Problem Set 1
Welfare Economics

Solutions

1. Consider a pure exchange economy with two goods, $h = 1, 2$, and two consumers, $i = 1, 2$, with utility functions u_1 and u_2 respectively, and total endowment, $e = (e_1, e_2) \gg 0$. An attainable allocation is said to be weakly Pareto-efficient if it is impossible to make every agent better off and strongly Pareto-efficient if it is impossible to make some agent better off without making other agents worse off. For each of the following cases, derive an expression for the set of (strongly) Pareto-efficient allocations and illustrate the set graphically. Also, in each case, determine whether the set of weakly Pareto-efficient allocations is different from the set of (strongly) Pareto-efficient allocations. If it is different, characterize and illustrate the set of weakly Pareto-efficient allocations.

(a) $u_1(x, y) = \alpha \ln x + (1 - \alpha) \ln y$ and $u_2(x, y) = \beta \ln x + (1 - \beta) \ln y$, where $\alpha < \beta$ and $\ln x$ denotes the natural logaritm of x, i.e., $\log_e x$.

Solution:

Let x_i be the amount of good x agent i is consuming. Define y_i similarly. Let us first examine boundary points. Obviously, the two allocations where one of the agents consume the total endowment is strongly efficient.[1] We next show that these two allocations are the only efficient allocations on the boundary of the Edgeworth box. That is, we shall show that any other boundary allocation is not weakly efficient.

To this end, assume for some i we have $x_i > 0$ and $y_i = 0$ (the argument for the case $x_i = 0, y_i > 0$ is identical). Since $x_i > 0$, we can transfer $x_i/2$ to consumer j. This will make j strictly better off. Now since j's utility is continuous we could transfer some amount of good y from j to i leaving j still in a better situation than before. Since i's utility is no longer $-\infty$, she is also clearly better off (see figure 1). Thus the only efficient allocations on the

[1] Any other attainable allocation would make the individual who is receiving all the endowment worse off.

boundary (weakly and strongly) are the allocations where one of the agents receives the total endowment of both goods.

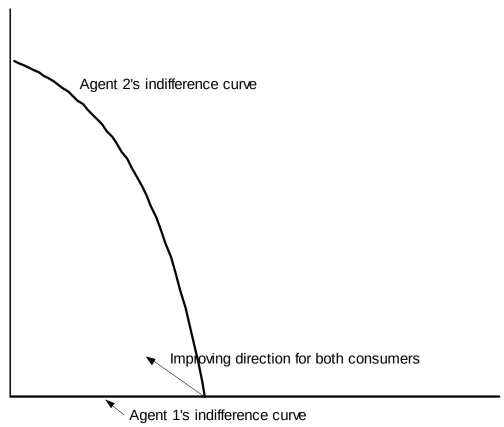

Figure 1: Boundary Case

Next, let's focus to the interior of the Edgeworth box. Note that both utilities are strongly monotone and continuous, and thus strongly efficient allocations coincide with weakly efficient allocations. Moreover, note that the sufficient conditions for Negishi's Theorem are satisfied. Thus for weights $\lambda \in (0,1)$, we can find the interior efficient allocations via

$$\max_{x_1,y_1,x_2,y_2} \lambda\left(\alpha \ln x_1 + (1-\alpha)\ln y_1\right) + (1-\lambda)\left(\beta \ln x_2 + (1-\beta)\ln y_2\right)$$
$$s.t.\ x_1 + x_2 = e_1 \text{ and } y_1 + y_2 = e_2.$$

Note that solving the maximization above is equivalent to solving the following two problems:

$$\max_{x_1,x_2} \lambda\alpha \ln x_1 + (1-\lambda)\beta \ln x_2$$
$$s.t.\ x_1 + x_2 = e_1$$

and

$$\max_{y_1,y_2} \lambda(1-\alpha)\ln y_1 + (1-\lambda)(1-\beta)\ln y_2$$
$$s.t.\ y_1 + y_2 = e_2.$$

The problems above are standard Cobb-Douglas problems, and thus

$$\begin{aligned} x_1 &= \frac{\lambda\alpha}{\lambda\alpha + (1-\lambda)\beta} e_1, \\ x_2 &= \frac{(1-\lambda)\beta}{\lambda\alpha + (1-\lambda)\beta} e_1, \\ y_1 &= \frac{\lambda(1-\alpha)}{\lambda(1-\alpha) + (1-\lambda)(1-\beta)} e_2, \\ y_2 &= \frac{(1-\lambda)(1-\beta)}{\lambda(1-\alpha) + (1-\lambda)(1-\beta)} e_2. \end{aligned}$$

The equations above are a possible characterization of our Pareto-efficient allocations as a function of λ. Alternatively we can use the first and third equation to write y_1 as a function of x_1 as

$$y_1 = (1-\alpha) e_2 \left(\frac{\beta x_1}{\alpha(1-\beta) e_1 - \alpha x_1 + \beta x_1} \right).$$

Moreover, it can be shown that

$$\frac{dy_1}{dx_1} > 0 \text{ and } \frac{d^2 y_1}{dx_1^2} < 0,$$

that is, y_1 is a strictly increasing, concave function of x_1. Finally, observe that

$$\begin{aligned} \frac{y_1}{x_1} &= \frac{\lambda(1-\alpha)}{\lambda(1-\alpha) + (1-\lambda)(1-\beta)} \frac{\lambda\alpha + (1-\lambda)\beta}{\lambda\alpha} \frac{e_2}{e_1} \\ &> \frac{(1-\alpha)}{\lambda(1-\alpha) + (1-\lambda)(1-\alpha)} \frac{\lambda\alpha + (1-\lambda)\alpha}{\alpha} \frac{e_2}{e_1} \\ &= \frac{(1-\alpha)}{(1-\alpha)} \frac{\alpha}{\alpha} \frac{e_2}{e_1} = \frac{e_2}{e_1}. \end{aligned}$$

Hence, agent 1's consumption bundle lies always above the diagonal of the Edgeworth box. We can thus characterize the set of efficient allocations graphically as in figure 2.

