

Financial Fragility, Liquidity, and Asset Prices ^{*} [†]

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Abstract

We define a financial system to be *fragile* if small shocks have disproportionately large effects. In a model of financial intermediation, we show that small shocks to the demand for liquidity cause either high asset-price volatility or bank defaults or both. Furthermore, as the liquidity shocks become vanishingly small, the asset-price volatility is bounded away from zero. In the limit economy, with no shocks, there are many equilibria. However, if banks face idiosyncratic liquidity shocks then the only equilibria that are

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robust to the introduction of small aggregate risk involve stochastic consumption as well as volatile asset prices.

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1 Excess sensitivity and sunspots

There are numerous historical examples of *financial fragility*, where shocks that are small in relation to the economy as a whole have a significant impact on the financial system. For example, Kindleberger (1978, pp. 107-108) argues that the immediate cause of a financial crisis

may be trivial, a bankruptcy, a suicide, a flight, a revelation, a refusal of credit to some borrower, some change of view which leads a significant actor to unload. Prices fall. Expectations are reversed. The movement picks up speed. To the extent that speculators are leveraged with borrowed money, the decline in prices leads to further calls on them for margin or cash, and to further liquidation. As prices fall further, bank loans turn sour, and one or more mercantile houses, banks, discount houses, or brokerages fail. The credit system itself appears shaky and the race for liquidity is on.

Schnabel and Shin (2004) document one such example, the financial crisis of 1763. They argue that the bankruptcy of the de Neufville brothers' banking house forced them to sell their stocks of commodities. In the short run, the liquidity of the market was limited and the commodity sales resulted in lower prices. The fall in prices in turn put other intermediaries under strain and forced them to sell. A collapse in commodity prices and a financial crisis followed. Schnabel and Shin draw a parallel between the 1763 crisis and the Long Term Capital Management (LTCM) episode in 1998. They suggest that, without the rescue coordinated by the Federal Reserve Bank of New York, there might have been a collapse in asset prices and a widespread financial crisis.

In this paper, we use the term *crisis* to mean a profound drop in the value of asset prices that affects the solvency of a large number of banks and their ability to meet their commitments to depositors. If the price movement is large enough, some banks are forced into liquidation; but there may also be crises in which banks avoid default, although their balance sheet is under extreme pressure. It is well known that sufficiently large *exogenous* shocks can cause a crisis. For example, Allen and Gale (1998) describe a model in which financial crises are caused by exogenous asset-return shocks. Following a large (negative) shock to asset returns, banks are unable to meet their commitments and are forced to default and liquidate assets. In the current paper we investigate *endogenous* crises, where small or negligible shocks set off self-reinforcing and self-amplifying price changes. We use a simple version of the general equilibrium model introduced in Allen and Gale (2004; henceforth AG). The key element of the model is the role of liquidity in determining asset prices. The supply of liquidity is determined by the banks' initial portfolio choices. Subsequently, small shocks to the demand for liquidity, interacting with the fixed supply, cause a collapse in asset prices.

We model liquidity preference in the standard way by assuming that consumers have stochastic time preferences. There are three dates, $t = 0, 1, 2$; contracts are drawn up at date 0, and all consumption occurs at dates 1 and 2. There are two types of consumers: *early consumers*, who only value consumption at date 1; and *late consumers*, who only value consumption at date 2. Consumers are identical at date 0 and learn their true type, “early” or “late”, at the beginning of date 1.

There are two types of investments in this economy: short-term investments, which yield a return after one period; and long-term investments, which take two periods to mature. There is a trade-off between liquidity and returns: long-term investments have higher returns but take longer to mature (are less liquid). By contrast, there is no risk–return trade-off: we assume that asset returns are nonstochastic in order to emphasize that, in the present model, financial crises are not driven by shocks to asset returns. For an example of a more general model of financial crises, including a risk–return trade-off, see AG.

Banks are modeled as risk-sharing institutions that pool consumers' endowments and invest them in a portfolio of long- and short-term investments. In exchange for their endowments, banks give consumers a deposit contract, which is an option to receive either a fixed amount of consumption at date 1 or a fixed

amount of consumption at date 2. The deposit contract provides consumers with insurance against liquidity shocks by intertemporally smoothing the returns paid on deposits. There is free entry into the banking sector: in a competitive equilibrium, banks must maximize the expected utility of depositors in order to attract customers.

Once the supply of liquidity is fixed by the banks' portfolio decisions, shocks to the demand for liquidity can cause substantial asset-price volatility and/or default. The supply of liquidity is fixed in the short run by the banks' portfolio decisions at date 0. In the absence of default, the demand for liquidity is perfectly inelastic in the short run. If the banks' supply of liquidity is sufficient to meet the depositors' demand when liquidity preference is high, there must be an excess supply of liquidity when liquidity preference is low. The banks will be willing to hold this excess liquidity between dates 1 and 2 only if the interest rate is zero (the price of date-2 consumption in terms of date-1 consumption is 1). A low interest rate implies that asset prices are correspondingly high. However, asset prices cannot be high in all states for then the short asset would be dominated at date 0 and no one would be willing to hold it. So, in the absence of default, there will be substantial price volatility. Note that this argument does not require large shocks to liquidity demand.

We define a financial system to be *fragile* if a small aggregate shock in the demand for liquidity leads to disproportionately large effects in terms of default or asset-price volatility. One test for financial fragility is to allow the shocks to become vanishingly small and see whether the effects disappear in the limit. If they do not, we can say that the system is fragile because the shocks are infinitesimal relative to the consequences. As we have argued so far, there must be either substantial asset-price volatility or default (or both), regardless of the size of the shocks. In fact, we can show that we must have positive asset-price volatility even in the case of default. Further, we can show that this volatility is bounded away from zero as the variance of the shocks becomes vanishingly small. In the limit, the volatility of asset prices is disproportionately large relative to the shocks—that is, the financial system is fragile.

In the limit, when the shocks become vanishingly small, there is no aggregate exogenous uncertainty, but this does not mean that there is no endogenous uncertainty. We distinguish two kinds of uncertainty. *Intrinsic* uncertainty is caused by stochastic fluctuations in the primitives or fundamentals of the economy.

Examples would be exogenous shocks that effect liquidity preferences or asset returns. *Extrinsic* uncertainty by definition has no effect on the fundamentals of the economy. An equilibrium with no extrinsic uncertainty is called a *fundamental* equilibrium, because the endogenous variables are functions of the exogenous primitives or fundamentals of the model (endowments, preferences, technologies). An equilibrium with extrinsic uncertainty is called a *sunspot* equilibrium,¹ because endogenous variables may be influenced by extraneous variables (sunspots) that have no direct impact on fundamentals. A crisis cannot occur in a fundamental equilibrium in the absence of exogenous shocks to fundamentals, such as asset returns or liquidity demands. In a sunspot equilibrium, by contrast, asset prices fluctuate in the absence of aggregate exogenous shocks, and crises appear to occur spontaneously.

There are multiple equilibria in the limit economy with no aggregate exogenous uncertainty. Some of these equilibria are characterized by financial crises and some are not. Which type of equilibrium is most likely to be observed? To test the robustness of a given equilibrium in the limit economy, we perturb the economy by introducing a small amount of aggregate uncertainty and ask whether there exists an equilibrium of the perturbed economy that is close to the given equilibrium. If the answer is “yes” then we describe the equilibrium as *robust*. Equivalently, we say that an equilibrium is robust if it is the limit of a sequence of equilibria, corresponding to a sequence of perturbed economies, as the shocks become vanishingly small. We already know which equilibria are robust in this sense. We have seen that any equilibrium of the perturbed economy is characterized by asset-price volatility that is bounded away from zero as the aggregate liquidity shocks converge to zero. Thus, a robust equilibrium of the limit economy must have extrinsic uncertainty. Conversely, the fundamental equilibrium of the limit economy is not robust.²

¹The theoretical analysis of sunspot equilibria began with the seminal work of Azariadis (1981) and Cass and Shell (1983), which gave rise to two streams of literature. The Cass–Shell paper is most closely related to work in a Walrasian, general equilibrium framework; the Azariadis paper is most closely related to the macroeconomic dynamics literature. For a useful survey of applications in macroeconomics, see Farmer (1999); for an example of the current literature in the general equilibrium framework, see Gottardi and Kajii (1995, 1999).

²The limit economy has two types of equilibria with extrinsic uncertainty. In a trivial sunspot equilibrium, prices are random but the allocation is essentially the same as in the fundamental equilibrium and no banks default. In a nontrivial sunspot equilibrium, the prices and the equilibrium allocation are random.

These results help us understand the relationship between two traditional views of financial crises. One view is that they are spontaneous events, unrelated to changes in the real economy. Historically, banking panics were attributed to “mob psychology” or “mass hysteria” (see e.g. Kindleberger 1978). The modern version of this theory explains banking panics as equilibrium coordination failures (Bryant 1980; Diamond and Dybvig 1983). An alternative view is that financial crises are a natural outgrowth of the business cycle (Gorton 1988; Calomiris and Gorton 1991; Calomiris and Mason 2000; Allen and Gale 1998, 2000a–c, 2004). Our model combines the most attractive features of both traditional approaches. Like the sunspot approach, it produces large effects from small shocks.³ Like the real business cycle approach, it makes a firm prediction about the conditions under which crises will occur.⁴

There are important differences between the present model of systemic or economywide *crises* and models of individual bank runs or *panics* (Bryant 1980; Diamond and Dybvig 1983). In our model a crisis is a *systemic event*. It occurs only if the number of defaulting banks is large enough to affect the equilibrium asset price. In the panic model, by contrast, bank runs are an idiosyncratic phenomenon. Whether a run occurs at a particular bank depends on the decisions taken by the bank’s depositors, independently of what happens at other banks. It is only by coincidence that runs are experienced by several banks at the same time.

Another difference between panics and crises concerns the reasons for the default. In the Bryant–Diamond–Dybvig story, bank runs are spontaneous events that depend on the decisions of late consumers to withdraw early. Given that almost all agents withdraw at date 1, early withdrawal is a best response for every agent; but if late consumers were to withdraw at date 2, then late withdrawal is a best response for every late consumer. So there are two “equilibria” of the coordination game played by agents at date 1, one with a bank run and one without. This kind of coordination failure plays no role in the present model. In

³Strictly speaking, much of the banking literature exploits multiple equilibria without addressing the issue of sunspots. We adopt the sunspots framework here because it encompasses the standard notion of equilibrium and allows us to address the issue of equilibrium selection.

