 0 \tag{5}$$

is satisfied. Agent i's behavior is characterized by a decision problem in which he chooses a money balance m_i and a commodity bundle x_i to maximize his utility subject to the budget constraint and the cash-in-advance constraint. Formally, he has to choose an ordered pair $(x_i, m_i) \in \mathbf{R}^\ell_+ \times \mathbf{R}_+$ to maximize $u_i(x_i)$ subject to the cash-in-advance constraint (4) and the budget constraint (5).

Because of the usual homogeneity properties, there is no essential loss of generality in assuming the money supply M is fixed in the sequel, so the CB's policy choice can be summarized by $(g, \Delta M)$.

An allocation is an array $(x, m) = ((x_i, m_i))$ such that $(x_i, m_i) \in \mathbf{R}^\ell_+ \times \mathbf{R}_+$ for each agent i. The allocation (x, m) is *attainable* if it satisfies the market-clearing conditions

$$\sum_{i=1}^{I} x_i + g = \sum_{i=1}^{I} e_i \tag{6}$$

and

$$\sum_{i=1}^{I} m_i = M. \tag{7}$$

The first condition (6) is the market-clearing condition for money in the first sub-period. The second condition (7) is the market-clearing condition for assets in the second sub-period.

We assume that the CB chooses its policy $(g, \Delta M)$ before the markets open and treat the policy $(g, \Delta M)$ as exogenous when defining an equilibrium. However, the CB has to anticipate the equilibrium to ensure that its budget constraint will be satisfied at the equilibrium prices and interest rate.

For a given a policy $(g, \Delta M)$, define an *equilibrium relative to the policy* $(g, \Delta M)$ to consist of an attainable allocation (x, m) and a vector of prices and an interest rate (p, r) such that, for every agent i, (x_i, m_i) solves

$$\begin{array}{ll} \max & u_i(x_i) \\ \text{s.t} & p \cdot (x_i - e_i)^+ \leq m_i, \\ & p \cdot (x_i - e_i) + rm_i \leq 0, \end{array}$$

and the CB's budget constraint

$$p \cdot g = \Delta M,$$

is satisfied. Note that attainability and the budget constraints imply that

$$rM = \Delta M$$

so the money supply in the final sub-period is just sufficient to repay the loans to the CB.

2.2 Price level determination

In the present model, we assume that the only cost of liquidity is the interest charged by the CB on loans. The positive interest rate $r > 0$ forces agents to economize on liquid balances and this in turn helps determine the price level. The importance of a positive interest rate for price determination can be illustrated by considering the limiting case of an economy in which the CB earns no seigniorage, $(g, \Delta M) = (0, 0)$, and consequently the interest rate $r = 0$.

To see that the price level is indeterminate, let $(x, m, p, 0)$ be an equilibrium and consider what happens if all prices are reduced in the same proportion, that is, replace the price vector p with λp for some $0 < \lambda < 1$. The homogeneity of demand ensures that assets markets continue to clear at the new price vector λp. The cash-in-advance constraints, which were satisfied before, are now strictly satisfied, but that too is consistent with equilibrium because liquidity is costless. Thus, the price level can be reduced without affecting equilibrium in an essential way.

Moreover, if the cash-in-advance constraints are slack at the initial equilibrium $(x, m, p, 0)$, the price level can be increased without disturbing the equilibrium conditions. For example, the equilibrium vector p can be replaced by λp for any λ sufficiently close to 1, and λp will be an equilibrium price vector as well. This proves the equilibrium price level is indeterminate.

Proposition 1 *Let (x, m, p, r) be an equilibrium for the policy $(g, \Delta M) = (0, 0)$ that raises no seigniorage. Then $r = 0$ and $(x, m, \lambda p, 0)$ is an equilibrium for the same policy, for any $\lambda > 0$ such that*

$$\lambda p \cdot (x_i - e_i)^+ \leq m_i, i = 1, ..., I.$$

Note that one cannot multiply prices by just any positive constant and this is another way in which the indeterminacy differs from the indeterminacy of nominal prices expressed in terms of some abstract unit of account. Nonetheless it is a violation of local uniqueness which is the usual meaning of "determinateness."

The argument preceding Proposition 1 makes it clear that the price level is indeterminate because the cash-in-advance constraints are not binding. To ensure that each agent's cash-in-advance constraint is binding, liquidity must be costly. If the interest rate r is positive, each agent will choose to hold the minimum money balance m_i that allows him to satisfy his cash-in-advance constraint. So a positive interest rate is sufficient for individual cash-in-advance constraints to hold as equations in equilibrium. Then summing the individual constraints yields an aggregate cash-in-advance constraint

$$\sum_{i=1}^{I} p \cdot (x_i - e_i)^+ = M. \tag{8}$$

We can think of equation (8) as a version of classical monetary theory's Quantity Equation. The additional equation, together with the other equilibrium conditions, allows us to determine the equilibrium price level.

As was pointed out earlier, in a modern payments system, the intraday interest cost of liquidity is vanishingly small, so it is interesting to consider what happens as the CB's seigniorage and the associated interest rate both converge to zero. The next result shows that, in the limit, the equilibrium corresponds to a classical competitive equilibrium. However, since the individual cash-in-advance constraints hold as equations for any positive interest rate, they must also hold in the limit.

Theorem 2 *Let (x^n, m^n, p^n, r^n) be a sequence of equilibria corresponding to a sequence of policies $(g^n, \Delta M^n)$. Suppose that $(x^n, m^n, p^n, r^n) \to (x^0, m^0, p^0, r^0)$ as $(g^n, \Delta M^n) \to (0, 0)$. For each i, assume that u_i is continuous and locally non-satiable and*

$$p^0 \cdot e_i > 0.$$

Then (x^0, m^0, p^0, r^0) is an equilibrium for the policy $(0, 0)$ and

$$p^0 \cdot (x_i^0 - e_i)^+ = m_i^0$$

for $i = 1, ..., I$. Furthermore, (p^0, x^0) is a Walrasian equilibrium.

Proof. See Section 6. ■

Thus, in contrast to the earlier examples of equilibria in the limit economy with $r = 0$, the limit of a sequence of equilibrium with $r > 0$ must satisfy the Quantity Equation, so we have an extra equation to determine the price level. The next result shows that if the cash-in-advance constraint is binding for each agent, then the equilibrium price level is determinate in the limit economy.

Theorem 3 DETERMINATENESS OF THE PRICE LEVEL. *Let (x, m, p, r) be an equilibrium for the policy $(g, \Delta M) = (0, 0)$ and suppose that for each agent $i = 1, ..., I$,*

$$p \cdot (x_i - e_i)^+ = m_i.$$

If the Walrasian equilibria of the exchange economy are locally unique, then the equilibrium (x, m, p, r) is locally unique among the set of equilibria with binding cash constraints.

Proof. See Section 6. ■

To sum up the story so far, if liquidity is costly, then individual cash-in-advance constraints are binding, the aggregate cash-in-advance constraint plays the role of the Quantity Equation in classical monetary theory, and the money supply 'determines' the price level.

2.3 The fiscal theory of the price level

The use of the CB's open market operations to determine the price level is reminiscent of the *fiscal theory of the price level* (FTPL). Some of the issues that have been raised in the FTLP may also apply to the model presented here so it is worth reviewing them here.

The basic idea behind the FTPL is that government fiscal policy determines the dynamic path of the price level. In its *weak form* (this terminology is borrowed from Carlstrom and Fuerst (2000) this simply refers to the influence of fiscal policy on monetary policy, as when a budget deficit leads to monetary expansion. Sargent and Wallace (1981) is an example of this kind of argument. This version of the FTPL is consistent with traditional monetary theory because an increase in the price level still occurs as a result of an increase in the money supply. In the *strong form*, however, fiscal policy has an effect on the price level even if the rate of expansion of the money supply is taken as given. This is because the indeterminacy of the price level in dynamic macroeconomic models allows fiscal policy to be used to select an equilibrium (Woodford (1995), Kocherlakota and Phelan (1999), Carlstrom and Fuerst (2000), Christiano and Fitzgerald (2000), Cochrane (2001)). Buiter (1998, 1999, 2002, 2004) has criticized this theory on the grounds that it treats the government's budget constraint as an equilibrium condition rather than a constraint that must be satisfied both in and out of equilibrium. According to this view, the FTPL takes as given the government's planned expenditures and tax revenues and assumes the price level will adjusts to ensure that in equilibrium the government's budget constraint is satisfied. But if the government is required to satisfy its budget constraint in all eventualities, a different price level might force the government to change its planned expenditures and taxes, thus leading to a different equilibrium price level.

