Existence

October 11, 2000

1 The model

There are three dates $t = 0, 1, 2$ and a single good at each date. There is a finite set of states of nature $\eta \in H$. The true state is revealed at the beginning of date 1. There is a finite set of ex ante types $i = 1, ..., n$ and a measure $\mu_i > 0$ of each type. For each ex ante type i there is a finite set Θ_i of ex post types revealed at date 1. The measure of ex post types θ_i in state η is denoted by $\lambda_i(\theta_i, \eta)$.

The commodity space X consists of functions like $x : \Theta \times H \to \mathbf{R}^2_+$. Each type i is represented by a closed, non-empty, consumption set $X_i \subset X$ and a continuous utility function $u_i : X_i \to \mathbf{R}$. Each type i has an initial endowment of μ_i units of the good at date 0.

There are two assets, a short asset and a long asset. One unit of the good invested in the short asset at date t produces one unit of the good at date $t + 1$. One unit of the good invested in the long asset at date 0 produces $R(\eta) > 0$ units of the good at date 2 in state η (and nothing at date 1).

2 Equilibrium

An *allocation* specifies a consumption bundle $x_i \in X_i$ and a portfolio $y_i \in [0, \mu_i]$ for each type $i = 1, ..., n$. A *random allocation* is a finite set of numbers $\{\rho_j\}$ and allocations $\{(x_i^j, y_i^j)\}$ such that $\rho_j \geq 0$ and $\sum_j \rho_j = 1$. A random allocation is *attainable* if the expected value $\{(x_i, y_i)\} = \sum_j \rho_j \left\{\left(x_i^j, y_i^j\right)\right\}$ satisfies the market-clearing conditions

$$\sum_i \sum_{\theta_i} \lambda_i(\theta_i, \eta) x_{i1}(\theta_i, \eta) \leq \sum_i y_i, \forall \eta \tag{1}$$

and

$$\sum_i \sum_{\theta_i} \lambda_i(\theta_i,\eta)(x_{i1}(\theta_i,\eta)+x_{i2}(\theta_i,\eta)) = \sum_i y_i + (\mu_i - y_i)R(\eta), \forall \eta. \qquad (2)$$

Note that $\{(x_i, y_i)\}$ may not be an allocation even if each $\left\{\left(x_i^j, y_i^j\right)\right\}$ is.

There are complete markets for commodities distinguished by the date of delivery date $t = 1, 2$ and by the state of delivery $\eta \in H$. A price system is a function $p : H \to \mathbf{R}_+^2$, where $p_t(\eta)$ is the price of one unit of the good delivered in state η at date t. An *equilibrium* consists of a price system p and an attainable random allocation $\{\rho^j, (x_i^j, y_i^j)\}$ such that, for every type i and for every allocation j the choice of (x_i^j, y_i^j) solves the decision problem

$$\begin{array}{ll} \max & u_i(x_i) \\ \text{s.t.} & x_i \in X_i; \\ & \sum_\eta p(\eta) \cdot \left(\sum_{\theta_i} \lambda(\theta_i,\eta) x_i(\theta_i,\eta)\right) \leq \sum_\eta p(\eta) \cdot (y_i, (\mu_i - y_i)R(\eta)). \end{array}$$

3 Demand

Assumption: (Local non-satiation) For any ex ante type $i = 1, ..., n$, for any consumption bundle $x_i \in X_i$, and for any $\varepsilon > 0$, there exists a bundle $x_i' \in X_i$ within a distance ε of x_i such that $u_i(x_i') > u_i(x_i)$.

Assumption: (Non-empty interior) For any ex ante type $i = 1, ..., n$, for any consumption bundle $x_i \in X_i$ and price system $p \in P$, and for any $\varepsilon > 0$, either

$$\sum_\eta p(\eta) \cdot \left(\sum_{\theta_i} \lambda(\theta_i,\eta) x_i(\theta_i,\eta)\right) = 0$$

or there exists a bundle x_i' within a distance ε of x_i such that

$$\sum_\eta p(\eta) \cdot \left(\sum_{\theta_i} \lambda(\theta_i,\eta) x_i'(\theta_i,\eta)\right) < \sum_\eta p(\eta) \cdot \left(\sum_{\theta_i} \lambda(\theta_i,\eta) x_i(\theta_i,\eta)\right).$$

Auxiliary Assumption: The consumption sets X_i are compact for $i = 1, ..., n$.

Let

$$P = \left\{ p : H \to \mathbf{R}_+^2 | \sum_\eta \sum_t p_t(\eta) = 1, p_1(\eta) \geq p_2(\eta), \forall \eta \right\}$$

denote the set of admissible prices and let G_i denote the graph of the optimal choice correspondence for type i, that is, let G_i is the set of ordered triples (p, x_i, y_i) such that (x_i, y_i) solves type i's optimization problem for the given p. The set G_i is closed. To see this, let $\{(p^q, x_i^q, y_i^q)\}$ denote a sequence in G_i converging to a point (p^0, x_i^0, y_i^0). Since X_i is closed and the budget constraint is continuous, the choice (x_i^0, y_i^0) is clearly feasible for the given p^0. If (x_i^0, y_i^0) is not optimal, then there must exist a choice $(\hat{x}_i, \hat{y}_i)$ that is feasible at p^0 and preferred to (x_i^0, y_i^0). For each value of q the choice of y_i^q maximizes the expression on the right hand side of the budget constraint, so

$$\begin{aligned}\sum_\eta p^q(\eta) \cdot (y_i^q, (\mu_i - y_i^q)R(\eta)) &= \max\left\{\sum_\eta p_1^q(\eta), \sum_\eta p_2^q(\eta)R(\eta)\right\}\mu_i \\ &\geq \max\left\{\sum_\eta p_1^q(\eta), \sum_\eta p_2^q(\eta)\right\}\mu_i \\ &\geq \mu_i/2 > 0.\end{aligned}$$