⁴Models of sunspot phenomena typically have many equilibria, including as a special case the fundamental equilibria in which extrinsic uncertainty has no effect on endogenous variables. Thus, financial fragility remains a possibility but not a necessity.

fact, coordination failure is explicitly ruled out: a bank run occurs only if the bank cannot simultaneously satisfy its budget constraint and its incentive constraint.⁵ From the point of view of a single, price-taking bank, default results from an exogenous shock. When bankruptcy does occur, it is the result of low asset prices. Asset prices are endogenous, of course, and there is a “self-fulfilling” element in the relationship between asset prices and crises. Banks are forced to default and liquidate assets because asset prices are low, and asset prices are low as a result of mass bankruptcy and the associated liquidation of bank assets.

An interesting feature of the model is the role of mixed equilibria, in which ex ante identical banks must choose different strategies. For some parameter specifications, we can show that one group of banks will follow a risky strategy by investing almost all of their funds in the long asset. They meet their demands for liquidity by selling the asset in the market. Another group of banks follow a safe strategy and hold a large amount of the short asset. The safe banks provide liquidity to the risky banks by purchasing the risky banks’ long-term assets. Safe banks also provide liquidity to each other: because there are idiosyncratic shocks to liquidity demand, the safe banks with a high demand for liquidity sell long-term assets to those with a low demand.

Our work is related to the wider literature on general equilibrium with incomplete markets (GEI). As is well known, sunspots do not matter when markets are complete (for a precise statement, see Shell and Goenka 1997). The incompleteness in our model reveals itself in two ways. First, sunspots are assumed to be noncontractible—that is, the deposit contract is not explicitly contingent on the sunspot variable. In this respect we are simply following the incomplete contracts literature (see e.g. Hart 1995). Second, there are no markets for Arrow securities contingent on the sunspot variable, so financial institutions cannot insure themselves against asset-price fluctuations associated with the sunspot variable. This is the standard assumption of the GEI literature (see e.g. Geanakoplos 1990 or Magill and Quinzii 1996).

⁵This is a refinement of the equilibrium concept. We assume that late consumers withdraw at the last date whenever it is incentive-compatible for them to do so. Bank runs occur only when it is impossible for the bank to meet its obligations in an incentive-compatible way. Such runs are called “essential” in AG to distinguish them from the coordination failures in Diamond and Dybvig (1983).

There is a small but growing literature related to financial fragility. Financial multipliers were introduced by Bernanke and Gertler (1989). In the model of Kiyotaki and Moore (1997), the impact of illiquidity at one link in the credit chain has a large impact further down the chain. Chari and Kehoe (2000) show that herding behavior can cause a small information shock to have a large effect on capital flows. Lagunoff and Schreft (2001) show how overlapping claims on firms can cause small shocks to lead to widespread bankruptcy. Bernardo and Welch (2004) develop a model of runs on financial markets and asset-price collapses based on the anticipation of liquidity needs.

Postlewaite, Mailath and Samuelson (2003) study a model in which transactions in a competitive market are preceded by fixed investments. They show that, in the absence of forward contracts, equilibrium spot prices are highly volatile even when the degree of uncertainty is very small.

In AG the authors provide a more general model of which the present model is a very special case. The focus of that paper is on the welfare properties of equilibrium rather than financial fragility. For example, it is shown that, with complete markets for aggregate risks, the laissez-faire equilibrium is either incentive-efficient or constrained-efficient—depending on whether intermediaries can use general incentive-compatible contracts or incomplete contracts. By contrast, if markets are incomplete then there may be too much or too little liquidity, and government regulation may be welfare-improving. There is no discussion of the sensitivity of equilibrium to small shocks, which is the central point of the present paper.

The rest of this paper is organized as follows. Section 2 lays out the basic assumptions of the model. Section 3 describes the optimal contracts offered by banks and the rules governing default and liquidation. Section 4 defines equilibrium. A special case of the model and a numerical example are considered in Section 5. Section 6 contains a full analysis of the equilibria of the economy. We first consider the limit economy with no aggregate uncertainty and then perturb the economy by introducing aggregate uncertainty. We show that, in any equilibrium of the economy with aggregate uncertainty, crises occur with positive probability. We also show that the limit of a sequence of equilibria corresponding to a sequence of perturbed economies is an equilibrium in the limit economy, and we characterize these limit equilibria. Section 7 contains a discussion. Proofs are gathered in Section 8.

2 Assets and preferences

The model we use is a special case of that in AG.

Dates. There are three dates ($t = 0, 1, 2$) and a single good at each date. The good can be used for consumption or investment.

Assets. There are two assets, a short-term asset (the *short* asset) and a long-term asset (the *long* asset).

- The short asset is represented by a storage technology. Investment in the short asset can take place at date 1 or date 2. One unit of the good invested at date t yields one unit at date $t + 1$ for $t = 0, 1$.
- The long asset takes two periods to mature and is more productive than the short asset. Investment in the long asset can take place only at date 0. One unit invested at date 0 produces $r > 1$ units at date 2.

Consumers. There is a continuum of ex ante identical consumers, whose measure is normalized to unity. Each consumer has an endowment $(1, 0, 0)$ consisting of one unit of the good at date 0 and nothing at subsequent dates. There are two (ex post) types of consumers at date 1: *early consumers*, who value consumption only at date 1; and *late consumers*, who value consumption only at date 2. If η denotes the probability of being an early consumer and c_t denotes consumption at date $t = 1, 2$, then the consumer's ex ante utility is

$$u(c_1, c_2; \eta) = \eta U(c_1) + (1 - \eta)U(c_2).$$

The period utility function $U : \mathbf{R}_+ \rightarrow \mathbf{R}$ is twice continuously differentiable and satisfies the usual neoclassical properties: $U'(c) > 0$, $U''(c) < 0$, and $\lim_{c \searrow 0} U'(c) = \infty$.

Uncertainty. There are three sources of intrinsic uncertainty in the model. First, each individual consumer faces idiosyncratic uncertainty about her preference type (early or late consumer). Second, each bank faces idiosyncratic uncertainty about the number of early consumers among the bank's depositors. For example, different banks could be located in regions subject to independent liquidity shocks. Third, there is aggregate uncertainty about the fraction of early consumers in the economy. Aggregate uncertainty is represented by

a state of nature θ , a nondegenerate random variable with finite support and density function $f(\theta)$. The bank's idiosyncratic shock is represented by a random variable α , with finite support and density function $g(\alpha)$. The probability of being an early consumer at a bank in state (α, θ) is denoted by $\eta(\alpha, \theta)$, where

$$\eta(\alpha, \theta) \equiv \alpha + \varepsilon\theta$$

and $\varepsilon \geq 0$ is a constant. We adopt the usual “law of large numbers” convention and assume that the fraction of early consumers at a bank in state (α, θ) is identically equal to the probability $\eta(\alpha, \theta)$. The economywide average of α is assumed to be constant and equal to the mean $\bar{\alpha} = \sum \alpha g(\alpha)$. Thus, there is aggregate intrinsic uncertainty only if $\varepsilon > 0$.

Information. All uncertainty is resolved at date 1. The true value of θ is publicly observed,⁶ the true value of α for each bank is publicly observed, and each consumer learns his type—that is, whether he is an early consumer or a late consumer.

Asset markets. There are no asset markets for hedging against aggregate uncertainty at date 0; for example, there are no Arrow securities contingent on the state of nature θ . At date 1, there is a market in which future (date-2) consumption can be exchanged for present (date-1) consumption. If $p(\theta)$ denotes the price of future consumption in terms of present consumption at date 1, then one unit of the long asset is worth $p(\theta)r$ at date 1 in state θ .

Markets are incomplete at date 0 but complete at date 1. We assume that market participation is incomplete: financial institutions such as banks can participate in the asset market at date 1, but individual consumers cannot.⁷

⁶It is not strictly necessary to assume that θ is observed. In equilibrium, all that agents need to know is the equilibrium price $p(\theta)$, which may or may not reveal θ .

⁷As Cone (1983) and Jacklin (1986) showed, if consumers have access to the capital market then it is impossible for banks to offer risk sharing that is superior to the market.

3 Banking

Banks are financial institutions that provide investment and liquidity services to consumers. They do this by pooling the consumers' resources, investing them in a portfolio of short- and long-term assets, and offering consumers future consumption streams with a better combination of asset returns and liquidity than individual consumers could achieve by themselves. Banks also have access to the interbank capital market, where they can buy or sell the long asset at date 1 and from which consumers are excluded.

As noted in AG, one could imagine a world in which a single “universal” intermediary offering contingent contracts could act as a central planner and implement the first-best allocation of risk. There would be no reason to resort to markets at all. Our world view is based on the assumption that transaction costs preclude this kind of centralized solution and that decentralized intermediaries are restricted in the number of different contracts they can offer. Here we further assume that intermediaries are restricted to offering deposit contracts.

Banks compete by offering deposit contracts to consumers in exchange for their endowments, and consumers respond by choosing the most attractive of the contracts offered. Free entry ensures that banks earn zero profits in equilibrium. The deposit contracts offered in equilibrium must maximize consumers' welfare subject to the zero-profit constraint. Otherwise, a bank could enter and make a positive profit by offering a more attractive contract.

Anything a consumer can do, the bank can do. So there is no loss of generality in assuming that consumers

deposit their entire endowment in a bank at date 0.⁸ Consumers cannot diversify by spreading their money across more than one bank.⁹ The bank invests y units per capita in the short asset and $1 - y$ units per capita in the long asset and offers each consumer a deposit contract, which allows the consumer to withdraw either d_1 units at date 1 or d_2 units at date 2. Without loss of generality, we set $d_2 = \infty$. This ensures that consumers receive the residue of the bank's assets at date 2. Then the deposit contract is characterized by the middle-period payment $d_1 \equiv d$.

If $p(\theta)$ denotes the price of future consumption at date 1 in state θ , then the value of the bank's assets at date 1 is $y + p(\theta)r(1 - y)$.