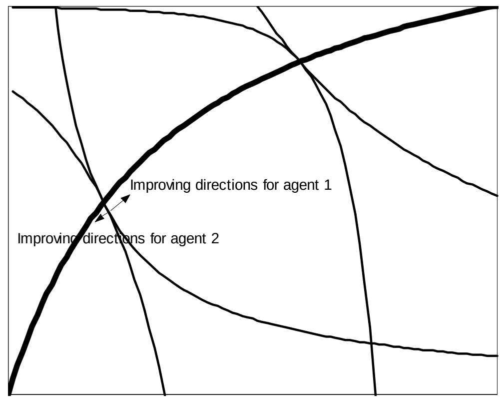

Figure 2: Pareto-Efficient Allocations

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(b) For $i = 1, 2$, $u_1 = u_2$ is strictly concave, smooth, and homothetic.

Solution:

First, note that the allocations $(e_1, e_2), (0, 0)$ and $(0, 0), (e_1, e_2)$ are strongly Pareto Efficient. Next, let's show that any allocation $(te_1, te_2), ((1-t)e_1, (1-t)e_2)$ for $t \in (0, 1)$ is also strongly Pareto Efficient. Note that the conditions for Negishi are satisfied, and thus for some weight $\lambda > 0$, the efficient allocations can be found by solving

$$\max_{x_1, y_1, x_2, y_2} u(x_1, y_1) + \lambda u(x_2, y_2)$$
$$s.t.\ x_1 + x_2 = e_1 \text{ and } y_1 + y_2 = e_2.$$

Since u is differentiable and concave, necessary and sufficient conditions for optimality are

$$\frac{\partial u(x_1^*, y_1^*)}{\partial x} / \frac{\partial u(x_1^*, y_1^*)}{\partial y} = \frac{\partial u(x_2^*, y_2^*)}{\partial x} / \frac{\partial u(x_2^*, y_2^*)}{\partial y}.$$

Since u is homothetic and strictly concave, however, in order for the MRS's to be equal, it must be that

$$\frac{x_1^*}{y_1^*} = \frac{x_2^*}{y_2^*} = \frac{e_1}{e_2},$$

where the second equality follows from the feasibility constraints. Note that for any $t \in (0,1)$, the allocation $(te_1, te_2), ((1-t)e_1, (1-t)e_2)$ satisfies this condition, and thus is strongly Pareto Efficient.

Next, note that any other interior, feasible allocation, say $\left(x_1^{'}, y_1^{'}\right), \left(x_2^{'}, y_2^{'}\right)$, will not satisfy $\frac{x_1^{'}}{y_1^{'}} = \frac{x_2^{'}}{y_2^{'}}$, and thus

$$\frac{\partial u\left(x_1', y_1'\right)}{\partial x} / \frac{\partial u\left(x_1', y_1'\right)}{\partial y} \neq \frac{\partial u\left(x_2', y_2'\right)}{\partial x} / \frac{\partial u\left(x_2', y_2'\right)}{\partial y},$$

again by homotheticity and strict concavity. Since the equality of the MRS's is necessary for optimality of the planner's problem, we can conclude that $\left(x_1^{'}, y_1^{'}\right), \left(x_2^{'}, y_2^{'}\right)$ is not Pareto Efficient.

Finally, let's check the remaining boundary points. Consider the feasible point $\left(x_1^{''}, y_1^{''}\right), \left(x_2^{''}, y_2^{''}\right)$, and assume without loss of generality that $x_1^{''} = 0$ and $y_1^{''} > 0$. By strict concavity, we can find an $\epsilon > 0$ and $\delta > 0$ such that both agents are stictly better off under the allocation $\left(\epsilon, y_1^{''} - \delta\right), \left(x_2^{''} - \epsilon, y_2^{''} + \delta\right)$, and thus these boundary allocations cannot be Pareto Efficient. See figure 3 for a graphical representation.

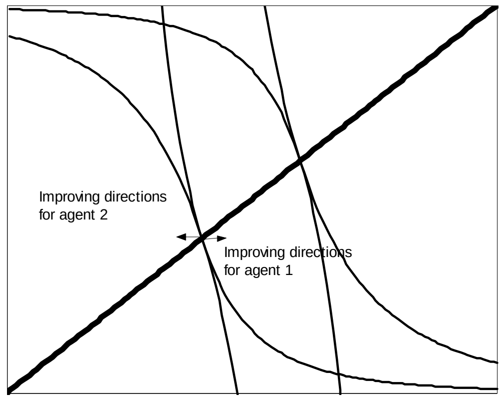

Figure 3: Homothetic Utility Functions

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(c) For $i = 1, 2$, $u_i(x, y) = x + g(y)$, where g is an increasing and strictly concave function.

Solution:

Again, continuity and monotonicity imply equivalence of strong efficiency and weak efficiency. Hence, to find the efficient allocations we can solve the following Negishi problem:

$$\max_{x_1, y_1} \lambda (x_1 + g(y_1)) + (1 - \lambda)(e_1 - x_1 + g(e_2 - y_1))$$
$$s.t.\ 0 \leq x_1 \leq e_1 \text{ and } 0 \leq y_1 \leq e_2.$$

As in part (a), the problem can be separated into the following two problems:

$$\max_{x_1} \lambda x_1 + (1 - \lambda)(e_1 - x_1)$$
$$s.t.\ 0 \leq x_1 \leq e_1$$

and

$$\max_{y_1} \lambda g(y_1) + (1 - \lambda) g(e_2 - y_1)$$
$$s.t.\ 0 \leq y_1 \leq e_2.$$

For the first problem, if $\lambda < 1/2$ then $x_1 = 0$, if $\lambda > 1/2$ then $x_1 = 1$ and if $\lambda = 1/2$ then $x_1 \in [0,1]$.