In an attempt to answer Buiter's criticism, Bassetto (2002) has provided an example of an extensive-form game in which prices and quantities are determined by the actions of the players, including government, and shows that there exists a complete strategy for government that does determine the price level. The government's strategy in Bassetto's model is more complicated than the reduced form required by the theory of competitive equilibrium, but it is feasible both in and out of equilibrium.

The model developed above differs in a number of respects from the variety of models found in the FTLP literature. In the first place, we only consider very short-term influences and in the context of a finite-horizon model. Secondly, the counterpart of "fiscal policy" in the present model is the expenditure that is funded by seigniorage and that are necessary to clear markets if the interest rate is positive. Thirdly, there does not appear to be a counterpart to the slack cash-in-advance constraint in the FTLP. The indeterminacy of the price level that appears in the FTLP comes from the dynamic tradeoff between levels and rates of exchange. The FTLP does not even require fiat money: the theory works just as well with money as a numeraire or abstract unit of account, whereas in the current model there could be no meaningful CB policy without fiat money.

Despite these differences, some of the same issues arise. In particular, it may be asked whether Buiter's critique of the FTPL applies to the present model. Certainly, we have not followed Bassetto's approach and modeled the CB's interaction with the market as a complete, extensive-form game. To that extent, we have left open a number of questions: What are the appropriate choice variables for the CB? Is it appropriate to treat the choice of these variables as completely exogenous? In what sense can the CB control of these variables "determine" the price level? Some of these questions are treated in the next subsection.

What the controversy over the FTPL highlights is the difference between establishing the determinateness of equilibrium in a mathematical sense and explaining how the government controls the economy through the choice its policy choices. What the critics are complaining about is the assumption, implicit in all equilibrium theory, that one may safely ignore the question of how equilibrium comes about and simply study those states that satisfy equilibrium conditions. The assumption may seem more egregious in the case of the FTPL, where it requires us to believe that the economy's equilibrium path over the infinite future is being determined by current policy choices, but this is essentially the same assumption we make everyday whenever we use equilibrium theory to study policy.

2.4 Specifications of CB policy

There are four variables that represent potential policy instruments for the CB, the initial money supply M, the interest rate r, the open-market purchases g, and the monetary injection ΔM. Homogeneity allows us to normalize $M = 1$ without loss of generality, so the set of policy instruments is reduced to three, r, g, and ΔM. The CB's policy choices are further constrained by its budget constraint and by the equilibrium conditions. For this reason, we have represented the CB's policy by the ordered pair $(g, \Delta M)$ and treated the interest rate r as being determined endogenously "by the market". Even so, the pair $(g, \Delta M)$ has to satisfy the CB's budget constraint $p \cdot g = \Delta M$ at the equilibrium prices p. Alternatively, we could have treated g and r as choice variables and allowed ΔM to be determined by the budget constraint $p \cdot g = \Delta M$. As usual, the CB can choose either the money supply $(M, \Delta M)$ or the interest rate r, but not both. Equilibrium requires that $rM = \Delta M$ in any case, so the equivalence of the two policy specifications is clear.

Whichever policy instruments the CB chooses, we assume that the CB's policy is determined first and that the endogeneous equilibrium variables, the prices p (and interest rate r if money supply is the policy instrument) and the quantities $x = \{x_i\}$ are determined afterwards, taking the policy as given. If the equilibrium, including the price level, is locally unique for the given policy, we say that the the policy determines the price level. This does not, of course, address the question of how the equilibrium is achieved (a "disequilibrium" question that equilibrium theory by definition cannot answer). Moreover, it begs a subtle question about the proper specification of feasible policies, which in turn may affect the interpretation of the "determinateness" of equilibrium.

As Buiter has insisted in his critique of the FTLP, a budget constraint should hold in all situations and not just in equilibrium. In the current case, this means that the constraint $p \cdot g = \Delta M$ should hold for any price vector p and not just for the equilibrium price vector. (Actually, we are not assuming that the CB is a price-taker and so it may be unreasonable to require the budget constraint to hold for all price vectors, but let us assume it does for the sake of argument). Then the demand for goods and assets and the supply of money are functions of prices and interest rates and the budget constraint,

$$p \cdot g(p, r) \equiv \Delta M(p, r),$$

is identically satisfied for each ordered pair (p, r). In order to have an equilibrium, it is still necessary that

$$\Delta M(p, r) = rM$$

so putting $g(p, r)$ equal to a constant vector γ gives us exactly the same mathematical structure as before. It may be objected that the determinateness of the price level is being imposed by brute force: once it is assumed that the CB can dictate the level of seigniorage to be collected, everything else follows of necessity. We should think of this representation of CB policy as a reduced form of a more detailed theory in which the structure of the financial and banking systems and the nature of the CB's intervention in the market are more explicitly laid out, possibly as an extensive-form game (cf. Bassetto (2002)). In such a theory, it should be possible to identify the limits of the CB's control of the price level, if any. Of course, if the CB adopts an *accommodating* policy, with seigniorage accommodating changes in the price level, the indeterminacy would be reinstated.

2.5 Existence

Given a Walrasian equilibrium corresponding to the policy $(g, \Delta M) = (0, 0)$, existence of equilibrium corresponding to a policy $(g, \Delta M)$ in some neighborhood of $(0, 0)$ (i.e., in some neighborhood of a Walrasian equilibrium) follows from the implicit function theorem under the usual conditions plus the assumption that $x_{ih} \neq e_{ih}$ for every good h and every agent i.

Let (p^*, x^*) be a Walrasian equilibrium such that $p^* \gg 0$ and $x_i^* \gg 0$ for every i. Suppose further that $x_{ih}^* \neq e_{ih}$ for $h = 1, ..., \ell$ and every i. Let $H_i = \{h : x_{ih}^* > e_{ih}\}$. If u_i is strictly quasi-concave then, for any (p, r) in some neighborhood N_i of $(p^*, 0)$, there is a unique solution $f_i(p, r)$ to the problem of maximizing $u_i(x_i)$ subject to the budget constraint

$\pi_i(p,r) \cdot (x_i - e_i) \leq 0$, where

$$\pi_{ih}(p,r) \equiv \begin{cases} p_h & \text{if } h \notin H_i, \\ (1+r)p_h & \text{if } h \in H_i. \end{cases}$$

Assume that f_i is C^1 on N_i. Then consider the system of equations

$$\sum_{i=1}^{I} f_i(p,r) + g = \sum_{i=1}^{I} e_i.$$

For every g sufficiently small, there is at least one solution (p^g, r^g) to these equations, and we can clearly choose M^g and ΔM^g so that (p^g, r^g) is an equilibrium relative to $(g, M^g, \Delta M^g)$.

3 The price level under uncertainty

The analysis in the preceding section assumes that the price level is known with certainty. This may seem a purely theoretical concern, but it actually has important implications about the mechanism by which the price level is determined. In fact, it allows us to distinguish between alternative theories of price level determination.

If the price level is assumed to be known with certainty, a binding cash-in-advance constraint provides the additional equation we need, along with the other equilibrium conditions, to determine the price level. If, however, there is uncertainty about the price level, a positive cost of liquidity ensures that the cash-in-advance constraint is binding in at least one state, but the constraint may still be slack in other states. Obviously, the cash-in-advance constraint cannot help to determine the price level in states in which it is a strict inequality.

Fortunately, there is another set of equilibrium relations that does serve to determine the price level. The CB supplies money in exchange for goods and assets in each state. This allows the CB to determine that rate of exchange between money, on the one hand, and goods and assets on the other. This process is analogous to a gold-exchange system, in which the CB can fix the rate at which it will buy and sell gold in exchange for its currency or the open-market operations that allow the bank to determine the rate at which government bonds are exchanged for money (and hence determines the yield on bonds). Here we generalize this notion to allow the CB to fix the price level, but in practice it could fix the price level by fixing the price of any single good or asset. We can conclude that the (stochastic) price level is determinate, but the explanation depends on the active involvement of the CB and not on the cash-in-advance constraint (Quantity Equation). This is an important distinction, but the reader who is eager to get on to asset markets can skip this section and go straight to Section 4.