A consumer's type is private information. An early consumer cannot misrepresent his type because he needs to consume at date 1; but a late consumer can claim to be an early consumer, withdraw d at date 1, store it until date 2, and then consume it. The deposit contract is *incentive-compatible* if and only if the residual payment to late consumers at date 2 is at least d . Because the late consumers are residual claimants

⁸This is not simply an application of the Modigliani–Miller theorem. The consumer may do strictly better by putting all his “eggs” in the bank's “basket”. Suppose that the deposit contract allows the individual to hold m units in the short asset and deposit $1 - m$ units in the bank. The bank invests y in the short asset and $1 - m - y$ in the long asset. If the bank does not default at date 1, then early consumers receive $d + m$ and late consumers receive

$$\frac{p(\theta)r(1 - m - y) + y - \eta(\theta)d}{(1 - \eta(\theta))p(\theta)} + m.$$

If the bank defaults, both early and late consumers receive

$$p(\theta)r(1 - m - y) + y + m.$$

Suppose that $m > 0$ and consider a reduction in m and an increase in y and d of the same amount. It is clear that early consumers' consumption is unchanged. So is late consumers' consumption if the bank defaults. The change in the late consumers' consumption when the bank does not default is

$$\frac{\Delta y - \eta(\theta)\Delta d}{(1 - \eta(\theta))p(\theta)} + \Delta m = \frac{-\Delta m}{p(\theta)} + \Delta m \geq 0$$

because $p(\theta) \leq 1$ and $\Delta m < 0$. Thus, it is optimal for the bank (and for the consumer) to choose $m = 0$.

⁹Note that an investor would like to diversify his holdings across intermediaries if, for example, there is a positive risk of default, so the assumption of no diversification does have bite. However, the equilibrium implications of allowing diversification are unclear because the objective function of the intermediary will change as a result of changing the composition of its customers, and so the behavior of the intermediary will also change. This remains an interesting question for future research.

at date 2, it is possible to give them at least d units of consumption if and only if

$$\eta(\alpha, \theta)d + (1 - \eta(\alpha, \theta))p(\theta)d \leq y + p(\theta)r(1 - y). \quad (1)$$

The left-hand side is a lower bound for the present value of consumption when early consumers are given d and late consumers are given at least d . The right-hand side is the value of the portfolio. Thus, condition (1) is necessary and sufficient for the deposit contract d to satisfy incentive compatibility and the budget constraint simultaneously. We often refer to the inequality in (1) as the *incentive constraint* for short.

In what follows, we assume that bank runs occur only if they are unavoidable. In other words, late consumers will withdraw at date 2 as long as the bank can satisfy the incentive constraint. If (1) is violated, then all consumers will want to withdraw at date 1. In the event of bankruptcy, the bank is *required* to liquidate its assets in an attempt to provide the promised amount d to the consumers who withdraw at date 1. Whatever withdrawal decisions consumers make, those who withdraw at date 2 will receive less than those who withdraw at date 1. Hence, in equilibrium, all consumers must withdraw at date 1. Then each consumer receives the liquidated value of the portfolio $y + p(\theta)r(1 - y)$.

Let $x_t(d, y, \alpha, \theta)$ denote the consumption at date t if the bank chooses (d, y) and the bank is in state (α, θ) at date 1. Let $x = (x_1, x_2)$, where:

$$x_1(d, y, \alpha, \theta) = \begin{cases} d & \text{if (1) is satisfied,} \\ y + p(\theta)r(1 - y) & \text{otherwise;} \end{cases}$$

$$x_2(d, y, \alpha, \theta) = \begin{cases} \frac{y + p(\theta)r(1 - y) - \eta d}{(1 - \eta)p(\theta)} & \text{if (1) is satisfied;} \\ y + p(\theta)r(1 - y) & \text{otherwise.} \end{cases}$$

Here $\eta = \eta(\alpha, \theta)$. Using this notation, the bank's decision problem can be written as

$$\begin{aligned} \max \quad & E[u(x(d, y, \alpha, \theta), \eta(\alpha, \theta))] \\ \text{s.t.} \quad & 0 \leq d, 0 \leq y \leq 1. \end{aligned} \quad (\text{DP1})$$

An ordered pair (d, y) is *optimal* for the given price function $p(\cdot)$ if it solves (DP1).

4 Equilibrium

The bank's decision problem is "nonconvex". To ensure the existence of equilibrium, we take advantage of the convexifying effect of large numbers and allow for the possibility that ex ante identical banks will choose different deposit contracts d and portfolios y . Each consumer is assumed to deal with a single bank, and each bank offers a single contract. In equilibrium, consumers will be indifferent between banks offering different contracts. Consumers allocate themselves to different banks in proportions consistent with equilibrium.

To describe an equilibrium, we need some additional notation. A *partition* of consumers at date 0 is defined by an integer $m < \infty$ and an array $\rho = (\rho_1, \dots, \rho_m)$ of numbers $\rho_i \geq 0$ such that $\sum_{i=1}^m \rho_i = 1$. Consumers are divided into m groups, and each group i contains a measure ρ_i of consumers. We impose an arbitrary bound m on the number of groups to rule out pathological cases.¹⁰ The banks associated with group i offer a deposit contract d_i and a portfolio y_i , both expressed in per-capita terms. An *allocation* consists of a partition (m, ρ) and an array $(d, y) = \{(d_i, y_i)\}_{i=1}^m$ such that $d_i \geq 0$ and $0 \leq y_i \leq 1$ for $i = 1, \dots, m$.

To define the market-clearing conditions we need some additional notation. Let $\bar{x}(d_i, y_i, \alpha, \theta) = (\bar{x}_1(d_i, y_i, \alpha, \theta), \bar{x}_2(d_i, y_i, \alpha, \theta))$ denote the bank's demand for goods. If the bank is solvent then a fraction $\eta(\alpha, \theta)$ of consumers are paid $x_1(d_i, y_i, \alpha, \theta)$ at date 1 and a fraction $(1 - \eta(d_i, y_i, \alpha, \theta))$ are paid $x_2(d_i, y_i, \alpha, \theta)$ at date 2. Then the bank's total demand for goods is given by

$$\bar{x}(d_i, y_i, \alpha, \theta) = (\eta(\alpha, \theta)x_1(d_i, y_i, \alpha, \theta), (1 - \eta(d_i, y_i, \alpha, \theta))x_2(d_i, y_i, \alpha, \theta)).$$

If the bank is bankrupt, on the other hand, then everyone gets paid $x_1(d_i, y_i, \alpha, \theta) = x_2(d_i, y_i, \alpha, \theta)$ at date 1 and the bank's total demand for goods is

$$\bar{x}(d_i, y_i, \alpha, \theta) = (x_1(d_i, y_i, \alpha, \theta), 0).$$

An allocation (m, ρ, d, y) is *attainable* if it satisfies the market-clearing conditions

$$\sum_i \rho_i E[\bar{x}_1(d_i, y_i, \alpha, \theta)] \leq \sum_i \rho_i y_i \tag{2}$$

and

¹⁰In general, two groups are sufficient for existence of equilibrium.

$$\sum_i \rho_i \{E[\bar{x}_1(d_i, y_i, \alpha, \theta) + \bar{x}_2(d_i, y_i, \alpha, \theta)]\} = \sum_i \rho_i \{y_i + r(1 - y_i)\}$$

for any state θ . In the market-clearing conditions, we take expectations with respect to α because the cross-sectional distribution of idiosyncratic shocks is assumed to be the same as the probability distribution. The first inequality says that the total demand for consumption at date 1 is less than or equal to the supply of the short asset. The inequality may be strict, because an excess supply of liquidity can be reinvested in the short asset and consumed at date 2. The second condition states that total consumption at date 2 is equal to the return from the investment in the long asset *plus* the amount invested in the short asset at date 1, which is the difference between the left- and right-hand sides of (2).

In equilibrium, it must be the case that $p(\theta) \leq 1$. Otherwise banks could make an arbitrage profit at date 1 by selling goods forward and investing the proceeds in the short asset. If $p(\theta) < 1$, then no one is willing to invest in the short asset at date 1 and so (2) must hold as an equation. A price function $p(\cdot)$ is *admissible* (for the given allocation) if it satisfies the following complementary slackness condition:

For any state θ , $p(\theta) \leq 1$ and $p(\theta) < 1$ imply that (2)
holds as an equation.

Now we are ready to define an equilibrium.

An *equilibrium* consists of an attainable allocation (m, ρ, d, y) and an admissible price function $p(\cdot)$ such that, for every group $i = 1, \dots, m$, (d_i, y_i) is optimal given the price function $p(\cdot)$.

5 A first look at equilibrium

In this section we use a simple parametric example to illustrate the properties of the model. We will see that relatively small shocks to liquidity demand have substantial effects on asset-price volatility and the possibility of default. Asset-price volatility and bank defaults have real effects on the equilibrium allocations and the welfare of individuals. Furthermore, these real effects remain substantial even as the liquidity shocks that cause them become vanishingly small. This is a clear example of financial fragility, defined as excess

sensitivity to small shocks (and, in the limit, to no shocks at all).

Recall that the fraction of early consumers in any bank is $\eta(\alpha, \theta) = \alpha + \varepsilon\theta$, where α is an idiosyncratic (bank-specific) shock and θ is an aggregate shock. For simplicity, we assume that the random variables α and θ have two-point supports. That is:

$$\alpha = \begin{cases} \alpha_H & \text{w. pr. } 0.5, \\ \alpha_L & \text{w. pr. } 0.5, \end{cases}$$

where $0 < \alpha_L < \alpha_H < 1$; and

$$\theta = \begin{cases} 0 & \text{w. pr. } \pi, \\ 1 & \text{w. pr. } (1 - \pi), \end{cases}$$

where $0 < \pi < 1$. Because there are only two values of θ , the price of future consumption at date 1 takes at most two values, $p(0)$ and $p(1)$.

Safe banks

Suppose that a bank chooses a deposit contract d and a portfolio y such that it never has to default in equilibrium. Then, in each state θ , the early consumers will receive the promised payment $c_1(\alpha, \theta) = d$ at date 1 and the late consumers will receive the residue of the bank's portfolio at date 2. The budget constraint implies that the consumption of the late consumers is

$$c_2(\alpha, \theta) = \frac{y + p(\theta)r(1 - y) - \eta(\alpha, \theta)d}{p(\theta)(1 - \eta(\alpha, \theta))}.$$

By assumption, the incentive constraint $c_2(\alpha, \theta) \geq d$ is satisfied in every state (α, θ) . Thus, the bank's decision problem is to choose the ordered pair (d, y) so as to

$$\begin{aligned} \max \quad & \mathbf{E} [\eta U(d) + (1 - \eta)U(c_2(\alpha, \theta))] \\ \text{s.t.} \quad & c_2(\alpha, \theta) \geq d \quad \forall (\alpha, \theta). \end{aligned}$$

In the sequel, we call a bank that never defaults a *safe bank* and denote the *safe strategy* it chooses by (d_S, y_S) .

Equilibrium without default

One of our objectives is to show that even a small amount of aggregate uncertainty, as represented by a small but positive value of ε , can have large effects on equilibrium values. Specifically, it can cause either high volatility of asset prices or a nonnegligible probability of default (or both). We begin by assuming that there is no default in equilibrium and show that this implies a high volatility of asset prices.