By assumption, we can find a consumption bundle $\tilde{x}_i \in X_i$ such that $\tilde{x}_i$ is arbitrarily close to $\hat{x}_i$ and either $\hat{x}_{it}(\eta) = 0$ or $0 < \tilde{x}_{it}(\eta)$. Then $\tilde{x}_i$ is preferred to x_i^0 and

$$\sum_\eta p^0(\eta) \cdot \left(\sum_{\theta_i} \lambda(\theta_i, \eta)\tilde{x}_i(\theta_i, \eta)\right) < \sum_\eta p^0(\eta) \cdot \left(y_i^0, \left(\mu_i - y_i^0\right) R(\eta)\right),$$

which means that $(\tilde{x}_i, y_i^0)$ is feasible for p^q and q sufficiently large. Since x_i^q converges to x_i^0, $\tilde{x}_i$ is preferred to x_i^q for q sufficiently large, a contradiction. This contradiction proves that $(p^0, x_i^0, y_i^0) \in G_i$, so G_i is closed.

The optimal excess demand correspondence for type i is denoted by ζ_i and defined by putting $\zeta_i(p)$ equal to the set of all vectors $z = \{z(\eta)\}$ having the form

$$z(\eta) = \left(\sum_{\theta_i} \lambda(\theta_i, \eta)x_i(\theta_i, \eta) - (y_i, (\mu_i - y_i)R(\eta))\right), \forall \eta,$$

for some (x_i, y_i) such that $(x_i, y_i, p) \in G_i$. The set $\zeta_i(p)$ is non-empty, for every p, because the choice set is compact and the objective function is continuous. Since G_i is closed, the graph of ζ_i is closed.

Define the aggregate excess demand by

$$\zeta(p) = \sum_i \mu_i \zeta_i(p).$$

Then ζ is well defined and the graph of ζ is closed. The range of ζ is clearly compact.

4 Equilibrium

Let Z denote a compact convex set containing the range of ζ. For any $z \in Z$ let

$$\pi(z) = \arg\max_{p\in P} \left\{ \sum_\eta p_1(\eta) z_1^*(\eta) + p_2^*(\eta) \left(z_2^*(\eta) - z_1^*(\eta) \right) \right\}$$

and consider the correspondence $\psi : P \times Z \rightrightarrows P \times Z$ defined by putting

$$\psi(p, z) = \pi(z) \times \text{conv } \zeta(p), \forall (p, z) \in P \times Z,$$

where conv $\zeta(p)$ denotes the convex hull of $\zeta(p)$. Then ψ satisfies the conditions of the Kakutani fixed point theorem and thus has a fixed point $(p^*, z^*) \in \psi(p^*, z^*)$. We claim that, for any state η,

$$z_1^*(\eta) \leq 0 = z_2^*(\eta)$$

and

$$z_1^*(\eta) < 0 \Longrightarrow p_1^*(\eta) = p_2^*(\eta).$$

The budget constraints imply that

$$\sum_\eta p_1(\eta) z_1^*(\eta) + p_2^*(\eta) \left(z_2^*(\eta) - z_1^*(\eta) \right) \leq 0, \forall p \in P.$$

Clearly, we cannot have $z_1^*(\eta) > 0$ so suppose that $z_1^*(\eta) \leq 0$ for all η and that $z_2^*(\eta) > 0$ for some $\eta = \eta_0$. Choose p so that $p_1(\eta_0) = p_2(\eta_0) = 1/2$ and $p(\eta) = 0$ for $\eta \neq \eta_0$. Then

$$\begin{aligned} 0 &\geq \sum_\eta p_1(\eta) z_1^*(\eta) + p_2(\eta) \left(z_2^*(\eta) - z_1^*(\eta) \right) \\ &= \frac{1}{2} \left(z_1^*(\eta_0) + \left(z_2^*(\eta) - z_1^*(\eta) \right) \right) \\ &= \frac{1}{2} z_2^*(\eta) > 0, \end{aligned}$$

a contradiction. Thus, $z^* \leq 0$. From local non-satiability, the budget constraint is binding at an optimum, so

$$\begin{aligned} 0 &= \sum_\eta p_1^*(\eta) z_1^*(\eta) + p_2^*(\eta)\left(z_2^*(\eta) - z_1^*(\eta)\right) \\ &\leq \sum_\eta p_2^*(\eta) z_1^*(\eta) + p_2^*(\eta)\left(z_2^*(\eta) - z_1^*(\eta)\right) \\ &= \sum_\eta p_2^*(\eta) z_2^*(\eta) \leq 0. \end{aligned}$$

From this it follows that $p_2^*(\eta) = 0$ if $z_2^*(\eta) < 0$, but this is inconsistent with optimality. Thus, $z_2^* = 0$.

5 Extensions

To remove the auxiliary boundedness assumption, consider sequences of economies with the bound on X_i gradually relaxed. For each element of the sequence there is an equilibrium and the sequence of equilibria has a convergent subsequence. To show that the limit of this subsequence is the desired equilibrium, one uses the usual argument for *u.h.c.* of the choice correspondence to show that the limiting choice is optimal for each type.

The number of different optimal choices is finite by Carathéodory's Theorem (Rockafellar, 1970, p. 155). This completes the proof of existence.

References

Rockafellar, R. (1970). *Convex Analysis.* Princeton, NJ: Princeton University Press.