3.1 Extrinsic uncertainty

The argument sketched above applies to any kind of uncertainty, but the simplest way to illustrate these ideas is to extend the model of an exchange economy to allow for *extrinsic* uncertainty. Suppose there is a finite set of states, $s = 1, ..., S$, with a common-knowledge prior probability distribution $\pi = (\pi_1, ..., \pi_S) \gg 0$. In the first sub-period, the state is unknown but agents know the true probability distribution of the state. At the beginning of the second sub-period, i.e., after agents have chosen their money demands and before trade in assets begins, the true state is revealed.

As before, we assume that the initial money supply $M > 0$ is fixed in the first sub-period and that the CB demands a bundle of assets $g(s) \in \mathbf{R}_+^\ell$ and issues a quantity of money ΔM in each state s in the second sub-period. Note that, although the uncertainty represented by the state of nature s is extrinsic in the sense that it does not affect preferences or endowments, we do allow the CB's demand for assets $g(s)$ to depend on the state. However, the injection of ΔM units of money in the second sub-period is independent of s. As in the certainty case, this is a requirement for equilibrium. Since the initial money supply M and the interest rate r are independent of s, the final demand for money to repay loans in the last sub-period $(1 + r)M$ is also independent of s. Market clearing requires $rM = \Delta M$, so the injection of money in the middle sub-period must be independent of s also.

Agent i borrows $m_i \geq 0$ units of money from the CB in the first sub-period and demands a bundle of assets $x_i(s)$ in each state s in the second sub-period.

Let $p(s) \in \mathbf{R}_+^\ell$ denote the equilibrium vector of assets prices in state s. Then agent i satisfies the budget constraint

$$p(s) \cdot (x_i(s) - e_i) + rm_i \leq 0$$

and the cash-in-advance constraint

$$p(s) \cdot (x_i(s) - e_i)^+ \leq m_i$$

in each state s in the second sub-period.

An allocation (x, m) is *attainable* if

$$\sum_{i=1}^{I} x_i(s) + g(s) = \sum_{i=1}^{I} e_i, \forall s,$$

and

$$\sum_{i=1}^{I} m_i = M.$$

For a given policy $(g, \Delta M)$, an *equilibrium relative to the policy* $(g, \Delta M)$ consists of an attainable allocation (x, m) and a ordered pair $(p, r) : S \to \mathbf{R}_+^\ell \times \mathbf{R}_+$ such that, for every agent i, (x_i, m_i) solves

$$\begin{array}{ll} \max & \sum_{s=1}^{S} \pi_s u_i(x_i(s)) \\ \text{s.t} & p(s) \cdot (x_i(s) - e_i) + rm_i \leq 0, \forall s, \\ & p(s) \cdot (x_i(s) - e_i)^+ \leq m_i, \forall s. \end{array}$$

and the CB's policy satisfies the budget constraint

$$p(s) \cdot g(s) = \Delta M, \forall s.$$

The budget constraints together with the market-clearing conditions imply that $rM = \Delta M$, so the agents have the correct amount of money to repay their loans at the last date.

If preferences are locally non-satiable, then $r > 0$ implies that, for each agent i,

$$\max_{s=1,\ldots,S} \left\{ p(s) \cdot (x_i(s) - e_i)^+ \right\} = m_i.$$

But this equation, together with the usual market-clearing conditions, is not enough to determine the price level, even locally. The easiest way to see this is to consider the limiting case in which $(g, \Delta M) = (0, 0)$ and contrast the effects of assuming the cash-in-advance constraint is binding, with and without uncertainty. First, suppose there is no uncertainty and let (p^*, x^*) denote a Walrasian equilibrium. Define money balances by putting $m_i = p^* \cdot (x_i^* - e_i)^+$ for each i. By appropriately scaling prices, we can ensure that $\sum_{i=1}^{I} m_i = 1$. Then (p^*, r, x^*, m) is an equilibrium with $r = 0$ for the CB policy $(g, \Delta M) = (0, 0)$. Now suppose there is (extrinsic) uncertainty represented by the states of nature $s = 1, ..., S$. We can define an equilibrium by putting $x_i(s) = x_i^*$ for every i and s and putting $p(s) = \lambda(s) p^*$, where $\lambda(s) \leq 1$, for every s. By inspection, (p, r, x, m) satisfies all the equilibrium conditions and the cash-in-advance constraint is satisfied exactly as long as $\lambda(s) = 1$ for some s. We can choose $0 < \lambda(s) < 1$ arbitrarily for other values of s, however, so the price level is clearly not determinate.

3.2 Budget constraints

If the cash-in-advance constraint cannot "determine" the price level, what can? In this case, the equilibrium price level is "determined" by the budget constraints

$$p(s) \cdot g(s) = \Delta M, \forall s, \tag{9}$$

which, in addition to the other equilibrium conditions, provide an extra equation for each state and price level. This answers the question about the determinateness of the price level when $r > 0$, but not in the limit when $r = 0$. When $g(s) = 0 = \Delta M$, both sides of equation (9) are identically zero and the equation tells us nothing about the price level. As an alternative, consider what happens in the limit as $(g, \Delta M) \to (0, 0)$. We must pay particular attention to the way in which the CB's policy approaches the limit, because different limiting price levels can be obtained depending on how the limiting policy is approached. In other words, small changes in the CB's policy can have large effects on the price level.

For concreteness, consider a sequence of equilibria $\{(x^n, m^n, p^n, r^n)\}$ relative to a corresponding sequence of policies $\{(g^n, \Delta M^n)\}$ and suppose the policies take the form

$$(g^n, \Delta M^n) = \frac{1}{n}(\gamma, \mu),$$

for each n. Then (9) reduces to

$$p^n(s) \cdot \gamma(s) = \mu, \forall s,$$

for each n and each state s and, in the limit, as $(x^n, m^n, p^n, r^n) \to (x^0, m^0, p^0, r^0)$, we have S additional equations,

$$p^0(s) \cdot \gamma(s) = \mu, \forall s,$$

which, together with the other equilibrium conditions, suffice to determine the price level.

Theorem 4 *Let $\{(x^n, m^n, p^n)\}$ be a sequence of equilibria relative to the sequence of corresponding policies $\{(g^n, \Delta M^n)\} = \left\{\frac{1}{n}(\gamma, \mu)\right\}$. Suppose that $(x^n, m^n, p^n, r^n) \to (x^0, m^0, p^0, r^0)$ as $n \to \infty$. For each i, assume that u_i is continuous and locally non-satiable and*

$$p^0(s) \cdot e_i > 0.$$

Then (x^0, m^0, p^0, r^0) is an equilibrium for the policy $(g^0, \Delta M^0) = (0, 0)$ and

$$p^0(s) \cdot \gamma(s) = \mu, \forall s. \tag{10}$$

Furthermore, $(p^0(s), x^0(s))$ is a Walrasian equilibrium for each $s = 1, ..., S$. If the Walrasian equilibria of the exchange economy are locally unique, then the equilibrium (x^0, m^0, p^0, r^0) is locally unique among the set of equilibria in which the "budget constraints" (10) are satisfied.

Proof. See Section 6. ■

To illustrate how the budget constraints suffice to determine the equilibrium price levels, consider the following example.

Assume there are two agents $i = 1, 2$, two assets $h = 1, 2$, and endowments $e_1 = (e, 0)$ and $e_2 = (0, e)$. The agents have identical Cobb-Douglas utility functions $u_i(x_i) = x_{i1}^{1/2} x_{i2}^{1/2}$. The CB demands $\gamma(s)$ units of each good in each state s, that is, $g(s) = (\gamma(s), \gamma(s))$, for each s. The money supply M is normalized to equal 2.

The symmetry of the economy implies the existence of a symmetric equilibrium for $\gamma(s)$ sufficiently small. In a symmetric equilibrium, the two agents each hold $m = 1$ units of money and the prices of the two assets are equal in every state. Denote the price level in state s by $p(s)$ and denote each agent's demand for the non-endowment good by $z(s)$ in state s. Then

$$z(s) = \min\left\{\frac{p(s)e - rm}{2p(s)}, \frac{m}{p(s)}\right\}$$

for each s and the agent chooses m to maximize

$$E\left[\ln\left(e - \frac{rm}{p(s)} - z(s)\right) + \ln z(s)\right].$$

When the cost of liquidity is small, but positive, the cash-in-advance constraints will be binding in the states with a high price level but not in states with a low price level. Let

$\gamma^q(s) = \frac{1}{q}\gamma(s)$ for every $q = 1, 2, ...$ and let $p^q(s)$ and $z^q(s)$ denote the corresponding equilibrium price levels and excess demands for every state s. For every state s, $z^q(s)$ converges to $e/2$ as $q \to 0$, so for q sufficiently large and any states s and s',

$$p^q(s) < p^q(s') \Longrightarrow p^q(s) \cdot z^q(s) < p^q(s') \cdot z^q(s') \leq m.$$

Thus, the cash-in-advance constraint holds in state s' but not in state s.