Examining the decision problem of the safe bank (just described), it is clear that this is a convex programming problem. Since the utility functions are strictly concave, the decision problem has a unique solution and so there is no loss of generality in assuming that all banks choose the same contract (d_S, y_S) .

Since α represents purely idiosyncratic risk, the fraction of early consumers in the economy is $\eta(\bar{\alpha}, \theta)$, where $\bar{\alpha} = (\alpha_H + \alpha_L)/2$. Then, at date 1, the total demand for consumption is $\eta(\bar{\alpha}, \theta)d$, the total supply is y , and the market-clearing condition requires that

$$\eta(\bar{\alpha}, \theta)d \leq y \quad \forall \theta,$$

with the complementary slackness condition

$$\eta(\bar{\alpha}, \theta)d < y \implies p(\theta) = 1.$$

By definition, $\eta(\bar{\alpha}, 0) < \eta(\bar{\alpha}, 1)$, so clearly the market-clearing condition $\eta(\bar{\alpha}, 1)d \leq y$ implies that $\eta(\bar{\alpha}, 0)d < y$. Then the complementary slackness condition implies that $p(0) = 1$.

Equilibrium requires that the short asset must be held at date 0; otherwise, there can be no consumption at date 1. But since $p(0) = 1$, the short asset will be dominated by the long asset at date 0 unless $p(1) < 1/r$. This shows that, in the absence of default, asset-price volatility is unavoidable in equilibrium. Furthermore, asset-price volatility is bounded away from zero for any value of $\varepsilon > 0$ however small. Thus, only a very small amount of aggregate uncertainty is needed to generate substantial price volatility.

The fact that all banks choose the same behavior is not essential to this argument. As long as there is no default, the same argument applies when there are multiple groups of banks. The essential feature of an equilibrium with no default is that demand for and supply of liquidity are both inelastic. Then a small shift in demand or supply can require a large change in the market-clearing price.

We may sum up the discussion so far as follows.

Default or Volatility: *In equilibrium with $\varepsilon > 0$, there must be either default or substantial asset-price volatility (or both).*

Risky banks

Avoiding default is costly. To satisfy the incentive constraint $c_2(\alpha, \theta) \geq d$ means either choosing a small value of d , which distorts the intertemporal consumption stream, or choosing a high value of y , which means foregoing the higher returns on the long-term asset. If the costs of avoiding default are too great, it will be optimal for the bank to choose a policy (d, y) that results in default with positive probability. So in practice, there may also be some banks that choose to default.

A bank defaults only if it violates the incentive constraint $c_2(\alpha, \theta) \geq d$ in some state. Since there are four possible combinations of the variables (α, θ) , there are several possible states in which a bank may default. We must consider all the possible combinations of default states in order to figure out which is the optimal contract for the bank. For the sake of illustration, suppose that (a) the dominant factor in the bank's decision is the variation in asset prices and (b) a bank finds it optimal to default if and only if $\theta = 1$ regardless of the value of α . Then, if $\theta = 0$, the consumption of early consumers is d and the consumption of late consumers is $c_2(\alpha, 0)$. If $\theta = 1$, then the bank defaults and everyone receives the liquidated value of the bank's portfolio $w(1) = y + p(1)r(1 - y)$. Thus, the decision problem of a bank anticipating default when $\theta = 1$ is to maximize

$$\pi \mathbf{E} [\alpha U(d) + (1 - \alpha)U(c_2(\alpha, 0))] + (1 - \pi)U(y + p(1)r(1 - y)),$$

subject to the incentive constraint

$$c_2(\alpha, 0) \geq d, \quad \alpha = \alpha_H, \alpha_L.$$

We call this bank a *risky bank* and denote the *risky strategy* by (d_R, y_R) .

Equilibrium with default

Given the assumption that the risky banks default in state $\theta = 1$ regardless of the value of α , there must be some banks that choose the safe strategy in equilibrium. Otherwise, all the banks in the economy will

default when $\theta = 1$ and all will have to liquidate their assets. There will be no solvent banks left to purchase assets and the asset price $p(1)$ will fall to zero. Clearly, this cannot be an equilibrium because, anticipating the price $p(1) = 0$, the banks would find it more profitable at date 0 to choose the safe strategy and make an unboundedly large arbitrage profit at date 1. So, under our assumption that all risky banks default in state $\theta = 1$, any equilibrium with default is of necessity a *mixed* equilibrium—that is, one with positive proportions of safe and risky banks.

To illustrate the possibility of a mixed equilibrium with two types of banks, one safe and the other risky, we consider a numerical example with the following parameters:

$$U(c) = \log_e c, \quad \varepsilon = 0.1, \quad \pi = 0.65, \quad (\alpha_H, \alpha_L) = (0.85, 0.75), \quad r = 1.5.$$

The equilibrium values are listed in Table 1.

Table 1

$\begin{bmatrix} E[U_S] \\ E[U_R] \end{bmatrix}$	$\begin{bmatrix} y_S \\ y_R \end{bmatrix}$	$\begin{aligned} &(d_S, c_2^S(\alpha_L, 0), c_2^S(\alpha_H, 0)) \\ &(d_S, c_2^S(\alpha_L, 1), c_2^S(\alpha_H, 1)) \\ &(d_R, c_2^R(\alpha_L, 0), c_2^R(\alpha_H, 0)) \\ &(w_R(1), w_R(1), w_R(1)) \end{aligned}$	$\begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$
$\begin{bmatrix} 0.077 \\ 0.077 \end{bmatrix}$	$\begin{bmatrix} 0.809 \\ 0 \end{bmatrix}$	$\begin{aligned} &(0.995, 1.414, 1.624) \\ &(0.995, 1.700, 1.366) \\ &(1.360, 1.502, 1.502) \\ &(0.678, 0.678, 0.678) \end{aligned}$	$\begin{bmatrix} 0.940 \\ 0.430 \end{bmatrix}$

Depositors must be indifferent between the safe and the risky bank in equilibrium. The first column of Table 1 shows that the two types of banks yield the same expected utility, as required. The second column shows that the safe bank holds a large amount of liquidity in the form of the short asset, whereas the risky bank goes to the other extreme and holds no liquidity at all.

The safe bank gives early consumers the same consumption d_S in all states; late consumers face risk. When the idiosyncratic shock is low ($\alpha = \alpha_L$), the safe bank has excess liquidity and can make profits buying up the long asset at a low price in state $\theta = 0$. Hence late consumers do better when the price is low ($\theta = 1$) than when it is high ($\theta = 0$). When the idiosyncratic shock is high ($\alpha = \alpha_H$), the safe bank has insufficient liquidity and must sell the long asset to meet the demands of early consumers. So the late consumers in the safe bank do better when the price is high ($\theta = 0$) than when it is low ($\theta = 1$).

Depositors in the risky bank generally do well when the price is high ($\theta = 0$). The early consumers receive the constant amount d_R and the late consumers bear the risk of variations in α . When the price is low ($\theta = 1$), the bank defaults and all depositors receive the liquidated value of the bank's portfolio, which is low because of the fall in the asset price.

Notice that, although the price is high when $\theta = 0$, it does not equal unity—as it would if there were no default in $\theta = 1$. Default introduces some elasticity into the demand for liquidity, so the quantity of liquidity demanded can be the same in both states.

Financial fragility

So far, we have illustrated an equilibrium in which there is high asset-price volatility and a high probability of default. Is this the same thing as “financial fragility”? We think of financial fragility as meaning that small shocks can produce a large effect on the system. If it takes very large shocks to produce these effects, the financial system would be robust rather than fragile. But what is large? And what is small? The usual way to establish that small shocks have large consequences is to let ε become vanishingly small and then show that the impact of aggregate uncertainty does not disappear in the limit.

We have already shown that, in the absence of default, there must be high asset-price volatility for any value of $\varepsilon > 0$. Thus, we must have either high asset-price volatility or default (or both) when $\varepsilon > 0$. We can use this fact to demonstrate that the asset-price volatility does not disappear as $\varepsilon \rightarrow 0$. Therefore, in the limit as $\varepsilon \rightarrow 0$, the consequences of small shocks become disproportionately large and we have an example of financial fragility. The proof of this result is by contradiction. Suppose, contrary to what we want to prove, that asset-price volatility becomes vanishingly small as $\varepsilon \rightarrow 0$. By our earlier result (Default or Volatility),

this implies that there must be default in equilibrium for arbitrarily small values of $\varepsilon \rightarrow 0$. Then we can find a subsequence of equilibria, corresponding to values of $\varepsilon \rightarrow 0$, such that (i) asset prices converge to a constant and (ii) it is optimal to choose default in each of the equilibria in the sequence. In Section 6 we show that the limit of equilibria as $\varepsilon \rightarrow 0$ is an equilibrium of the limit economy. We will also show that $p(0) = p(1) = 1/r$ if there is no volatility in the limit equilibrium. Note that, if it is optimal for a bank to choose default when $\varepsilon > 0$, it will be optimal in the limit. However, if there is no asset-price volatility, then the only reason for defaulting is because α is too high. Comparing the utility of the safe and the risky bank, we can check that the risky bank's strategy is not optimal. This contradicts our claim that default is optimal as $\varepsilon \rightarrow 0$, and this contradiction forces us to conclude that asset-price volatility does not disappear in the limit.

The financial fragility of equilibrium as $\varepsilon \rightarrow 0$ is really the central result of this paper, but the interpretation of this result requires some care. What we have established so far is that small liquidity shocks cause a large amount of asset-price volatility. This does not necessarily imply that small liquidity shocks have a large impact on the allocation of consumption and risk and hence on expected utility. In fact, the impact of price variations on consumption and risk depends on the idiosyncratic shocks α . Examining the bank's budget constraints (above), it is clear that a change in asset prices has a wealth effect if and only if there is an imbalance between the bank's supply of liquidity y and the depositors' demand for liquidity $\eta(\alpha, \theta)d$. If α is random, then for any choice of d and y there is a positive probability that $\alpha d \neq y$ and prices have a wealth effect.

By contrast, if α is a nonstochastic constant then, in the limit as $\varepsilon \rightarrow 0$, the bank can achieve the optimal expected utility while remaining autarkic. Price changes have no effect because the bank does not have a net excess demand for liquidity at date 1.