For the second problem, if $\lambda = 0$ then $y_1 = 0$ and if $\lambda = 1$ then $y_1 = e_2$. If $\lambda = 1/2$, then the strict concavity of g implies that the solution is $y_1 = e_2/2$. Since g is increasing in y and continuous, then for $\lambda \in (0, 1/2)$, there exists a unique $y_1 \in (0, e_2/2)$ that solves the maximization problem, and similarly for $\lambda \in (1/2, 1)$. Moreover, by the continuity of g, the optimal set for y_1 (as a function of $\lambda \in [0,1]$) is onto $[0, e_2]$.

Consequently, the efficient allocations are: $x_1 = 0$, $y_1 \in [0, e_2/2)$ for some $\lambda \in [0, 1/2)$; $x_1 \in [0,1]$, $y_1 = e_2/2$ for $\lambda = 1/2$; and $x_1 = 1$, $y_1 \in (e_2/2, e_2]$ for some $\lambda \in (1/2, 1]$. See figure 4 for a graphical representation.

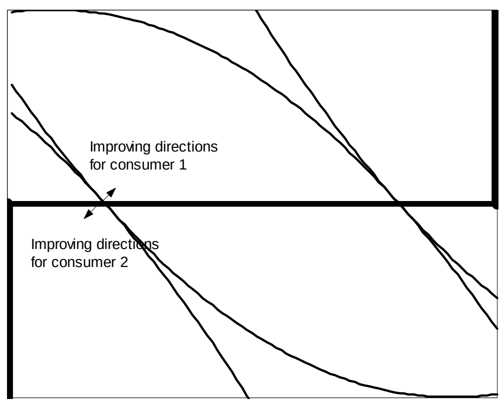

Figure 4: Quasi-linear Utility Functions

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(d) $u_1(x,y) = \min\{x, 2y\}$, $u_2(x,y) = \min\{2x, y\}$ *and* $e_1 = e_2$.

Solution:

To find the Pareto-efficient allocations for this problem, we will investigate four cases:

Case 1: $x_1 < 2y_1$ and $2x_2 > y_2$

Such an allocation is not weakly Pareto-efficient. To see this, let $\delta > 0$ and $\varepsilon > 0$ be such that

$$x_1 + \delta < 2(y_1 - \varepsilon) \text{ and } 2(x_2 - \delta) > y_2 + \varepsilon.$$

It follows that

$$u_1(x_1+\delta, y_1-\varepsilon) = \min\{x_1+\delta, 2(y_1-\varepsilon)\} = x_1+\delta > x_1 = \min\{x_1, 2y_1\} = u_1(x_1, y_1)$$

and

$$u_2(x_2-\delta, y_2+\varepsilon) = \min\{2(x_2-\delta), y_2+\varepsilon\} = y_2+\varepsilon > y_2 = \min\{2x_2, y_2\} = u_2(x_2, y_2).$$

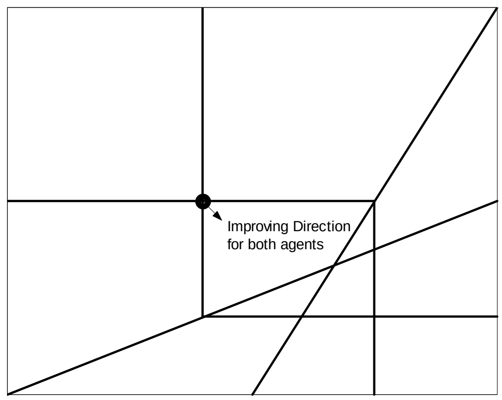

Figure 5: Case 1

Case 2: $x_1 > 2y_1$ and $2x_2 < y_2$

Such an allocation is not weakly Pareto-efficient. To see this, let $\delta > 0$ and $\varepsilon > 0$ be such that

$$x_1 - \delta > 2(y_1+\varepsilon) \text{ and } 2(x_2+\delta) < y_2 - \varepsilon.$$

But then we have

$$u_1(x_1-\delta, y_1+\varepsilon) = \min\{x_1-\delta, 2(y_1+\varepsilon)\} = 2(y_1+\varepsilon) > 2y_1 = \min\{x_1, 2y_1\} = u_1(x_1, y_1)$$

and

$$u_2(x_2+\delta, y_2-\varepsilon) = \min\{2(x_2+\delta), y_2-\varepsilon\} = 2(x_2+\delta) > 2x_2 = \min\{2x_2, y_2\} = u_2(x_2, y_2).$$

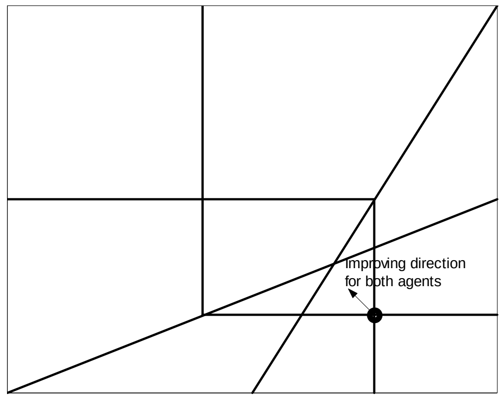

Figure 6: Case 2

Case 3: $x_1 \geq 2y_1$ and $2x_2 \geq y_2$

Such an allocation is strongly Pareto-efficient. To see this, note that to increase the utility of either consumer you'll have to give her more of good y, but this will decrease the utility of the other consumer.