To see how price levels are determined, let $\Delta M^q = \mu/q$ and note that, for each q, $p^q(s) \cdot \gamma(s) = \mu$ so in the limit $2p^0(s)\gamma(s) = \mu$ for each s. Then we can solve the seigniorage equation for the limiting prices

$$p^0(s) = \frac{\mu}{2\gamma(s)}$$

Thus, control of the money supply and real seigniorage allows the CB to control the price level. Even in the limit, where there is no seigniorage, the possibility of raising small amounts of seigniorage allows us to explain how the price level is determined by the CB's demand decisions. The crucial assumption, of course, is that CB policy is exogenous to the model.

Finally, note that although the preceding discussion has dealt only with extrinsic uncertainty, the same ideas extend immediately to economies with intrinsic uncertainty.

3.3 Complete markets

It may be thought that incomplete markets have something to do with the indeterminateness of the price level; but we can introduce Arrow securities without changing these results. In fact, Arrow securities will ensure that $x(s)$ is independent of s, as required for Pareto efficiency, but, as the last theorem shows, even when the allocation is Walrasian there can be price level "indeterminacy" in the limit.

In this model, Arrow securities have two functions: they re-allocate wealth across states and they re-allocate liquidity across states. As long as these two functions are linked, the analysis of this section is essentially unchanged. The introduction of Arrow securities changes the real allocation—the possibility of re-allocating wealth across states ensures that the equilibrium allocation is Pareto-efficient—but it does not change the nature of the cash-in-advance constraint. An example will make this clear. In order to relax the cash-in-advance constraint as much as possible, we suppose that Arrow securities are settled before there is any trade in assets, so agents have the liquidity they need before the assets markets open. we must be careful to specify whether they affect only the budget constraint or the budget constraint and the cash-in-advance constraint. There are four sub-periods. In the first, agents borrow money from the CB and trade Arrow securities; in the second sub-period the state of nature is realized and agents receive the payoffs from their Arrow securities; in the third, agents trade money and assets; and in the fourth, they settle their debts with the CB.

If $q(s)$ denotes the price of one unit of money in state s and agent i purchases $z_i(s)$ of money for delivery in sub-period 2 in state s, that is, $z_i(s)$ units of the corresponding Arrow security, then the budget constraint in the first sub-period is

$$\sum_{s\in S} q(s)z_i(s) \leq 0.$$

Note that I am assuming there is no trade in money and no cash-in-advance constraint in the first sub-period. This does not change anything. In the second sub-period, if state s is observed, the budget constraint is

$$m_i + z_i(s) \geq 0,$$

and there is no need for an additional cash-in-advance constraint. At the beginning of the third sub-period, the agent has $m_i + z_i(s)$ units of money and he must end the fourth sub-period with m_i units (we assume that $r = 0$ for simplicity). So the budget constraint in the third sub-period is essentially

$$p(s) \cdot (x_i(s) - e_i) \leq z_i(s)$$

and the cash-in-advance constraint is

$$p(s) \cdot (x_i(s) - e_i)^+ \leq m_i + z_i(s).$$

If $r = 0$, the shadow price of the cash-in-advance constraint is zero, so, assuming the agent is non-satiable, the budget constraint holds in every state. Then the budget constraints are equivalent to

$$\sum_{s\in S} q(s)p(s) \cdot (x_i(s) - e_i) \leq 0$$

and the efficiency of equilibrium implies that $x_i(s)$ is independent of s. If we write $\bar{x}_i$ for the commodity bundle demanded in equilibrium in each state, the cash-in-advance constraint reduces to

$$p(s) \cdot (\bar{x}_i - e_i)^+ \leq m_i,$$

since $z_i(s) = 0$ in each state. Then the consumer is effectively choosing $(\bar{x}_i, m_i)$ to maximize $u_i(\bar{x}_i)$ subject to the constraints

$$\hat{p} \cdot (\bar{x}_i - e_i) \leq 0$$

and

$$\max_{s\in S} \left\{ p(s) \cdot (\bar{x}_i - e_i)^+ \right\} \leq m_i,$$

where $\hat{p} = \sum_{s\in S} q(s)p(s)$. The source of indeterminacy is exactly the same in this set as in the model without Arrow securities: the agent holds enough money to satisfy his cash-in-advance constraint in the state where his demand for liquidity is the greatest, but the constraint can be slack in other states. The determination of the price level is essentially the same as in the previous model.

The structure outlined above is not the only one. We could, for example, consider the alternative sequence of events, in which Arrow securities are traded, uncertainty is resolved and assets are traded for money, trades in Arrow securities are settled, and, finally, loans from the CB are settled. As long as the two functions of Arrow securities, transferring wealth and transferring liquidity, are bundled, the results will be the same.

Note that there are two potential sources of incompleteness here: due to liquidity cost and due to unhedged risk. If $r > 0$, one does not necessarily get efficient risk sharing even with a full set of Arrow securities.

4 Asset price volatility

After dealing with the CB's ability to control the price level, we can return to the question of the impact of monetary policy on asset markets. In the preceding sections, we focused on the limit as $r \to 0$, for two main reasons. First, the limiting case corresponds most closely to the classical model of Walrasian equilibrium; the frictionless Walrasian equilibrum is a natural starting place for our analysis. Secondly, the limiting case reflects the fact that liquidity costs are vanishingly small for daily transactions executed through a modern payments system. When we consider the impact of liquidity on asset markets, however, a longer time-frame is needed and here it may be appropriate to assume $r > 0$.

4.1 From nominal to relative prices

It is well known that asset-price volatility is very high. A relatively large movement in the market can take place in a matter of hours. Any such movement presents an opportunity for arbitrage, which raises the question of why arbitrage does not eliminate large fluctuations in prices. The answer is that changes in prices are typically unexpected. If an arbitrageur wants to be ready to profit from a sharp drop in price, he will need the liquidity to make a purchase at a moment's notice. This may require him to hold low-yield liquid assets for a considerable period of time. If the arbitrageur thinks assets are underpriced and wants to profit when they revert to their long-run equilibrium value, he may have to buy now and hold them for a very long time. In both cases, intertemporal arbitrage may require holding assets for days or even weeks and the carrying costs may be a significant factor in determining whether the arbitrage is profitable or not. If it is not, then intertemporal arbitrage will not successfully smooth out fluctuations in asset prices.

In the present model, we represent the cost of maintaining a liquid position by the interest rate charged by the CB on cash balances. If we think of seigniorage as a source of revenue for government expenditures, a positive interest rate is necessary in order to finance a positive level of expenditure. The short-run cost of liquidity is determined by the rate at which the CB is willing to lend (the discount rate). The long-run cost of liquidity is determined not by the rate at which the CB is willing to lend, but rather by the opportunity cost of funds for investment purposes. If arbitrageurs have the opportunity of investing liquid funds in assets that yield a positive return, then any other use of those funds must earn a comparable rate of return. In these circumstances, arbitrage can never be completely successful in eliminating fluctuations in asset prices, because perfect arbitrage eliminates the profit that is needed to justify the use of liquidity for this purpose.

In "real" models (Allen and Gale, 1994, 1998, and 2003b), the result of a shortage of liquidity is what we call "cash-in-the-market" pricing. Some investors who hold a long-term investment asset have a sudden need for liquidity that forces them to sell their assets. Buyers need liquidity to purchase assets, so their demand is constrained by the amount of liquidity in their portfolios. So the price at which assets can be sold is determined in part by the liquidity in the market. In fact, a small liquidity shock may lead to a substantial price movement. On the one hand, if the liquidity in the market is fixed in the short run, this limits the elasticity of demand for assets. On the other hand, if the sellers of assets need a fixed amount of liquidity, the supply of assets may be "backward bending": the more the price falls the more assets the asset holders must supply. A small shift in demand or supply may result in a large change in the market-clearing price. As we have argued, one possible stabilizing force comes from speculators who hold liquid assets in order to buy up other assets when the price falls; but their willingness to do so depends on the expectation of capital gains. The speculator must weigh the low returns he receives on the liquid assets against the probability and size of a price movement. To compensate for the low returns to holding the liquid asset, there has to be an expected fall in price. Thus, in equilibrium, there will always be too little liquidity to eliminate asset-price volatility altogether.