Another point to note about equilibrium in the limit as $\varepsilon \rightarrow 0$ is that, although it is optimal for banks to default, none of the banks chooses to do so. In other words: when α is nonstochastic, the proportion of risky banks converges to zero. This is a very special case, but it warns us against jumping to conclusions about the incidence of default in equilibrium.

The following result sums up our discussion.

Financial Fragility: *Asset-price volatility is bounded away from zero in the limit as $\varepsilon \rightarrow 0$.*

The effect of volatility depends on the presence of idiosyncratic risk: this asset price volatility has an effect on the allocation of consumption and risk in the limit if and only if α is stochastic.

Robustness of financial fragility

In addition to showing that small shocks can have large consequences, we can use the analysis of the limiting behavior of equilibria to test the robustness of certain other kinds of equilibria.

In the limit economy, where $\varepsilon = 0$, there is no aggregate uncertainty and yet the endogenous variables such as prices exhibit aggregate (extrinsic) uncertainty. This is nothing but a sunspot equilibrium of the limit economy. We can therefore think of our analysis of what happens in the limit as providing some justification for sunspot equilibria. The limit economy has other kinds of equilibria, but these are not the limit of equilibria as $\varepsilon \rightarrow 0$. In other words, financial fragility is a robust phenomenon.

The presence of multiple equilibria of different types makes the analysis of what happens in the limit technically difficult. We offer a classification of these equilibria in the following section.

6 Analysis

In this section we proceed to characterize the set of equilibria of the model by considering several different cases. We begin by analyzing the equilibria of the limit economy with no aggregate uncertainty ($\varepsilon = 0$), and here we distinguish two subcases. In Section 6.2, we consider the economy in which there are no idiosyncratic or bank-specific shocks to liquidity demand. This is the case where α is nonstochastic. Then, in Section 6.3, we consider the case where banks are subject to idiosyncratic shocks to liquidity demand. This is the case where α is stochastic. The difference between these two cases is that asset-price fluctuations necessarily have real effects if and only if α is stochastic. Then we consider economies with aggregate uncertainty ($\varepsilon > 0$). Our two main results, Default or Volatility (Theorem 3) and Financial Fragility (Theorem 5), are contained in Section 6.4.

6.1 Equilibria with $\varepsilon = 0$

In this section we first characterize the equilibria of the limit economy in which $\varepsilon = 0$. There is no aggregate intrinsic uncertainty in the model, but there may still be aggregate extrinsic uncertainty (sunspots). We first classify equilibria in the limit economy according to the impact of extrinsic uncertainty. An equilibrium (m, ρ, d, y, p) in the limit economy is a *fundamental equilibrium* (F) if $x(d_i, y_i, \alpha, \theta)$ is constant for each i and α and if $p(\theta)$ is constant. In that case, the sunspot variable θ has no influence on the equilibrium values. An equilibrium (m, ρ, d, y, p) of the limit economy is a *trivial sunspot equilibrium* (T) if $x(d_i, y_i, \alpha, \theta)$ is constant for each i and α and if $p(\theta)$ is not constant. In this case, the sunspot variable θ has no effect on the allocation of consumption, but it does affect the equilibrium price $p(\theta)$. An equilibrium (m, ρ, d, y, p) that is neither an F nor a T is called a *nontrivial sunspot equilibrium* (N); that is, an N is an equilibrium in which the sunspot variable has some nontrivial impact on the allocation of consumption.

We can also classify equilibria according to the variety of choices made by different groups of banks. An equilibrium (m, ρ, d, y, p) is *pure* if each group of banks makes the same choice:

$$(d_i, y_i) = (d_j, y_j) \quad \forall i, j = 1, \dots, m.$$

An equilibrium (m, ρ, d, y, p) is *semi-pure* if the consumption allocations are the same for each group of banks:

$$x(d_i, y_i, \alpha, \theta) = x(d_j, y_j, \alpha, \theta) \quad \forall i, j = 1, \dots, m.$$

Otherwise, (m, ρ, d, y, p) is a *mixed* equilibrium where different types of banks provide different allocations (but the same ex ante expected utility). The role of mixed equilibria is to ensure the existence of equilibrium in the presence of nonconvexities.

6.2 Equilibrium with $\varepsilon = 0$ and nonstochastic α

In the special case with α constant, the following theorem partitions the equilibrium set into two cases with distinctive properties.

Theorem 1 *Suppose that α is constant, and let (ρ, m, x, y, p) be an equilibrium of the limit economy in*

(i) (ρ, m, x, y, p) is a semi-pure, fundamental equilibrium F

in which the probability of default is zero; or

which $\varepsilon = 0$. There are two possibilities:

(ii) (ρ, m, x, y, p) is a pure, trivial sunspot equilibrium T

in which the probability of default is zero.

Proof. See Section 8. ■

By definition, an equilibrium must be either an F , T , or N . What Theorem 1 shows is that an N does not occur, and each of the remaining cases is associated with distinctive properties in terms of symmetry and probability of default.

Because the incentive constraint is not binding in either the F or T , both achieve the first-best or Pareto-efficient allocation. No equilibrium can do better. Any bank can guarantee this level of utility by choosing $\alpha d_i = y_i$, where d_i is the deposit contract chosen in the F . For this choice of (d_i, y_i) , prices have no effect on the bank's budget constraint and the depositors will receive the first-best consumption. In an N , by contrast, agents receive noisy consumption allocations. Because these agents are risk-averse, the noise in their consumption allocations is inefficient. The equilibrium condition that depositors must be indifferent between the two types of banks is violated, since an individual bank can choose an allocation with the same mean consumption and no risk.

6.3 Equilibrium with $\varepsilon = 0$ and stochastic α

In economies with stochastic α , individual banks face idiosyncratic risk. For some choices of (d_i, y_i) a bank may be forced to default in some states. More importantly, the bank may find it optimal to choose (d_i, y_i) so that default occurs with positive probability, because the costs of avoiding default are greater than the benefits. In order to distinguish crises caused by aggregate extrinsic uncertainty from defaults caused by idiosyncratic shocks, in the sequel *we assume that the parameters of the model are such that default is never optimal in the F* . In that case, the bank's optimal choice of (d_i, y_i) must satisfy the incentive constraint, so the bank's optimal decision problem can be written as follows. At the equilibrium price $\bar{p} = 1/r$, the value

of the bank's assets at date 1 is $y_i + \bar{p}r(1 - y_i) = 1$, independently of the choice of y_i . In the absence of default, the budget constraint implies that the consumption at date 2 is given by $r(1 - \alpha d_i)/(1 - \alpha)$. The incentive constraint requires that $r(1 - \alpha d_i)/(1 - \alpha) \geq d_i$. Thus, the decision problem can be written as:

$$\begin{aligned} \max \quad & E[\alpha U(d_i) + (1 - \alpha)U(r(1 - \alpha d_i)/(1 - \alpha))] \\ \text{s.t.} \quad & r(1 - \alpha d_i)/(1 - \alpha) \geq d_i, \forall \alpha. \end{aligned}$$

This is a convex programming problem that has a unique solution for d_i . As noted, y_i is indeterminate, but the equilibrium allocation must satisfy the market-clearing condition

$$\sum_i \rho_i \bar{\alpha} d_i = \sum_i \rho_i y_i,$$

where $\bar{\alpha} = E[\alpha]$. Note that there is a single *pure* F (ρ, m, d, y, \bar{p}) , in which $y_i = E[\alpha d_i]$ for every $i = 1, \dots, m$.

In the case of idiosyncratic shocks, the following theorem partitions the equilibrium set into two cases, F and N.

Theorem 2 *Let (ρ, m, d, y, p) be an equilibrium of the limit economy, in which α is stochastic and $\varepsilon = 0$.*

(i) (ρ, m, d, y, p) is a semi-pure, fundamental equilibrium F

in which the probability of default is zero; or

There are two possibilities:

(ii) (ρ, m, d, y, p) is a non-trivial sunspot equilibrium N

that is pure if the probability of default is zero.

Proof. See Section 8. ■

The fundamental equilibrium is semi-pure for the usual reasons, and there is no default by assumption. Unlike the case with no idiosyncratic shocks (nonstochastic α), there can be no trivial sunspot equilibrium. To see this, suppose that some bank group i with measure ρ_i chooses (d_i, y_i) and that default occurs with positive probability. Then consumption for both early and late consumers is equal to $y_i + p(\theta)r(1 - y_i)$, which is independent of $p(\theta)$ only if $y_i = 1$. In that case, there is no point in choosing $d > 1$ and hence no need for default. If there is no probability of default, then the consumption of the late consumers is

$$\frac{y_i + p(\theta)r(1 - y_i) - \alpha d_i}{(1 - \alpha)p(\theta)}.$$

This is independent of $p(\theta)$ only if $y_i = \alpha d_i$, which cannot hold unless α is a constant. Thus, there can be no T.

The only remaining possibility is an N. If the probability of default is zero, then the bank's decision problem is a convex programming problem and the usual methods suffice to show uniqueness of the optimum choice of (d, y) under the maintained assumptions.

The properties of equilibria in the limit economy when α is stochastic are summarized in the following table.

Symmetry	Type	Default
Semi-pure	F	No
Pure	T	No
Mixed or pure	N	Possible

6.4 The limit of equilibria as $\varepsilon \searrow 0$

According to the definition of equilibrium in Section 4, prices $p(\theta)$ and consumption $x(\cdot, \theta)$ are functions of θ . When $\varepsilon = 0$, a change in θ has no effect on preferences, so any dependence of $p(\theta)$ or $x(\cdot, \theta)$ on θ represents extrinsic uncertainty. When $\varepsilon > 0$, on the other hand, a change in θ has an effect on preference, so any dependence on θ of either $p(\theta)$ or $x(\cdot, \theta)$ represents intrinsic uncertainty. Thus, an equilibrium for an economy with aggregate uncertainty, $\varepsilon > 0$, is by definition “fundamental” in the sense that endogenous variables depend only on real exogenous shocks. To avoid confusion with the fundamental equilibrium in the limit economy with $\varepsilon = 0$, we use the letter A to denote equilibria with *aggregate uncertainty*.

The next theorem characterizes the properties of equilibria in perturbed economies.