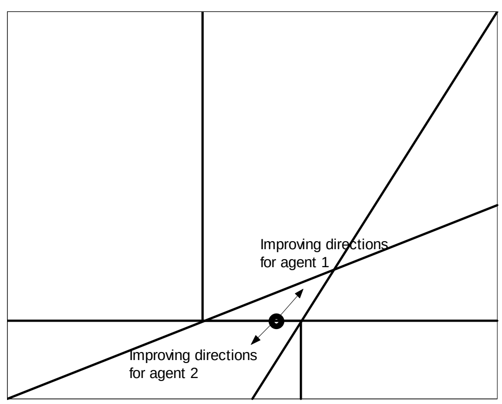

Figure 7: Case 3

Case 4: $x_1 \leq 2y_1$ and $2x_2 \leq y_2$

Such an allocation is strongly Pareto-efficient. To see this, note that to increase the utility of either consumer you'll have to give her more of good x, but this will decrease the utility of the other consumer.

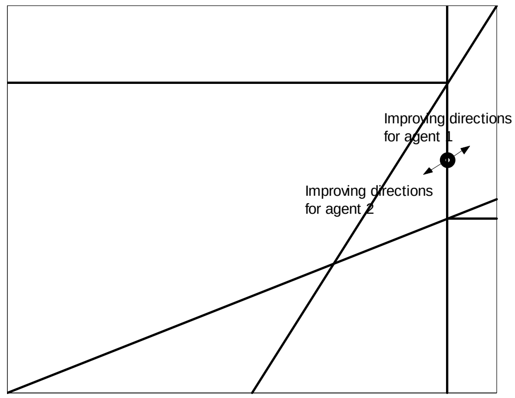

Figure 8: Case 4

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(e) $u_1(x,y) = \max\{x, 2y\}$, $u_2(x,y) = \max\{2x, y\}$ *and* $e_1 = e_2$.

Solution:

First note that the allocation $(x_1, y_1) = (0, e)$, $(x_2, y_2) = (e, 0)$ gives the maximum possible utility for both consumers. Since it's impossible to increase the utility of any of the two consumers, this allocation is trivially strongly Pareto-efficient. Moreover, there is no other strongly efficient allocation: any other allocation will give less than maximum utility to at least one of the consumers.

Moreover, note that if $y_1 = e$ it is impossible to increase the utility of consumer 1, so all these allocations are weakly Pareto-efficient. Similarly, allocations where $x_2 = e$ are also weakly Pareto-efficient. Finally it's clear that no other allocation can be weakly Pareto-efficient, since we always have the opportunity to move to $(0, e), (e, 0)$ and make both consumers better off.

So we can represent the set of efficient allocations as

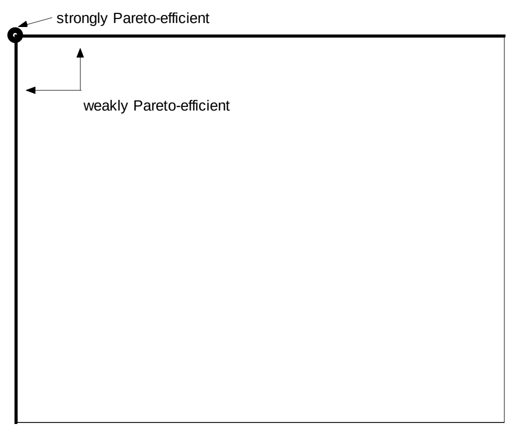

Figure 9: Max Utility

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5. Consider a pure exchange economy with two goods, $h = 1, 2$, and two consumers, $i = 1, 2$, with utility functions u_1 and u_2 respectively, and total endowment, $e = (e_1, e_2) \gg 0$. For each of the following cases, determine which of the Pareto-efficient allocations can be decentralized as competitive equilibrium with lump sum transfers. Briefly describe the equilibrium prices and transfers for each Pareto-efficient allocation.

(a) $u_1(x, y) = \alpha \ln x + (1-\alpha) \ln y$ and $u_2(x, y) = \beta \ln x + (1-\beta) \ln y$, where $\alpha < \beta$ and $\ln x$ denotes the natural logaritm of x, i.e., $\log_e x$.

Solution:

In this case, for any Pareto-efficient allocation $(x_1^*, y_1^*), (x_2^*, y_2^*)$, we know that at least one of the two agents is consuming a positive amount of the two goods, say agent i. So we can choose p_x, p_y such that

$$\frac{p_x}{p_y} = \frac{\partial u_i(x_i^*, y_i^*)/\partial x}{\partial u_i(x_i^*, y_i^*)/\partial y}.$$

Observe that if the allocation is interior we have

$$\frac{p_x}{p_y} = \frac{\alpha}{1-\alpha} \frac{y_1^*}{x_1^*} = \frac{\beta}{1-\beta} \frac{y_2^*}{x_2^*}.$$

So the first order conditions for the utility maximization problem of both consumers are being satisfied at the given bundles and prices. If we are at one of the corner allocations, since both prices are positive the consumer who has no wealth is optimally consuming nothing. By construction p_x, p_y are chosen so that the first order conditions for the problem of the other consumer are satisfied.

Finally, let e_x^i be the endowment of good x consumer i is receiving. Define e_y^i in a similar way. Clearly the transfers that make the prices and allocations above an equilibrium are

$$t_1 = p_x x_1^* + p_y y_1^* - p_x e_x^1 - p_y e_y^1$$

and

$$t_2 = p_x x_2^* + p_y y_2^* - p_x e_x^2 - p_y e_y^2.$$

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(b) For $i = 1, 2$, $u_1 = u_2$ is strictly concave, smooth, and homothetic.

Solution:

Again we know that at least one of the consumers is consuming a positive amount of both goods, say consumer i. So we can choose p_x, p_y such that

$$\frac{p_x}{p_y} = \frac{\partial u_i\left(x_i^*, y_i^*\right)/\partial x}{\partial u_i\left(x_i^*, y_i^*\right)/\partial y}.$$

But in this case, because of the homotheticity of u, we also know that

$$\frac{p_x}{p_y} = \frac{\partial u_j\left(x_j^*, y_j^*\right)/\partial x}{\partial u_j\left(x_j^*, y_j^*\right)/\partial y} = \frac{\partial u\left(e_1, e_2\right)/\partial x}{\partial u\left(e_1, e_2\right)/\partial y}.$$

The transfers are the same as in (a).