The models in Allen and Gale (1994, 1998, and 2003b) are "real" models—in which contracts, prices, and payoffs are denominated in assets—and hence they do not take into account a number of issues that arise in a monetary model. When we discuss the impact of monetary policy on asset prices, it is essential to distinguish *nominal* asset prices from *real* or *relative* asset prices. If we are only interested in the impact of monetary policy on nominal asset prices, the model described in Section 2 is sufficient. We can simply interpret the 'goods' in that model as 'assets' and observe that the general level of asset prices is determined by the CB's policy with respect to money supply and seigniorage. However, when we talk about the impact of monetary policy on asset prices, what we normally have in mind is a change in the general level of asset prices relative to the prices of goods. We have already mentioned the difficulty of making sense of the idea of a limited amount of liquidity when liquidity takes the form of fiat money. First, a reduction in the general price level (an equi-proportionate reduction in goods and asset prices) leaves relative prices unchanged but increases real balances and hence increases liquidity. Secondly, since money is a free good, the CB can always create more. So, in order to make sense of the idea of an insufficient amount of money, there must be some other frictions that prevent equi-proportionate changes in prices from relaxing the liquidity constraint. The two essential ingredients in the model that follows are, first, the money supply is be determined before the CB and the agents have complete information about liquidity shocks (represented here by preference shocks); and, second, that the market for goods is segregated from the market for assets, so that liquidity shocks may have a differential effect on the two markets.

Suppose that goods and assets are traded on separate markets. When an agent enters the asset market, he trades assets for money and money for assets. His trades in money and assets must satisfy a budget constraint and a cash-in-advance constraint. Similarly, when he enters the goods market, his trades in money and goods must satisfy both a budget constraint and a cash-in-advance constraint. Because of the separate cash-in-advance constraints in each market, a liquidity shock in the asset market will not have a direct effect on the goods market and hence on the goods price level. For example, if the volume of trades in the asset market is unusually high, the demand for assets is constrained by a shortage of liquidity but the supply is not and a temporary excess supply will depress asset prices. Goods prices are not directly affected because the demand for and supply of liquidity in the goods market are unchanged. This allows for the possibility that nominal asset prices can fluctuate independently of nominal goods prices, with the result that monetary policy can change real asset prices measured in terms of goods. By contrast, if all trades were aggregated in a single cash-in-advance constraint, a liquidity shock from whatever source would affect the prices of goods and assets similarly. When there is an increase in the volume of assets agents want to trade, agents can draw liquidity away from the demand for goods and spend it on demanding assets instead. The impact of the cash-in-advance constraint will be felt on prices of goods and assets equally.

How are we to interpret the assumption of segregated markets? In classical models of competitive equilibrium, all economic agents participate in all markets all the time. In reality, participation in markets is limited because economic agents do not have the required expertise, or because they have limited attention, or because there are insitutional barriers or other costs of entry. The result is that, at any point in time, the economic agents who are capable of participating in a particular market is a small subset of all the agents in the economy. The liquidity available to the market at a point in time depends on who is participating in the market and the amount of liquid assets in their portfolios (Allen and Gale, 1994). In the present model, we avoid the complications of distinguishing between the agents who are and are not able participate in a market at a point in time by assuming that markets clear sequentially and hence have separate cash-in-advance constraints; but we could achieve the same results by modeling incomplete participation explicitly.

4.2 Market segregation

In this section, we extend the description of equilibrium from Section 3 to deal with segregated trading of assets and goods and then show how costly liquidity influences the pricing of assets.

We start by explicitly distinguishing goods from assets. For simplicity, suppose there is a finite number of tradeable assets, indexed by $k = 1, ..., K$, and a single all-purpose good.

Time is divided into four periods. We retain the "period" terminology from the earlier sections, but stress that here we are thinking about longer periods of time, in which carrying costs are substantial. In the first period, agents decide how much money to borrow from the CB. In the second period, assets are traded in exchange for money. In the third period, goods are traded for money and, in the final period, the money borrowed is returned to the CB with interest.

There is a finite number of states of nature, indexed by $s = 1, ..., S$, with probabilities $\pi(s) > 0$. A crucial assumption is that the true state is revealed at the beginning of the second period, that is, after agents have decided how much money to borrow from the CB and before they have chosen their trades of assets and the good.

There are I agents indexed by $i = 1, ..., I$. Each agent i has an endowment of assets $\bar{\theta}_i \in \mathbf{R}_+^K$ and an endowment of goods $e_i \in \mathbf{R}_+$. An agent's utility depends on his portfolio of assets $\theta_i(s)$ and consumption of the good $x_i(s)$ as well as the state s. Agent i's preferences are represented by a von Neumann-Morgenstern utility function $u_i : \mathbf{R}_+^K \times \mathbf{R}_+ \times S \to \mathbf{R}$. The state-dependence of preferences allows for liquidity-preference shocks as well as asset-return shocks. For example, if we assume that

$$u_i(\theta_i, x_i, s) = U_i(x_i) + \beta(s) E\left[U_i\left(\sum_{k=1}^{K} \theta_{ik} \tilde{R}_k\right)\right],$$

then the time-preference parameter $\beta(s)$ represents a liquidity shock. If we assume that asset k's return $\tilde{R}_k$ has a cumulative distribution function $F_k(R; s)$ that depends on the state s, then we can put

$$u_i(\theta_i, x_i, s) = E\left[U_i\left(x_i, \sum_{k=1}^{K} \theta_{ik} \tilde{R}_k\right) \middle| s\right]$$

and the realization of s represents a shock to asset returns. Or we could combine preference shocks and asset-return shocks.

We assume that the CB takes its seigniorage in the form of purchases of goods and hence is in a position to control the goods price level, but does not engage in open market operations for assets. There are two reasons for adopting this CB policy rule. The first is practical. We observe in reality that asset prices are more volatile than goods prices. We would like our model to have this feature and the easiest way to do that in the current setup is to assume that the CB chooses a stable price level as its target. The second reason is theoretical. Since the CB is a large player, its demand for assets will clearly affect relative asset prices if it engages in open market operations. Because we want to see the effect of liquidity on asset prices if the CB does not deliberately intervene in the asset markets, the natural benchmark to choose is a laisser faire regime where the CB activity is restricted to stabilizing the price level of goods, not the price level of assets.

As before, we take the money supply M as fixed. In the third period, the CB policy demands $g(s) \geq 0$ units of the good in state s and supplies ΔM units of money in exchange.

Note, again, that the amount of money supplied by the CB at date 1 is assumed to be independent of the state s. Market-clearing in the money market is impossible unless $rM = \Delta M$, which implies that ΔM must be independent of s in equilibrium.

4.3 Equilibrium with segregated markets

To describe equilibrium, we require the following notation. Agent i borrows $m_i \geq 0$ units of money from the CB in the first period. In the second period he observes the state of nature

s and demands a portfolio of assets $\theta_i(s) \in \mathbf{R}^\ell_+$. In the third period, he demands a quantity of the good $x_i(s) \geq 0$ in each state s.

The sequential market structure we have adopted here is not crucial. One can assume that the goods market clears before the asset market or that they clear simultaneously. What is crucial is that the markets have separate budget and cash-in-advance constraints.

Let $q(s) \in \mathbf{R}^\ell_+$ denote the equilibrium vector of asset prices in state s and let $p(s) \geq 0$ denote the price of the good in state s. Then agent i satisfies the budget constraint

$$q(s) \cdot (\theta_i(s) - \bar{\theta}_i) + m'_i(s) - m_i \leq 0 \tag{11}$$

in the asset market in the second period, where $m'_i(s)$ is the amount of money carried forward to the third period, and satisfies the budget constraint

$$p(s)(x_i(s) - e_i) + (1 + r)m_i \leq m'_i(s) \tag{12}$$

in the goods market in the third period. In addition, he satisfies the cash-in-advance constraint

$$q(s) \cdot (\theta_i(s) - \bar{\theta}_i)^+ \leq m_i \tag{13}$$

in each state s in the asset market. The second-period cash-in-advance constraint (13) ensures that there exists a $m'_i(s) \geq 0$ that is satisfies the second-period budget constraint (11). So the two budget constraints (11) and (12) can be integrated into a single budget constraint covering the second and third periods

$$q(s) \cdot (\theta_i(s) - \bar{\theta}_i) + p(s)(x_i(s) - e_i) + rm_i \leq 0. \tag{14}$$

Since there is only one good traded for money in the third period, the budget constraint in the third period implies that the cash-in-advance constraint is satisfied in that period. So it is sufficient to require agent i to satisfy the integrated budget constraint (14) and the second-period cash-in-advance constraint (13).