Theorem 3 DEFAULT OR VOLATILITY *Let (m, ρ, d, y, p) denote an A. Then either there is a positive probability of default or there is nontrivial price volatility; that is, $p(\theta) = 1$ with positive probability and $p(\theta) < 1/r$ with positive probability.*

Proof. See Section 8. ■

We note several other properties of equilibrium with $\varepsilon > 0$. First, there cannot be a state in which all banks are in default, for this would imply $p(\theta) = 0$, which is inconsistent with equilibrium. Hence, any equilibrium with default must be mixed. Second, as we have seen, if there is no default then there must be a positive probability that $p(\theta) = 1$. Finally, in the absence of default: for any value of $\varepsilon > 0$, the volatility of asset prices, as measured by the variance of $p(\cdot)$, is bounded away from zero. Both assets are held at date 0 in equilibrium and this requires that the high returns to the long asset, associated with $p(\theta) = 1$, must be balanced by low returns associated with $p(\theta) < 1/r$. These properties are all preserved in the limit as $\varepsilon \rightarrow 0$.

The next result shows that the limit of a sequence of equilibria is an equilibrium in the limit.

Theorem 4 *Consider a sequence of perturbed economies corresponding to $\varepsilon = 1/q$, where q is a positive integer, and let $(m^q, \rho^q, d^q, y^q, p^q)$ be the corresponding As. For some convergent subsequence $q \in Q$, let*

$$(m^0, \rho^0, d^0, y^0, p^0) = \lim_{q \in Q} (m^q, \rho^q, d^q, y^q, p^q).$$

If $d_i^0 > 0$ and $0 < y_i^0 < 1$ for $i = 1, \dots, m^0$, then $(m^0, \rho^0, x^0, y^0, p^0)$ is an equilibrium of the limit economy.

Proof. See Section 8. ■

Theorem 4 shows that, under certain conditions, the limit of a sequence of equilibria as $\varepsilon \rightarrow 0$ is an equilibrium of the limit economy where $\varepsilon = 0$. We are also interested in the opposite question—namely, which equilibria of the limit economy in which $\varepsilon = 0$ are limits of equilibria from the perturbed economy in which $\varepsilon > 0$? This requirement of lower hemi-continuity is a test of *robustness*: if a small perturbation of the limit economy causes an equilibrium to disappear, we argue that the equilibrium is not robust. Since there are many equilibria of the limit economy, it is of interest to see whether any of these equilibria can be eliminated by being shown to be nonrobust.

We have shown in Theorem 3 that equilibria of the perturbed economy are characterized by default or nontrivial asset-price uncertainty. Furthermore, because the random variable θ has a finite support, the probability of these events is bounded away from zero, uniformly in ε . The fundamental equilibrium of the limit economy with $\varepsilon = 0$ has none of these properties. However, this does not by itself prove that the fundamental equilibrium is not robust. It could be the limit of a sequence of equilibria of the perturbed

economy if the fraction of banks that defaults in equilibrium converges to zero as $q \rightarrow \infty$. However, if the fundamental equilibrium were the limit referred to in Theorem 4, then it must be the case that (a) default is optimal in the limit and (b) there is no price volatility in the limit. These two properties can be shown to be inconsistent. This contradiction shows that asset-price volatility is bounded away from zero in the limit as the shocks become vanishingly small. In other words, the model exhibits financial fragility. As we have seen already, this asset-price volatility has real effects if and only if there is idiosyncratic risk (iff α is stochastic).

Theorem 5 FINANCIAL FRAGILITY *If $(m^0, \rho^0, x^0, y^0, p^0)$ is the equilibrium of the limit economy mentioned in Theorem 4, then $(m^0, \rho^0, x^0, y^0, p^0)$ is not an F of the limit economy. In fact, the limiting equilibrium $(m^0, \rho^0, x^0, y^0, p^0)$ will be a T if α is a constant or an N if α is a non-degenerate random variable.*

Proof. Suppose that, contrary to what we wish to prove, $(m^0, \rho^0, x^0, y^0, p^0)$ is the fundamental equilibrium. Then $p^0(\theta) = 1/r < 1$ with probability 1 and hence $p^q(\theta) < 1$ for all θ and all q sufficiently large. From Theorem 3 we know that this implies some group i defaults with positive probability for all sufficiently large q . Thus, in the limit, default must be optimal for group i and $\rho_i^q \rightarrow \rho_i^0 = 0$. However, we assumed in Section 6.3 that default is not optimal for stochastic α , and it is clear that it is not optimal with nonstochastic α because

$$U(1) < \max_{\alpha c_1 + (1-\alpha)c_2/r=1} \{\alpha U(c_1) + (1-\alpha)U(c_2)\}.$$

This contradiction proves that $(m^0, \rho^0, x^0, y^0, p^0)$ is not an F, i.e., the F is not robust. This implies that the limiting equilibrium is either an N or a T. Which case obtains depends on the idiosyncratic shocks α .

If α is a constant, the first best can be achieved by an autarkic bank when $\varepsilon = 0$, and this implies that the equilibrium allocation must be nonstochastic in the limit. So $(m^0, \rho^0, x^0, y^0, p^0)$ is a T if α is a constant.

Consider then the case where α is a nondegenerate random variable. For each n , either asset-price volatility is bounded away from zero (uniformly in n) or default is optimal. Suppose, contrary to what we want to prove, that $(m^0, \rho^0, x^0, y^0, p^0)$ is a T. Then the limiting allocation is non-stochastic. This implies that $p^0(\theta) = 1/r$ with probability 1 and that default is optimal in the limit. But we have assumed that variations in α are not sufficient to make default optimal at this price. This contradiction implies that p^0 is not a

constant. When α is random, any changes in prices have real effects so the equilibrium $(m^0, \rho^0, x^0, y^0, p^0)$ cannot be a T—that is, it must be an N. ■

Thus, we have shown that F is not robust (not the limit of a sequence of equilibria of the perturbed economy) and so conversely, there must be uncertainty in any robust equilibrium. Hence this approach of regarding sunspot equilibria as a limiting case selects the sunspot equilibria as the only robust equilibria. Furthermore, the intrinsic uncertainty is nontrivial (i.e., has real effects) except in the special case where α is a constant.

Figure 1 illustrates these results. The key issue is whether banks face idiosyncratic liquidity risk. If they do then α is stochastic and, as shown in Figure 1A, the only robust equilibria in the limit are nontrivial sunspot equilibria N, where the allocation of consumption is random. The reason for this is that prices have a volatility that is bounded away from zero and banks must trade at these prices if they face idiosyncratic liquidity risk. As a result, the extrinsic aggregate uncertainty has a real effect. In contrast, if banks do not face idiosyncratic liquidity risk so α is nonstochastic, then the banks can be autarkic and do not need to trade. In the limit, as shown in Figure 1B, the equilibrium is a T and has a non-random consumption allocation even though prices are varying.

Corresponding types of equilibria

— Place Figure 1 here —

7 Discussion

In this paper, we have defined financial fragility in terms of excess sensitivity to small shocks. We showed that small shocks can have large effects on prices and that this may lead banks to default. How small can the shocks be? This question led us to investigate the relationship between intrinsic and extrinsic uncertainty. Our general approach is to regard extrinsic uncertainty as a limiting case of intrinsic uncertainty. In our model, small shocks to the demand for liquidity are always associated with large fluctuations in asset prices. In this sense, the system is financially fragile. In the limit, as the liquidity shocks become vanishingly small,

the model converges to one with extrinsic uncertainty.

The limit economy has three kinds of equilibria:

- fundamental equilibria F , in which there is neither aggregate uncertainty nor a positive probability of crisis;
- trivial sunspot equilibria T , in which prices fluctuate but the real allocation is the same as in the fundamental equilibrium; and
- nontrivial sunspot equilibria N , in which prices fluctuate and financial crises can occur with positive probability.

Introducing small shocks into the limit economy destabilizes the first type of equilibrium, leaving the second and third as possible limits of equilibria of the perturbed economy. If α is a constant, then the limiting equilibrium as $\varepsilon \rightarrow 0$ is a trivial sunspot equilibrium in which price fluctuations have no real effects. If α is random, the limiting equilibrium as $\varepsilon \rightarrow 0$ is a non-trivial sunspot equilibrium, in which price fluctuations have a real effect on consumption and expected utility. We argue that only the sunspot equilibria are robust, in the sense that a small perturbation of the model causes a small change in these equilibria. This selection criterion provides an argument for the relevance of extrinsic uncertainty and the necessity of financial crises.

At the heart of our theory is a pecuniary externality: when one group of banks defaults and liquidates its assets, it forces down the price of assets and this may cause another group of banks to default. This pecuniary externality may be interpreted as a form of *contagion* that operates through the interbank asset market. Allen and Gale (2000a) describes a model of contagion in a multi-region economy where the nature of the contagion is rather different. Bankruptcy is assumed to be costly: long-term projects can be liquidated prematurely, but a fraction of the returns are lost. This deadweight loss from liquidation creates a spillover effect in the adjacent regions where the claims on the bankrupt banks are held. If the spillover effect is large enough, the banks in the adjacent regions will also be forced into default and liquidation. Each successive wave of bankruptcies increases the loss of value and strengthens the impact of the spillover effect on the next region. Under certain conditions, a shock to one small region can propagate throughout the economy. By

contrast, in the present model, a bank's assets are always marked to market. Given the equilibrium asset price p , bankruptcy does not change the value of the bank's portfolio. However, if a group of banks defaults, the resulting change in the price p may cause other banks to default, which will cause further changes in p , and so on. The "contagion" in both models is instantaneous.

Several features of the model are special and deserve further consideration.

Pecuniary externalities "matter" in our model because markets are incomplete: if banks could trade Arrow securities contingent on the states θ , they would be able to insure themselves against changes in asset values (AG). With a complete set of Arrow securities, risk sharing must be efficient, so in the absence of intrinsic uncertainty ($\varepsilon = 0$) the only possible equilibrium is the one we have called the fundamental equilibrium F. The equilibrium allocation is incentive-efficient, sunspots have no real impact, and there are no crises. Note that, although the existence of the markets for Arrow securities has an effect (by eliminating the other equilibria), there is no trade in Arrow securities in the fundamental equilibrium. Arrow securities are, of course, a convenient fiction that do not exist in reality, but it may be that other derivatives, such as options, would serve as well. However, derivatives are conditional on $p(\theta)$ rather than directly on θ , so it is not obvious how many derivatives or what kind would be required. This is an interesting topic for future research.

We have noted the importance of inelastic demand for liquidity in generating large fluctuations in asset prices from small demand shocks. The inelasticity of demand follows two assumptions. The first is the assumption that banks use demand deposits, which do not allow the payment at date 1 to be contingent on demand (or anything else). The second is the assumption of Diamond–Dybvig preferences, which rule out intertemporal substitution of consumption. We see both the Diamond–Dybvig preferences and the use of demand deposits as a counterpart to the empirical fact that financial contracts are typically written in a "hard" way that requires strict performance of precisely defined acts, independently of many apparently relevant contingencies. These hard contracts may be motivated by enforcement and incentive problems, but it would be too difficult to include them explicitly in the model. There seems little doubt that such factors are relevant in real markets and should be taken into account here.