■

(c) For $i = 1, 2$, $u_i(x, y) = x + g(y)$, where g is an increasing and strictly concave function.

Solution:

For simplicity, let's assume that g is differentiable. Recall that at least one of the agents consumes a positive amount of both goods, say consumer 1. So we can choose p_x, p_y such that

$$\frac{p_x}{p_y} = \frac{\partial u_1 \left(x_1^*, y_1^*\right) / \partial x}{\partial u_1 \left(x_1^*, y_1^*\right) / \partial y}.$$

We can thus write the expression above in a simplified way as

$$\frac{p_x}{p_y} = \frac{1}{\min \left\{g' \left(y_1^*\right), g' \left(y_2^*\right)\right\}}.$$

Therefore, the ratio of prices can be written as

$$\frac{p_x}{p_y} \geq \frac{1}{g' \left(y_1^*\right)}, \text{ with equality if } x_1^* > 0.$$

So, if we have $y_2^* > y_1^*$, which implies that $\frac{p_x}{p_y} > \frac{1}{g'\left(y_1^*\right)}$, we must have $x_1^* = 0$, and thus our price vector is giving us the correct allocations, even for the boundary cases. Again, the transfers are the same as in (a). ■

(d) $u_1 \left(x, y\right) = \min \left\{x, 2y\right\}$, $u_2 \left(x, y\right) = \min \left\{2x, y\right\}$ *and* $e_1 = e_2$.

Solution:

If $p_x, p_y > 0$ it's clear that in a solution of the problem of consumer 1 we will have $x_1^* = 2y_1^*$, and similarly we will have $2x_2^* = y_2^*$ in the problem of consumer 2. So in order to support an efficient allocation where one of the previous conditions is not true, we'll have to set one of the prices to zero. In fact, if $p_x = 0$, and $p_y > 0$ we can support all efficient allocations such that $x_1^* \geq 2y_1^*$ and $2x_2^* \geq y_2^*$. And if $p_x > 0$ and $p_y = 0$ we can support any efficient allocation such that $x_1^* \leq 2y_1^*$ and $2x_2^* \leq y_2^*$. If we set $p_x, p_y > 0$, the only equilibrium with transfers will be the allocation

$$\left(x_1^*, y_1^*\right) = \left(\frac{2}{3}e, \frac{1}{3}e\right) \text{ and } \left(x_2^*, y_2^*\right) = \left(\frac{1}{3}e, \frac{2}{3}e\right),$$

where $e := e_1 = e_2$.

Again, the transfers are the same as in (a). ■

(e) $u_1 \left(x, y\right) = \max \left\{x, 2y\right\}$, $u_2 \left(x, y\right) = \max \left\{2x, y\right\}$ *and* $e_1 = e_2$.

Solution:

It's clear that if $p_x, p_y > 0$ both consumers will consume only one of the goods. If one the two prices is equal to zero, the consumers will want to consume an infinite amount of one of the goods, so there is no equilibrium in this case. So the only allocation that we can support as an equilibrium with transfers is $(x_1^*, y_1^*) = (0, e)$, $(x_2^*, y_2^*) = (e, 0)$. To support this allocation we only need prices that make consumer 1 want to buy good y and makes consumer 2 want to buy good x. A sufficient condition for this to happen is

$$\frac{p_x}{2} \leq p_y \leq 2p_x.$$

Again, the transfers are the same as in (a). ■

6. Consider a pure exchange economy with two goods, $h = 1, 2$, and two consumers, $i = 1, 2$, with consumption sets $X_i = \mathbf{R}_+^2$, endowments $e_1 = (1, 0)$ and $e_2 = (1, 1)$, and utility functions

$$u_1(x_{11}, x_{12}) = (x_{11})^{1/2} + (x_{12})^{1/2}$$

and

$$u_2(x_{21}, x_{22}) = \min\{x_{21}, x_{22}\}$$

respectively. Show that the initial endowment can be decentralized as a quasi-equilibrium, that is, there exists a price vector p^* such that $(p^*, x^*) = (p^*, e)$ is a quasi-equilibrium. Is (p^*, x^*) a competitive equilibrium (with or without transfers)? Explain your answer with reference to the assumptions of the second theorem of welfare economics.

Solution:

Let $p_1 = 0$ and $p_2 > 0$. For consumer 1 we have that $p_1 \cdot 1 + p_2 \cdot 0 = 0$. So, since the cost of the bundle $(1, 0)$ is the minimum possible, the condition for a quasi-equilibrium is trivially satisfied - any allocation increeasing consumer 1's utility must cost at least 0. For consumer 2, any bundle such that $u_2(x_{21}, x_{22}) > u_2(1, 1)$ must satisfy $x_{22} > 1$ and, therefore, is more expensive than $(1, 1)$, and thus the condition for a quasi-equilibrium is also satisfied for consumer 2.

We shall now show that endowments cannot be a part of an equilibrium with transfers. Since marginal utility of good 2 for consumer 1 is ∞ at e_1, then consumer 1 could optimally choose e_1 only if she has no positive wealth. But since she has 1 unit of good one, this is possible only if $p_1 = 0$. But whenever $p_1 = 0$, she would choose $x_{11} = \infty$.