An allocation (θ, x, m) is *attainable* if

$$\sum_{i=1}^{I} \theta_i(s) = \sum_{i=1}^{I} \bar{\theta}_i, \forall s,$$

$$\sum_{i=1}^{I} x_i(s) + g(s) = \sum_{i=1}^{I} e_i, \forall s,$$

and

$$\sum_{i=1}^{I} m_i = M.$$

For a given policy $(g, \Delta M)$, an *equilibrium relative to the policy* $(g, \Delta M)$ consists of an attainable allocation (θ, x, m) and a price vector (p, q, r) such that, for every agent i, (θ_i, x_i, m_i) solves

$$\begin{array}{ll} \max & \sum_{s=1}^{S} \pi_s u_i(\theta_i(s), x_i(s), s) \\ \text{s.t} & p(s)(x_i(s) - e_i) + q(s) \cdot (\theta_i(s) - \bar{\theta}_i) + rm_i \leq 0 \\ & q(s) \cdot (\theta_i(s) - \bar{\theta}_i)^+ \leq m_i \end{array}$$

and the CB's policy satisfies the budget constraint

$$p(s)g(s) = \Delta M, \forall s.$$

As usual, the budget constraints together with the market-clearing conditions imply that $rM = \Delta M$, so there is the right amount of money to pay the loans at the last date.

4.4 Liquidity shocks and asset-price volatility

The usual exercise of letting $(g, \Delta M) \to 0$, so that $r \to 0$, is omitted here. The results are analogous to those obtained in Section 3: in the limit, the cash-in-advance constraint has a zero shadow price and agents' choices are not liquidity-constrained. The CB can control the price level $p^0(s)$, so the possibility of variations in the price level is still with us, but the usual homogeneity properties continue to hold and so the CB's policy has no effect on the equilibrium set.

To see the impact on equilibrium of a positive cost of liquidity, suppose that $(g, \Delta M) \gg 0$ and $r > 0$. Because the cost of holding money is constant, the agents are cash-constrained, that is, they do not have enough money to carry out their competitive demands at prevailing prices, in some states of nature. As a result, prices must fall when trading volume is high and rise when it is lower. However, because there are segregated markets for assets and goods, the fluctuations that occur in the asset market do not transfer to the goods market. As a result, there is volatility in asset prices relative to goods prices.

The calculation of equilibrium values is complicated by the fact that asset demands are cash-constrained, so we use to a parametric example and assume that there is a single asset ($\ell = 1$) and that agents have log-linear preferences. Agent i's utility function is

$$u_i(\theta_i(s), x_i(s), s) = \alpha_i(s) \log \theta_i(s) + (1 - \alpha_i(s)) \log x_i(s).$$

If $r > 0$, the agent's demand for goods or assets must be liquidity-constrained in some states of nature. Let $w_i(s) = q(s) \cdot \theta_i + p(s)e_i - rm_i$ denote agent i's income in state s. For a given agent i and state s there are two possibilities. If the notional competitive excess demand for assets is positive,

$$\frac{\alpha_i(s)w_i(s)}{q(s)} > \bar{\theta}_i,$$

then the agent is either unconstrained and can purchase the notional competitive demand for the asset or else he is constrained by the quantity of money he holds. In the unconstrained case, the demand for the asset is $\alpha_i(s)w_i(s)/q(s)$. In the constrained case it is $\bar{\theta}_i + m_i/q(s)$. Then optimality and the cash-in-advance constraint imply that the demand for the asset is given by the minimum of these two expressions

$$\theta_i(s) = \min\left\{\frac{\alpha_i(s)w_i(s)}{q(s)}, \bar{\theta}_i + \frac{m_i}{q(s)}\right\}.$$

The budget constraint implies that the demand for the good is the minimum of two quantities, one corresponding to the notional competitive demand and the other corresponding to the constrained amount:

$$\begin{aligned} x_i(s) &= \frac{w_i(s) - q(s)\left(\theta_i(s) - \bar{\theta}_i\right)}{p(s)} \\ &= \max\left\{\frac{(1-\alpha_i(s))w_i(s)}{p(s)}, e_i - \frac{(1+r)m_i}{p(s)}\right\}. \end{aligned}$$

In the other case, where the agent's notional competitive excess demand for the asset is negative,

$$\frac{\alpha_i(s)w_i(s)}{q(s)} < \bar{\theta}_i.$$

Since he is a seller of assets and a buyer of goods, the cash-in-advance constraint binds on the demand for the good, if at all. Then, by similar reasoning, his demand for goods is the minimum of his notional competitive demand and the constrained demand:

$$\theta_i(s) = \max\left\{\frac{\alpha_i(s)w_i(s)}{q(s)}, \bar{\theta}_i - \frac{m_i}{q(s)}\right\}. \tag{15}$$

His demand for assets is determined by the budget constraint as the minimum of two terms, corresponding to the notional competitive demand for goods and the constrained amount:

$$\min\left\{\frac{(1-\alpha_i(s))w_i(s)}{p(s)}, e_i + \frac{m_i}{q(s)}\right\}. \tag{16}$$

To illustrate the role of liquidity in asset-pricing volatility (Allen-Gale, 1994), consider the special case in which there is no uncertainty about the aggregate demand for the asset but there is uncertainty about the individual demands for the asset. In a model with complete markets zero liquidity costs, asset prices would be constant across states. However, the existence of costly liquidity means that the value of trades will be constrained by the liquidity held by the agents in the market. When volume of trade is high, the value of trades is constrained by the liquidity in the market. The only way this can be satisfied in equilibrium is if asset prices fall when the volume of trade is high. Thus, asset prices are determined by the amount of "cash in the market," rather than by primitive parameters such as asset returns and agents' risk aversion and discount rates.

Formally, we assume that agents are ex ante identical, that is, they have the same endowments

$$(e_i, \bar{\theta}_i) = (e, \bar{\theta}), \forall i,$$

their preference parameters $\alpha_i(s)$ have identical probability distributions, and aggregate demand for assets is constant, that is, there exists a constant $\bar{\alpha}$ such that

$$I^{-1}\sum_i \alpha(s) = \bar{\alpha}, \forall s.$$

We consider a symmetric equilibrium in which the prices (p, q, r) depend only on the aggregate distribution of parameter shocks $\{\alpha_i(s)\}$. Since all agents are ex ante identical, their choices in the first period will be the same and this implies that $m_i = m$ for each i. These assumptions imply that every agents' wealth in state s is given by

$$w(s) \equiv p(s)e + q(s)\bar{\theta} - rm$$

and substituting this expression in (15) and (16) gives us the appropriate demand functions.

For the sake of illustration, suppose that the CB's policy is to maintain $p(s) = 1$. If the demand for assets is not constrained by money balances at date 1 then the aggregate demand for assets in state s is

$$\begin{aligned}
\sum_i \theta_i(s) &= \sum_i \min\left\{\frac{\alpha_i(s)w(s)}{q(s)}, \frac{m}{q(s)} + \bar{\theta}\right\} \\
&= \sum_i \frac{\alpha_i(s)w(s)}{q(s)} \\
&= \frac{\bar{\alpha}I(e + q(s)\bar{\theta})}{q(s)}
\end{aligned}$$

where we assume that $r = 0$ (otherwise the constraint would be binding). The market-clearing condition is

$$I\bar{\theta} = \frac{\bar{\alpha}I(e + q(s)\bar{\theta})}{q(s)} = \bar{\alpha}I\left(\bar{\theta} + \frac{e}{q(s)}\right).$$

This equation has a unique solution $q(s) = \bar{q}$, independently of s. Thus, absent liquidity constraints, there is no asset price volatility. This is what will happen in the limit as $r \searrow 0$. However, if liquidity is costly, we have already seen that it pays to economize on liquidity. In other words, at the unconstrained optimum, a small reduction in the demand for assets has no first-order impact on utility but does have a first-order impact on liquidity costs. Thus, in some states, we expect that demand is constrained by cash balances for large values of $\alpha_i(s)$. Let $\hat{I}(s)$ denote the set of unconstrained agents in state s. Then the market-clearing condition is

$$\begin{aligned}
I\bar{\theta} &= \sum_i \min\left\{\frac{\alpha_i(s)w(s)}{q(s)}, \frac{m}{q(s)} + \bar{\theta}\right\} \\
&= \sum_{i \in \hat{I}(s)} \frac{\alpha_i(s)w(s)}{q(s)} + \sum_{i \notin \hat{I}(s)} \frac{m + q(s)\bar{\theta}}{q(s)} \\
&< \sum_i \frac{\alpha_i(s)w(s)}{q(s)} \\
&= \bar{\alpha}I\left(\bar{\theta} + \frac{e - rm}{q(s)}\right).
\end{aligned}$$

Comparing this inequality with the earlier market-clearing condition we see that the solution in the constrained case satisfies $q(s) < \bar{q}$.