An alternative justification for incomplete contracts is that they provide a way of modeling— within the standard, Walrasian, auction-market framework— some realistic features of alternative market-clearing mechanisms. In an auction market, prices and quantities adjust simultaneously in a *tatônement* process until a full equilibrium is achieved. An alternative mechanism is one in which quantities are chosen before prices are allowed to adjust. An example is the use of market orders in markets for company stocks. In the banking context, if depositors were required to make a withdrawal decision before the asset price was determined in the interbank market, then the same inelasticity of demand would be observed even if depositors had preferences that allowed for intertemporal substitution. There may be other institutional structures that have the qualitative features of our example. An investigation of these issues goes far beyond the scope of the present paper, but it is undoubtedly one of the most important topics for future research.

Our theory of banking is also rather special. Two assumptions should be noted. We have assumed that each bank offers a single type of contract and that each consumer can deal with only one bank. These assumptions are restrictive because they prevent banks and consumers from taking advantage of opportunities for diversification. For example, a consumer would like to hedge some of the idiosyncratic risk associated with a deposit contract by spreading his claims across several banks. As we have already noted, the pecuniary externalities associated with asset-price volatility have an impact on welfare only if there is some friction that prevents Pareto-efficient sharing of risk. Just as we assumed incomplete markets (no Arrow securities) in order to ensure that risk sharing was not efficient, we need to assume some limits on the ability of consumers and banks to contract and hedge risks. The assumptions adopted here can probably be relaxed somewhat, but if we want nontrivial crises then there must be some friction that prevents the economy from achieving the first-best outcome.

In this paper we have ignored the possibility of intervention by the central bank or government. A full understanding of the *laissez-faire* case should be seen as a prelude to the analysis of optimal intervention. In the same way, the analysis of a “real” model is a prelude to the introduction of fiat money into the model. These are both important topics for future research.

8 Proofs

8.1 Proof of Theorem 1

By definition, an equilibrium (ρ, m, d, y, p) must be either an F, T, or N. The theorem is proved by considering each case in turn.

Case (i). If (ρ, m, d, y, p) is an F then by definition the price $p(\theta)$ is constant and, for each group i , the consumption allocation $x(d_i, y_i, \alpha, \theta)$ is constant. In particular, $x_1(d_i, y_i, \alpha, \theta) = d_i$ with probability 1 so there is no default in equilibrium.

Let \bar{p} denote the constant price and let $c_i = (c_{i1}, c_{i2})$ be the consumption allocation chosen by banks in group i . The decision problem of a bank in group i is

$$\begin{aligned} \max \quad & [\alpha U(c_{i1}) + (1 - \alpha)U(c_{i2})] \\ \text{s.t.} \quad & c_{i1} \leq c_{i2}, \quad 0 \leq y_i \leq 1, \\ & \alpha c_{i1} + (1 - \alpha)\bar{p}c_{i2} \leq y_i + \bar{p}r(1 - y_i). \end{aligned}$$

Clearly, y_i will be chosen to maximize $y_i + \bar{p}r(1 - y_i)$. Then the strict concavity of $U(\cdot)$ implies that c_i is uniquely determined and independent of i . Thus, the equilibrium is semi-pure: $c_i = c_j$ for any i and j .

Case (ii). Suppose that (ρ, m, d, y, p) is a T. Then by definition $p(\theta)$ is not constant and, for each group i , the consumption allocation $x(d_i, y_i, \alpha, \theta)$ is constant. In particular, $x_1(d_i, y_i, \alpha, \theta) = d_i$ with probability 1 and so there is no default in equilibrium.

Let c_i denote the consumption allocation chosen by banks in group i . The budget constraint at date 1 reduces to

$$\alpha c_{i1} - y_i = -p(\theta)((1 - \alpha)\alpha c_{i2} - r(1 - y_i)) \text{ a.s.}$$

Since $p(\theta)$ is not constant, this equation can be satisfied only if

$$\alpha c_{i1} - y_i = (1 - \alpha)\alpha c_{i2} - r(1 - y_i) = 0.$$

Then the choice of (c_i, y_i) must solve the problem

$$\begin{aligned} \max \quad & E [\alpha U(c_{i1}) + (1 - \alpha)U(c_{i2})] \\ \text{s.t.} \quad & c_{i1} \leq c_{i2}, \quad 0 \leq y_i \leq 1, \\ & \alpha c_{i1} = y_i, \quad (1 - \alpha)c_{i2} = r(1 - y_i). \end{aligned}$$

The strict concavity of $U(\cdot)$ implies that this problem uniquely determines the value of c_i and hence y_i , independently of i . Consequently, the equilibrium is pure.

Case (iii). Suppose that (ρ, m, d, y, p) is an N. Then the allocation of consumption for group i is $x(d_i, y_i, \alpha, \theta)$, and the expected utility of each group is the same

$$E [u(x(d_i, y_i, \alpha, \theta), \alpha)] = E [u(x(d_j, y_j, \alpha, \theta), \alpha)] \quad \forall i, j.$$

The mean allocation $\sum_i \rho_i x(d_i, y_i, \alpha, \theta)$ satisfies the market-clearing conditions for every θ so consumption bundle $E [\sum_i \rho_i x(d_i, y_i, \alpha, \theta)]$ is feasible for the planner. Since agents are strictly risk-averse, it follows that

$$u \left(E \left[\sum_i \rho_i x(d_i, y_i, \alpha, \theta) \right], \alpha \right) > \sum_i \rho_i E [u(x(d_i, y_i, \alpha, \theta), \alpha)].$$

This contradicts the equilibrium conditions, since the individual bank could choose

$$y_0 = \alpha d_0 = \alpha E \left[\sum_i \rho_i x(d_i, y_i, \alpha, \theta) \right]$$

and achieve a higher utility.

8.2 Proof of Theorem 2

Again we let (ρ, m, d, y, p) be a fixed but arbitrary equilibrium and consider each of three cases in turn.

Case (i). If (ρ, m, d, y, p) is an F then $p(\theta)$ is constant and, for each group i and each α , the consumption allocation $x(d_i, y_i, \alpha, \theta)$ is constant. In particular, $x_1(d_i, y_i, \alpha, \theta) = d_i$ with probability 1, so there is no default in equilibrium.

Let \bar{p} denote the constant price and let $c_i(\alpha) = (c_{i1}(\alpha), c_{i2}(\alpha))$ be the consumption allocation chosen by

banks in group i . The decision problem of a bank in group i is

$$\begin{aligned} \max \quad & E[\alpha U(c_{i1}(\alpha)) + (1 - \alpha)U(c_{i2}(\alpha))] \\ \text{s.t.} \quad & c_{i1}(\alpha) \leq c_{i2}(\alpha), \quad 0 \leq y_i \leq 1, \\ & \alpha c_{i1}(\alpha) + (1 - \alpha)\bar{p}c_{i2}(\alpha) \leq y_i + \bar{p}r(1 - y_i). \end{aligned}$$

Clearly, y_i will be chosen to maximize $y_i + \bar{p}r(1 - y_i)$. Then the strict concavity of $U(\cdot)$ implies that $c_i(\alpha)$ is uniquely determined and independent of i (but not of α). Thus, the equilibrium is semi-pure: $c_i = c_j$ for any i and j .

Case (ii). Suppose that (ρ, m, d, y, p) is a T. Then by definition $p(\theta)$ is not constant and, for each group i and α , the consumption allocation $x(d_i, y_i, \alpha, \theta)$ is constant. In particular, $x_1(d_i, y_i, \alpha, \theta) = d_i$ with probability 1 and so there is no default in equilibrium.

Let $c_i(\alpha)$ denote the consumption allocation chosen by banks in group i . The budget constraint at date 1 reduces to

$$\alpha c_{i1}(\alpha) - y_i = -p(\theta)((1 - \alpha)\alpha c_{i2}(\alpha) - r(1 - y_i)) \text{ a.s.}$$

Since $p(\theta)$ is not constant, this equation can be satisfied only if

$$\alpha c_{i1}(\alpha) - y_i = (1 - \alpha)c_{i2}(\alpha) - r(1 - y_i) = 0.$$

Since α is not constant, this can only be true if $c_{i1}(\alpha) = c_{i2}(\alpha) = 0$ —contradiction. Thus, there cannot be a T when α is not constant.

Case (iii). The only remaining possibility is that (ρ, m, d, y, p) is an N. If there is no default in this equilibrium, then each bank in group i solves the problem

$$\begin{aligned} \max \quad & E \left[\alpha U(d_i) + (1 - \alpha)U \left(\frac{y_i + p(\theta)r(1 - y_i) - \alpha d_i}{(1 - \alpha)p(\theta)} \right) \right] \\ \text{s.t.} \quad & \frac{y_i + p(\theta)r(1 - y_i)}{(1 - \alpha)p(\theta)} \geq d_i. \end{aligned}$$

This is a convex programming problem, and it is easy to show that the strict concavity of $U(\cdot)$ uniquely determines (d_i, y_i) . Thus, an N without default is pure.

8.3 Proof of Theorem 3

Suppose that the probability of default in (ρ, m, d, y, p) is zero. Then, for each group i , $x_1(d_i, y_i, \alpha, \theta) = d_i$ and the market-clearing condition (2) implies

$$\sum_i E[\rho_i \eta(\alpha, \theta) d_i] \leq \sum_i \rho_i y_i. \quad (3)$$

There are two cases to consider. In the first case, $\sum_i \rho_i y_i = 0$. Then $d_i = 0$ for every i , and the utility achieved in equilibrium is

$$E \left[\eta(\alpha, \theta) U(0) + (1 - \eta(\alpha, \theta)) U \left(\frac{r}{1 - \eta(\alpha, \theta)} \right) \right].$$

By holding a small amount $\delta > 0$ of the short asset, positive consumption could be guaranteed at the first date. Optimality requires that

$$\begin{aligned} & E \left[\eta(\alpha, \theta) U \left(\frac{\delta}{\eta(\alpha, \theta)} \right) \right] + E \left[(1 - \eta(\alpha, \theta)) U \left(\frac{r(1 - \delta)}{1 - \eta(\alpha, \theta)} \right) \right] \\ & \leq E \left[\eta(\alpha, \theta) U(0) + (1 - \eta(\alpha, \theta)) U \left(\frac{r}{1 - \eta(\alpha, \theta)} \right) \right] \end{aligned}$$

for any $\delta > 0$. In the limit as $\delta \rightarrow 0$ we have

$$E[U'(0)] - E \left[U' \left(\frac{r}{1 - \eta(\alpha, \theta)} \right) \right] \leq 0,$$

which contradicts the assumption that $U'(0) = \infty$.