The condition we are missing here is the condition that there exists a cheaper bundle than the quasi-equilibrium bundles for both consumers. In the above depicted quasi-equilibrium, this condition is not satisfied for player 1. ■

8. Consider a pure exchange economy with two consumers $i = 1, 2$ and two commodities, $h = 1, 2$. The consumers have consumption sets $\mathbb{R}^2_+$ and endowments $e_1 = (4, 0)$ and $e_2 = (0, 3)$. Consumer 2 has preferences represented by the utility function $u_2(x_2, y_2) = min\{2x_2 + y_2, x_2 + 2y_2\}$. Illustrate a typical indifference curve for consumer 2. Consider the following possible utility functions for consumer 1:

$$u_1(x_1, y_1) = min\{x_1, 2y_1\};$$

$$u_1(x_1, y_1) = max\{x_1, 3y_1\};$$

$$u_1(x_1, y_1) = max\{x_1, y_1\};$$

$$u_1(x_1, y_1) = max\{min\{2x_1, y_1\}, min\{x_1, 2y_1\}\}.$$

In each case, illustrate the Pareto-efficient allocations in an Edgeworth Box diagram and find the set of Pareto-efficient allocations that can be supported as an equilibrium with transfers.

Solution:

Let's first study consumer 2's preferences. Note that preferences are strictly monotonic and if $x_2 > y_2$, then $u_2(x_2, y_2) = x_2 + 2y_2$, if $x_2 < y_2$, then $u_2(x_2, y_2) = 2x_2 + y_2$ and if $x_2 = y_2$, then $u_2(x_2, y_2) = 3x_2$. Consumer 2's indifference curves are as depicted below:

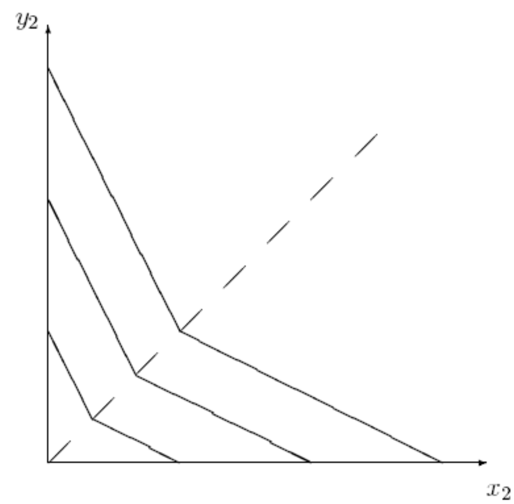

Figure 10: Consumer 2's Indifference Curves

(a) $u_1(x_1, y_1) = min\{x_1, 2y_1\}$.

Solution:

Any Strongly Pareto Efficient allocation must satisfy $x_1^* = 2y_1^*$. If $x_1 > 2y_1$, for instance, we could transfer ε of x_1 to consumer 2 and δ of y_2 to consumer 1 to make both consumers strictly better off. All of the points such that $x_1 = 2y_1$ are Strongly Pareto Efficient - to improve consumer 1, we must increase his consumption of either x_1 or y_1 (or both), which would decrease consumer 2's utility. Conversely, from the convexity of consumer 2's preferences, we must decrease 1's utility in order to increase consumer 2's welfare (see the graph below). Moreover, allocations $(e_1, y_1), (0, e_2 - y_1)$ for $y_1 \geq 2$ are (trivially) Weakly Pareto Efficient since consumer 1 cannot be made better off.

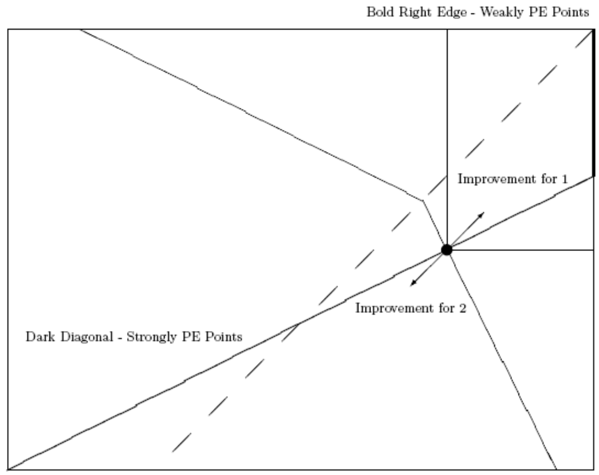

Figure 11: $u_1(x_1, y_1) = min\{x_1, 2y_1\}$

Every Strongly Pareto Efficient point can be decentralized. If $x_1^* \leq 2$, equilibrium can be supported at $\left(x_1^*, \frac{x_1^*}{2}\right), \left(e_1 - x_1^*, e_2 - \frac{x_1^*}{2}\right)$ with prices $p = (\alpha, 2\alpha)$ for any $\alpha > 0$, and if $2 < x_1^* \leq 4$, then $\left(x_1^*, \frac{x_1^*}{2}\right), \left(e_1 - x_1^*, e_2 - \frac{x_1^*}{2}\right)$ can be supported with prices $p = (2\alpha, \alpha)$ for any $\alpha > 0$. As in question 5, the transfers are

$$t_1 = p_x x_1^* + p_y y_1^* - p_x e_x^1 - p_y e_y^1$$

and

$$t_2 = p_x x_2^* + p_y y_2^* - p_x e_x^2 - p_y e_y^2.$$

(b) $u_1(x_1, y_1) = max\{x_1, 3y_1\}$.

Solution:

Clearly, any interior point is inefficient - if $0 < x_1 < y_1 < e_2$, then we could transfer x_1 to consumer 2 and δ of y_2 to consumer 1 to make both agents strictly better. Moreover, allocations $(x_1, 0), (e_1 - x_1, e_2)$ (along the bottom of the Edgeworth Box) are not Weakly Pareto Efficient since we can make both consumers strictly better off with the allocation $\left(0, \frac{x_1}{3} + \varepsilon\right), \left(e_1, e_2 - \left(\frac{x_1}{3} + \varepsilon\right)\right)$.