In fact, the more agents are constrained the lower the asset price. Consider two states s and s'. If $\{\alpha_i(s)\}$ is a mean preserving spread of $\{\alpha_i(s')\}$ then $q(s) \leq q(s')$.

In this exercise, we have assumed that the CB's policy is to keep the price of goods constant across states, so that seigniorage is constant. This implies that changes in the nominal price of assets are equivalent to changes in the relative price of assets, which is what we are interested in. Is it possible that by manipulating the price level, the real asset price could be stabilized? The answer is yes, but only by varying the real value of seigniorage across states. That is, making the price level $p(s)$ low when $q(s)$ is low means that the real value of the interest charge $rm/p(s)$ is higher. There will be a tradeoff between the value of stabilizing the asset price and destabilizing the real value of seigniorage.

To sum up: we have shown that nominal liquidity matters, in the sense that, other things being equal, variations in the volume of trade result in changes in asset price volatility when the supply of nominal liquidity is fixed, where the "other things" being held constant include the aggregate competitive demand for assets.

5 Discussion

What should we take away from these examples? What difference does money make in the theory of asset markets? Under classical conditions, the difference is not great. Under standard conditions, variations in the price level have no real effect. The exception is that, when there are pre-existing debts denominated in terms of money, changes in the price level have real effects and, in extreme cases, may cause a financial crisis. When there are significant frictions affecting the ability to trade in asset markets, however, we get results that are similar to those found in "real" models of liquidity-constrained equilibrium. More precisely, if markets are segregated and there are positive costs of obtaining the liquidity needed to trade in asset markets, then asset prices are determined in part by the supply of liquidity, independently of the factors such as risk aversion and asset returns that determine demand and supply in perfectly competitive models.

"Real" models have been used to explore a large number of issues related to liquidity and financial markets. The analysis of these issues may be very different when we recognize the role of fiat money. What we have attempted to do here is provide a basic framework in which some of these issues may be raised. To introduce money into a general-equilibrium theory of financial markets, we adopted the strategy of replacing a complex financial system by a single entity (the CB) and replacing a complex payments system with a single constraint (the cash-in-advance constraint). The resulting framework is abstract, but it allows one to begin to think about important issues revolving around the impact of monetary policy and liquidity on financial markets.

When debt contracts are denominated in terms of money, changes in nominal prices have real effects. Allen and Gale (1998, 2000c) and Gale and Vives (2002) explore the virtuous side of price changes and show that, by controlling the price level or the exchange rate, the

CB can improve risk sharing when deposit contracts are denominated in terms of money. For example, in Allen and Gale (1998), it is shown that if banks denominate demand deposits in money and the CB provides liquidity by lending money to banks at zero interest, there exists an equilibrium in which the first-best allocation of risk is achieved. Adjustments in the price level implement the optimal state-contingent risk-sharing scheme. There is no function for money in this "real" economy, other than as a unit of account, and hence nothing to determine the price level. If the CB cannot control the price level or the exchange rate, or if it gets the level wrong because of incomplete information, it might make things worse. We have seen that in some settings (Sections 2 and 3), the CB's ability to control prices is considerable; but in other settings (Section 4), because of segmented markets and limited information, its control is much less. This is the downside of the real impact of changes in the price level, which has been reflected in recent concern about deflation. So it is crucially important to understand the mechanism by which prices are controlled, in particular, asset prices, which are normally much more volatile than goods prices.

Perhaps the most important potential application of these ideas is to markets where default and financial crises can be triggered by changes in asset prices. Allen and Gale (2003a,b) describe a range of environments where crises may be benign or catastrophic, depending on the (in)completeness of markets, and where financial fragility may result from very small shocks to liquidity demand. This analysis was carried out mainly in the context of a real economy. How does this analysis change if contracts are denominated in terms of money and liquidity is supplied by the CB in the form of fiat money? The framework developed here provides us with the tools to begin answering this question. On the one hand, in the absence of complete markets, the possibility of intervention by the CB offers the hope that financial fragility may be eliminated by the appropriate policy: providing the right amount of liquidity, either by adjusting prices or injecting money, could prevent the wild fluctuations in interest that destabilize equilibrium in Allen and Gale (2003b). On the other hand, if the CB does not have the necessary information to adjust liquidity correctly, it could make things worse rather than better.

Another lesson from these exercises is the importance of timing and market participation. When liquidity is costless, agents can hold large amounts of liquid assets and ensure that the liquidity constraint is never binding. When the cost of liquidity is positive, it matters a great deal when the decision about how much liquidity to hold is made. If the decision is made before all the relevant information is available, unexpected changes in the price level will lead to binding liquidity constraints. Even if the CB is able to target some prices, the existence of segregated markets implies that other prices can be highly volatile. This is not to say that there is no policy that will stabilize prices, but it may require direct intervention in the markets rather than general control over prices. A case in point would be prevention of asset bubbles, where a policy of controlling inflation in goods prices may have no effect on inflation in asset prices.

Finally, a lot of attention has recently been focused on the question of whether the CB's job is to provide liquidity to the market as a whole or whether it should provide liquidity directly to distressed institutions. See, for example, Rochet and Vives (2003) and

Freixas, Parigi and Rochet (2003). If markets are functioning smoothly, it may be sufficient to provide liquidity to the market and allow the market to re-allocate it optimally. But market frictions, whether in the form of segmentation or the more traditional frictions such as asymmetric information or imperfect competition, may require more direct interventions. The models of Rochet and Vives (2002) and Freixas, Parigi and Rochet (2000) make the standard assumption, that contracts and asset prices are denominated in real terms. But if they are denominated in terms of money, the answer could well be different. For example, supplying liquidity to the entire system may simply raise all nominal prices and leave relative prices unchanged. Supplying liquidity to an individual bank may leave the general price level unchanged but increase the purchasing power of the particular bank.

The models needed to answer all these questions do not yet exist. There is a lot of work to be done.

6 Proofs

Proof of Theorem 2 It is clear that (x^0, m^0) is an attainable allocation because each (x^n, m^n) is attainable and $(x^n, m^n) \to (x^0, m^0)$.

To show that (x^0, m^0, p^0, r^0) is an equilibrium, it is sufficient to show that (x_i^0, m_i^0) solves agent i's decision problem given (p^0, r^0). The proof is by contradiction. Suppose that (x_i, m_i) is a feasible choice that is strictly preferred to (x_i^0, m_i^0). Since $p^0 \cdot e_i > 0$ and u_i is continuous, we can without loss of generality choose (x_i, m_i) so that the budget constraint and the cash-in-advance constraint are strictly satisfied. Then (x_i, m_i) is feasible and preferred to (x_i^n, m_i^n) for n sufficiently large, contradicting our hypothesis that (x^n, m^n, p^n, r^n) is an equilibrium.

The cash-in-advance constraint for agent i is satisfied as an equation in the limit because $r^n > 0$ implies that it is satisfied as an equation for every value of n.

Finally, the equilibrium condition $p^n \cdot g^n = rM$ implies that $p^0 \cdot 0 = rM$ or $r = 0$. Since liquidity is costless, the value of x_i^0 that solves agents i's decision problem also maximizes utility subject to the Walrasian budget constraint. It follows that (p^0, x^0) is a Walrasian equilibrium.

Proof of Theorem 3 Let (x, m, p, r) be an equilibrium relative to the policy $(g, \Delta M) = (0, 0)$ and suppose that the individual cash-in-advance constraints are all binding. If (x, m, p, r) is not locally unique among the set of equilibria with binding constraints, then for any $\varepsilon > 0$ we can find an equilibrium (x', m', p', r'), relative to the policy $(g, \Delta M) = (0, 0)$, ε-close to (x, m, p, r). Clearly, $r = r' = 0$ and (x, p) and (x', p') are Walrasian equilibria, so the local uniqueness of Walrasian equilibria implies that $x = x'$ and $p = kp'$ for some $k > 0$. Then the aggregate cash-in-advance constraint

$$\sum_{i=1}^{I} p \cdot (x_i - e_i)^+ = M = \sum_{i=1}^{I} p' \cdot (x_i' - e_i)^+$$

implies that $p = p'$ and the individual cash-in-advance constraints imply that $m_i = m'_i$ for each i.