In the second case, $\sum_i \rho_i y_i > 0$. Then the market-clearing condition (3) and the fact that $\eta(\alpha, \theta) = \alpha + \varepsilon \theta$ together imply that

$$\sum_i \rho_i \eta(\alpha, \theta) d_i < \sum_i \rho_i y_i$$

with positive probability. The complementary slackness condition implies that $p(\theta) = 1$ with positive probability, and the short asset will be dominated by the long asset at date 0 unless $p(\theta) < 1/r$ with positive probability. Thus, the price volatility is non-trivial if the probability of default is zero.

8.4 Proof of Theorem 4

Continuity and the convergence of $\{(\rho^q, m, d^q, y^q, p^q)\}$ immediately imply the following properties of the limit point $(\rho^0, m, d^0, y^0, p^0)$.

- (i) $\sum_i \rho_i^0 = 1$ and $\rho_i^0 \geq 0$ for every i , so (ρ^0, m) is a partition.
- (ii) For every i , $(d_i^0, y_i^0) \in \mathbf{R}_+ \times [0, 1]$. The market-clearing conditions

$$\sum_i \rho_i^0 E [\alpha \bar{x}_1(d_i^0, y_i^0, \alpha, \theta) | \theta] \leq \sum_i \rho_i^0 y_i^0$$

and

$$\sum_i \rho_i^0 E \{ \alpha \bar{x}_1(d_i^0, y_i^0, \alpha, \theta) + (1 - \alpha) \bar{x}_2(d_i^0, y_i^0, \alpha, \theta) | \theta \} = \sum_i \rho_i^0 \{ y_i^0 + r(1 - y_i^0) \}$$

are satisfied in the limit, and the complementary slackness condition holds. Thus, (d^0, y^0) is an attainable allocation.

It remains to show that (d_i^0, y_i^0) is optimal for each i . Let $W^q(d_i, y_i, \alpha, \theta)$ denote the utility associated with the pair (d_i, y_i) in the perturbed economy corresponding to $\varepsilon = 1/q$ when the price function is p^q and let $W^0(d_i, y_i, \alpha, \theta)$ denote the utility associated with the pair (d_i, y_i) in the limit economy corresponding to $\varepsilon = 0$, where the price function is p^0 . The function $W_i^0(\cdot)$ is discontinuous at the bankruptcy point, defined implicitly by the condition

$$(\alpha + (1 - \alpha)p^0(\theta))d_i = y_i + p^0(\theta)r(1 - y_i). \quad (4)$$

If (4) occurs with probability zero in the limit, then it is easy to see from the assumed convergence properties that

$$W^q(d_i, y_i, \alpha, \theta) \rightarrow W^0(d_i, y_i, \alpha, \theta) \text{ a.s.};$$

hence

$$\lim_{q \rightarrow \infty} E[W^q(d_i, y_i, \alpha, \theta)] = E[W^0(d_i, y_i, \alpha, \theta)].$$

Let (d_i^0, y_i^0) denote the pair corresponding to the limiting consumption allocation x_i^0 , and let $\{(d_i^q, y_i^q)\}$ denote the sequence of equilibrium choices converging to (d_i^0, y_i^0) . There may exist a set of states (α, θ) with positive

measure such that (d_i^q, y_i^q) implies default in state (α, θ) for arbitrarily large q but that (d_i^0, y_i^0) does not imply default in state (α, θ) . Then at least we can say that

$$\liminf_q W^q(d_i^q, y_i^q, \alpha, \theta) \leq W^0(d_i^0, y_i^0, \alpha, \theta) \text{ a.s.}$$

and this implies that

$$\liminf_q E[W^q(d_i^q, y_i^q, \alpha, \theta)] \leq E[W^0(d_i^0, y_i^0, \alpha, \theta)].$$

Now suppose, contrary to what we want to prove, that (d_i^0, y_i^0) is not optimal. Then there exists a pair (d_i, y_i) such that $E[W^0(d_i, y_i, \alpha, \theta)] > E[W^0(d_i^0, y_i^0, \alpha, \theta)]$. If $d_i = 0$ then (4) holds with probability zero and it is clear that, for some sufficiently large value of q , $E[W^0(d_i, y_i, \alpha, \theta)] > E[W^q(d_i^q, y_i^q, \alpha, \theta)]$, contradicting the equilibrium conditions. If $d_i > 0$, then either the critical condition (4) holds with probability zero or we can find a slightly lower value $d' < d$ that does satisfy the critical condition. To see this, note first that the critical condition uniquely determines the value of $p^0(\theta)$ as long as

$$(1 - \alpha)d \neq r(1 - y)$$

which is true for almost every value of d . Second, if the value of $p^0(\theta)$ for which the critical condition is satisfied is an atom, then we can always find a slightly smaller value $d'_i < d_i$ such that the value of $p^0(\theta)$ for which the critical condition is satisfied is not an atom. Furthermore, reducing d slightly will at most reduce the payoff by a small amount, so for $d'_i < d_i$ and close enough to d_i we still have $E[W^0(d'_i, y_i, \alpha, \theta)] > E[W^0(d_i^0, y_i^0, \alpha, \theta)]$. This then leads to a contradiction in the usual way.

References

- Allen, Franklin and Douglas Gale (1998). "Optimal Financial Crises." *Journal of Finance* 53, pp. 1245–1284.
- Allen, Franklin and Douglas Gale (2000a). "Financial Contagion." *Journal of Political Economy* 108, pp. 1–33.
- Allen, Franklin and Douglas Gale (2000b). "Optimal Currency Crises." *Carnegie Rochester Series on Public Policy* 53, pp. 177–230.

- Allen, Franklin and Douglas Gale (2000c). “Bubbles and Crises.” *The Economic Journal* 110, pp. 236–256
- Allen, Franklin and Douglas Gale (2004). “Financial Intermediaries and Markets.” *Econometrica* 72, pp. 1023–1061
- Azariadis, Costas (1981). “Self-Fulfilling Prophecies.” *Journal of Economic Theory* 25, pp. 380–96.
- Bernanke, Ben and Mark Gertler (1989). “Agency Costs, Net Worth, and Business Fluctuations.” *American Economic Review* 79, pp. 14–31.
- Bernardo, Antonio and Ivo Welch (2004). “Financial Market Runs.” *Quarterly Journal of Economics* 119, pp. 135–158.
- Bryant, John (1980). “A Model of Reserves, Bank Runs, and Deposit Insurance.” *Journal of Banking and Finance* 4, pp. 335–344.
- Calomiris, Charles and Gary Gorton (1991). “The Origins of Banking Panics, Models, Facts, and Bank Regulation.” In *Financial Markets and Financial Crises*, edited by Glenn Hubbard. Chicago, IL: University of Chicago Press.
- Calomiris, Charles and Joseph Mason (2000). “Causes of U.S. Bank Distress During the Depression.” NBER Working Paper W7919.
- Cass, David and Karl Shell (1983). “Do Sunspots Matter?” *Journal of Political Economy* 91, pp. 193–227.
- Chari, Varadarajan and Patrick Kehoe (2000). “Financial Crises as Herds.” Federal Reserve Bank of Minneapolis Working Paper.
- Cone, Kenneth (1983). *Regulation of Depository Institutions*. Unpublished Ph.D. dissertation, Stanford University.
- Diamond, Douglas and Philip Dybvig (1983). “Bank Runs, Deposit Insurance, and Liquidity.” *Journal of Political Economy* 91, pp. 401–419.

- Farmer, Roger (1999). *The Macroeconomics of Self-Fulfilling Prophecies*. Cambridge and London: MIT Press.
- Geanakoplos, John (1990). “An Introduction to General Equilibrium with Incomplete Asset Markets.” *Journal of Mathematical Economics* 19, pp. 1–38.
- Gorton, Gary (1988). “Banking Panics and Business Cycles.” *Oxford Economic Papers* 40, pp. 751–781.
- Gottardi, Piero and Atsushi Kajii (1995). “Generic Existence of Sunspot Equilibria: The Real Asset Case.” University of Pennsylvania, CARESS Working Paper 95/12.
- Gottardi, Piero and Atsushi Kajii (1999). “The Structure of Sunspot Equilibria: The Role of Multiplicity.” *Review of Economic Studies* 66, pp. 713–732.
- Hart, Oliver (1995). *Firms, Contracts and Financial Structure*. Oxford: Oxford University Press.
- Jacklin, Charles (1986). “Demand Deposits, Trading Restrictions, and Risk Sharing.” In *Contractual Arrangements for Intertemporal Trade*, edited by Edward Prescott and Neil Wallace. Minneapolis, MN: University of Minnesota Press.
- Kindleberger, Charles (1978). *Manias, Panics, and Crashes: A History of Financial Crises*. New York, NY: Basic Books.
- Kiyotaki, Nobu and John Moore (1997). “Credit Chains.” *Journal of Political Economy* 99, pp. 220–264.
- Lagunoff, Roger and Stacey Schreft (2001). “A Model of Financial Fragility.” *Journal of Economic Theory* 99, pp. 220–264.
- Magill, Michael and Martine Quinzii (1996). *Theory of Incomplete Markets, Volume I*. Cambridge and London: MIT Press.
- Postlewaite, Andrew, George Mailath and Larry Samuelson (2003). “Sunk Investments Lead to Unpredictable Prices.” University of Pennsylvania: <http://www.ssc.upenn.edu/~apostlew/>.

Schnabel, Isabel and Hyun Song Shin (2004). “Liquidity and Contagion: The Crisis of 1763.” *Journal of the European Economic Association*, this volume.

Shell, Karl and Aditya Goenka (1997). “When Sunspots Don’t Matter.” *Economic Theory* 9, pp. 169–178.

Figure 1A
Equilibrium correspondence
Stochastic α

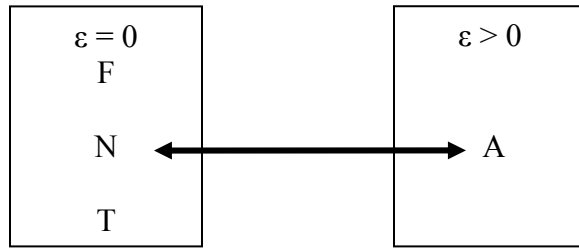


Figure 1B
Equilibrium correspondence
Nonstochastic α