Allocations such as $(e_1, y_1), (0, e_2 - y_1)$ (along the right of the Edgeworth Box) are not Weakly Pareto Efficient either since the point

$$\left(0, \max\{\frac{e_1}{3}, y_1\}\right), \left(e_1, e_2 - \max\{\frac{e_1}{3}, y_1\}\right)$$

is preferred by both consumers.

Allocations $(0, y_1), (e_1, e_2 - y_1)$ (along the left of the Edgeworth Box) are Strongly Pareto Efficient. Clearly, to improve consumer 2 we would have to reduce consumer 1's utility. To improve consumer 1, we either need to increase his consumption of y_1 (which would reduce consumer 2's utility) or set $x_1 > 3y_1$ (which would reduce consumer 2's utility and in some cases would not be feasible). Finally, allocations $(x_1, e_2), (e_1 - x_1, 0)$ (along the top of the Edgeworth Box) are (trivially) weakly Pareto Efficient since $e_1 < 3e_2$.

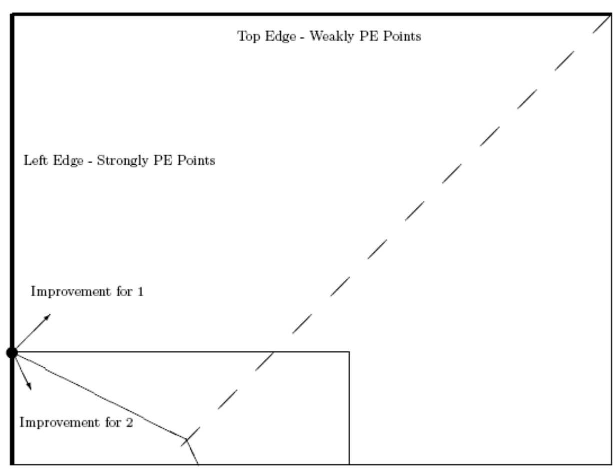

Figure 12: $u_1(x_1, y_1) = max\{x_1, 3y_1\}$

All of the Strongly Pareto Efficient points can be decentralized. For consumer 1 to be maximizing at $(0, y_1^*)$, it must be that $p_2 \leq 3p_1$; otherwise, consumer 1 could afford strictly more than $3y_1^*$ of the first commodity. Therefore, the price vector $p = (\alpha, 2\alpha)$ for any $\alpha > 0$ will support decentralization of the allocations $(0, y_1^*), (e_1, e_2 - y_1^*)$. The weakly Pareto Efficient allocations, however, cannot be supported, as for any price vector, consumer 1 would maximize his utility by consuming either only the first commodity or the second commodity. The corresponding transfers are identical to part (a).

(c) $u_1(x_1, y_1) = max\{x_1, y_1\}$.

Solution:

As in (b), all interior points are inefficient. Moreover, allocations such as $(0, y_1), (e_1, e_2 - y_1)$ (along the left of the Edgeworth Box) are not Weakly Pareto Efficient, as the allocation $(y_1 + \varepsilon, 0), (e_1 - (y_1 + \varepsilon), e_2)$ is a strict improvement for both consumers. Allocations such as $(x_1, e_2), (e_1 - x_1, 0)$ (along the top of the Edgeworth Box) are not Weakly Pareto Efficient either, since $(e_2 + \varepsilon, 0), (e_1 - (e_2 + \varepsilon), e_2)$ is a strict improvement for both consumers.

Points such as $(x_1, 0), (e_1 - x_1, e_2)$ (along the bottom of the Edgeworth Box) are Strongly Pareto Efficient. To increase consumer 2's utility, we clearly need to increase x_2, which would reduce consumer 1's utility. To increase consumer 1's utility, we would either need to increase x_1 or set $y_1^{'} > x_1$, both of which would decrease consumer 2's utility. Finally, the points $(e_1, y_1), (0, e_2 - y_1)$ (along the right of the Edgeworth Box) are (trivially) Weakly Pareto Efficient since $e_1 > e_2$.

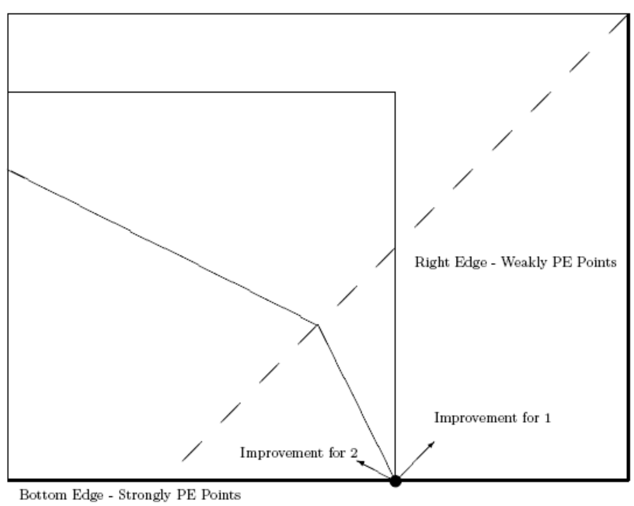

Figure 13: $u_1(x_1, y_1) = max\{x_1, y_1\}$

The only decentralizable allocations are $(x_1, 0), (e_1 - x_1, e_2)$ when $x_1 \leq 1$. As in (b), the Weakly Pareto Efficient allocations cannot be supported. The

Strongly Pareto Efficient points are maximal for consumer 1 if $p_1 \leq p_2$ - otherwise, he would purchase only commodity 2. When $x_1 \leq 1$, the prices $p = (\alpha, 2\alpha)$ for any $\alpha > 0$ will clearly constitute an equilibrium. When $1 < x_1 \leq 4$, however, in order for consumer 2 to be maximizing at $(e_1 - x_1, e_2)$, it must be that $p = (2\alpha, \alpha)$ for some $\alpha > 0$ (otherwise, he would be able to afford a better bundle). Clearly, $p_1 > p_2$ in this region, and thus equilibrium cannot be sustained. The corresponding transfers are identical to part (a).