Proof of Theorem4 The proof is similar to that for Theorem 2. It is clear that (x^0, m^0) is an attainable allocation because each (x^n, m^n) is attainable and $(x^n, m^n) \to (x^0, m^0)$.

To show that (x^0, m^0, p^0, r^0) is an equilibrium relative to $(0, 0)$, it is sufficient to show that (x_i^0, m_i^0) solves agent i's decision problem given (p^0, r^0). The proof is by contradiction. Suppose that (x_i, m_i) is a feasible choice that is strictly preferred to (x_i^0, m_i^0). Since $p^0(s) \cdot e_i > 0$ for every state s and u_i is continuous, we can without loss of generality choose (x_i, m_i) so that the budget constraint and the cash-in-advance constraint are strictly satisfied in every state s. Then (x_i, m_i) is feasible and preferred to (x_i^n, m_i^n) for n sufficiently large, contradicting our hypothesis that (x^n, m^n, p^n, r^n) is an equilibrium.

The cash-in-advance constraint is binding in the limit in at least one state because $r^n > 0$ implies that it is binding in at least one state for every value of n. Finally, the equilibrium condition $p^n(s) \cdot g^n(s) = rM$ implies that $p^0(s) \cdot 0 = rM$ or $r = 0$. Since liquidity is costless, the value of $x_i^0(s)$ that solves agents i's decision problem also maximizes utility in state s subject to the usual budget constraint. It follows that $(p^0(s), x^0(s))$ is a Walrasian equilibrium.

To show that (x^0, m^0, p^0, r^0) is locally unique, we use a variant of the argument used in the proof of Theorem 3. It is clear from the CB budget constraints that, in the limit, $p^0(s) \cdot \gamma(s) = \mu$ for every state s. Let (x, m, p, r) be an equilibrium relative to the policy $(g, \Delta M) = (0, 0)$. If (x^0, m^0, p^0, r^0) is not locally unique among the set of equilibria satisfying the CB budget constraints, then for any $\varepsilon > 0$ we can find an equilibrium (x, m, p, r), relative to the policy $(g, \Delta M) = (0, 0)$, that is ε-close to (x^0, m^0, p^0, r^0). Clearly, $r = r^0 = 0$ and $(x(s), p(s))$ and $(x^0(s), p^0(s))$ are Walrasian equilibria for every state s, so the local uniqueness of Walrasian equilibria implies that $x(s) = x^0(s)$ and $p(s) = k(s)p^0(s)$ for some $k(s) > 0$ and every state s. Then the aggregate budget constraints

$$p^0(s) \cdot \gamma(s) = \mu = p(s) \cdot \gamma(s)$$

implies that $k(s) = 1$ for every s. Finally, the individual cash-in-advance constraints imply that $m_i = m_i^0$ for each i.

References

Allen, F. and D. Gale (1994). "Liquidity Preference, Market Participation and Asset Price Volatility," *American Economic Review* **84**, 933-955.

— (1998). "Optimal Financial Crises," *Journal of Finance* **53**, 1245-1284.

— (2000a). "Bubbles and Crises," *The Economic Journal* **110**, 236-256.

— (2000b). "Financial Contagion," *Journal of Political Economy* **108**, 1-33.

— (2000c). "Optimal Currency Crises" *Carnegie-Rochester Conference Series on Public Policy* **53**, 177-230.

— (2003a). "Financial Intermediaries and Markets," *Econometrica*, forthcoming.

— (2003b). "Financial Fragility," *Journal of the European Economic Association*, forthcoming.

Balasko, Yves and D. Cass (1989). "The Structure of Financial Equilibrium with Exogenous Yields: The Case of Incomplete Markets," *Econometrica* **57**, 135-62.

Bassetto, M (2002). "A Game-Theoretic View of the Fiscal Theory of the Price Level," *Econometrica* **70**, 2167-95.

Buiter, W. (1998). "The Young Person's Guide to Neutrality, Price Level Determinacy, Interest Rate Peg," National Bureau of Economic Research, Inc, NBER Working Papers: 6396.

— (1999). "The Fallacy of the Fiscal Theory of the Price Level," National Bureau of Economic Research, Inc, NBER Working Papers: 7302.

— (2002). "The Fiscal Theory of the Price Level: A Critique," *Economic Journal* **112**, 459-80.

— (2004). "A Small Corner of Intertemporal Public Finance – New Developments in Monetary Economics," National Bureau of Economic Research, Inc, NBER Working Papers: 10524.

Carlstrom, C. and T. Fuerst (2000). "The Fiscal Theory of the Price Level," *Federal Reserve Bank of Cleveland Economic Review*, **36**, 22-32.

Cass, D (1984). "Competitive Equilibrium with Incomplete Financial Markets," University of Pennsylvania, CARESS Working Paper 84-09.

Christiano, L. and T. Fitzgerald (2000). "Understanding the Fiscal Theory of the Price Level," *Federal Reserve Bank of Cleveland Economic Review* **36**, 1-37.

Cochrane, J (2001). "Long-Term Debt and Optimal Policy in the Fiscal Theory of the Price Level," *Econometrica* **69**, 69-116.

Coleman, S. (2002). "The Evolution of the Federal Reserve's Intraday Credit Policies," *Federal Reserve Bulletin* 67-84.

Diamond, D. and P. Dybvig (1983). "Bank Runs, Deposit Insurance, and Liquidity," *Journal of Political Economy* **91**, 401-19

Duffie, D. (2001) *Dynamic Asset Pricing Theory*, 3rd edition. Princeton, NJ: Princeton University Press.

Fuerst, T. (1992). "Liquidity, Loanable Funds, and Real Activity," *Journal of Monetary Economics* **29**, 3-24.

Freixas, X., P. Parigi, and J.C. Rochet (2003). "The Lender of Last Resort: A 21st Century Approach," Autonomous University of Barcelona, unpublished.

Gale, D. and X. Vives (2002). "Dollarization, Bailouts, and the Stability of the Banking System," *Quarterly Journal of Economics* **117**, 467-502.

Geanakoplos, J. and P. Dubey (1992). "The Value of Money in a Finite-Horizon Economy: A Role for Banks," in Dasgupta, P. et al. (eds.) *Economic Analysis of Markets and Games.* Cambridge, MA: MIT Press.

Geanakoplos, J. and A. Mas-Colell (1989). "Real Indeterminacy with Financial Assets," *Journal of Economic Theory* **47**, 22-38.

Kocherlakota, N. and C. Phelan (1999). "Explaining the Fiscal Theory of the Price Level," *Federal Reserve Bank of Minneapolis Quarterly Review* **23**, 14-23.

Lucas, R. (1990). "Liquidity and Interest Rates," *Journal of Economic Theory* **50**, 237-264.

Magill, M. and M. Quinzii (1992). "Real Effects of Money in General Equilibrium," *Journal of Mathematical Economics* **21**, 301-342.

Martin, A. (2002). "Optimal Pricing of Intra-Day Liquidity," Federal Reserve Bank of Kansas City, unpublished.

Pigou, A. C. (1943). "The Classical Stationary State," *The Economic Journal*, **53**, 343-351.

Rochet, Jean-Charles and Xavier Vives (2002). "Coordination Failures and the Lender of Last Resort: Was Bagehot Right after all?" Université Sciences Sociales, Toulouse / INSEAD, Fontainebleau, unpublished.

Sargent, T. and N. Wallace (1981). "Some Unpleasant Monetarist Arithmetic," *Federal Reserve Bank of Minneapolis Quarterly Review* **4**, 1-17.

Werner, J. (1985). "Equilibrium in Economies with Incomplete Financial Markets," *Journal of Economic Theory* **36**, 110-19.

Woodford, M. (1995) "Price Level Determinacy Without Control of a Monetary Aggregate," National Bureau of Economic Research, Inc, NBER Working Papers: 5204.

Zhou, R. (2000). "Understanding Intraday Credit in Large-Value Payment Systems," *Federal Reserve Bank of Chicago: Economic Perspectives*, 29-44.