(d) $u_1(x_1, y_1) = max\{min\{2x_1, y_1\}, min\{x_1, 2y_1\}\}$.

Solution:

First, let's get a handle on consumer 1's preferences. If $x_1 > 2y_1$, then $u_1(x_1, y_1) = 2y_1$, if $y_1 > 2x_1$, then $u_1(x_1, y_1) = 2x_1$ and if $2x_1 \geq y_1$ and $2y_1 \geq x_1$ then $u_1(x_1, y_1) = \max\{x_1, y_1\}$. Graphically, we have:

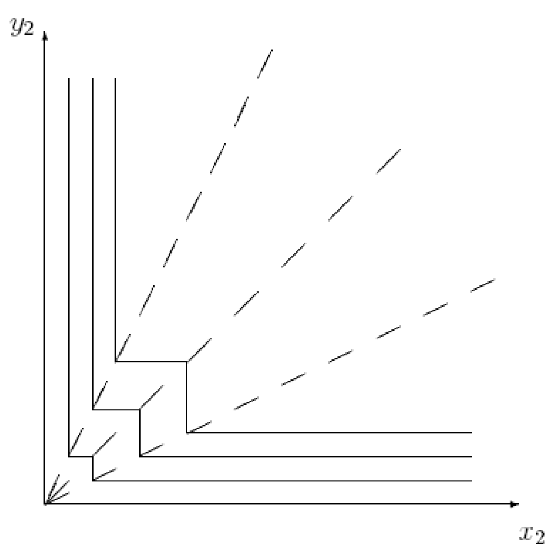

Figure 14: Consumer 1's Indifference Curves

Now let's examine the Pareto Efficient points. Clearly, allocations $(x_1, 0), (e_1 - x_1, e_2)$ and $(0, y_1), (e_1, e_2 - y_1)$ (along the bottom and left of the Edgeworth Box) are not Weakly Pareto Efficient since consumer 1 receives zero utility (we can simply move to the interior to improve both agents). Moreover, allocations $(x_1, e_2), (e_1 - x_1, 0)$ (along the top of the Edgeworth Box) are not Weakly Pareto Efficient either since the allocation

$$\left(x_1 + \varepsilon, \frac{x_1 + \varepsilon}{2}\right), \left(e_1 - (x_1 + \varepsilon), e_2 - \left(\frac{x_1 + \varepsilon}{2}\right)\right)$$

is an strict improvement for both consumers.

In the interior, the only (strongly and weakly) Pareto Efficient allocations are when $x_1 = 2y_1$ as in part (a). If $x_1 > 2y_1$, then transferring ε of x_1 to consumer 2 and δ of y_2 to consumer 1 will be a strict improvement for both. If $y_1 > 2x_1$, then moving to

$$\left(2x_1 + \varepsilon, \frac{2x_1 + \varepsilon}{2}\right), \left(e_1 - (2x_1 + \varepsilon), e_2 - \left(\frac{2x_1 + \varepsilon}{2}\right)\right)$$

will improve both consumers (see the figure below). Finally, if $2x_1 \geq y_1$ and $2y_1 > x_1$, then the allocations

$$\left(\max\{x_1, y_1\} + \varepsilon, \frac{\max\{x_1, y_1\} + \varepsilon}{2}\right), \left(e_1 - (\max\{x_1, y_1\} + \varepsilon), e_2 - \left(\frac{\max\{x_1, y_1\} + \varepsilon}{2}\right)\right)$$

will increase the utility of both consumers (again, see the figure below). The allocations $(e_1, y_1), (0, e_2 - y_1)$ for $y_1 \geq 2$ are weakly Pareto efficient, as consumer 1 cannot be made better off. Thus the Pareto points are identical to part (a).

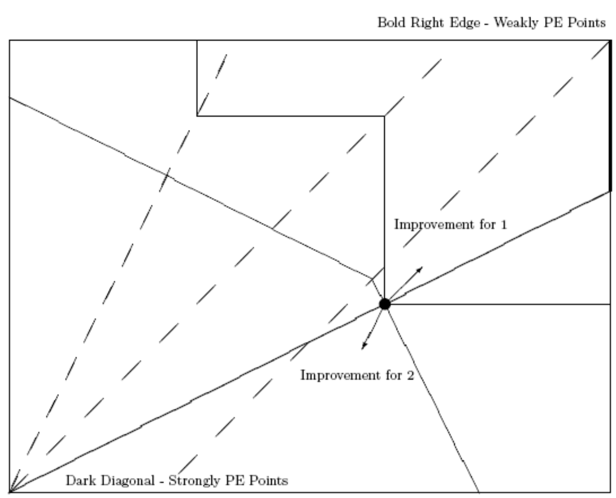

Figure 15: $u_1(x_1, y_1) = max\{min\{2x_1, y_1\}, min\{x_1, 2y_1\}\}$

The only decentralizable allocations are $\left(x_1, \frac{x_1}{2}\right), \left(e_1 - x_1, e_2 - \frac{x_1}{2}\right)$ when $x_1 \leq 2$. For these allocations, consumer 2 is maximizing his utility if prices are such that $p = (\alpha, 2\alpha)$ for any $\alpha > 0$ Under these prices, it is clear that consumer 1 is maximizing her utility. The Strongly Pareto Efficient allocations such that $2 < x_1 \leq 4$, however, are not decentralizable. At these allocations, prices must be such that $p = (2\alpha, \alpha)$ for some $\alpha > 0$ in order for consumer 2 to be maximizing over her budget set. Under these prices, however, consumer 1 can increase her utility by purchasing less of good 1 and more of good 2. Consequently, these points cannot be decentralized. The corresponding transfers are identical to part (a).