Economies with Production

December 5, 2007

1 Economies with production

Suppose there are ℓ commodities indexed $h = 1, ..., \ell$. The **commodity space** is $\mathbf{R}^\ell$. The economy consists of m **consumers** indexed $i = 1, ..., m$ and n **firms** indexed $j = 1, ..., n$. Each consumer i is characterized by

- a closed, nonempty **consumption set** X_i;
- an **endowment** $e_i \in X_i$;
- a **utility function** $U_i : X_i \to \mathbf{R}$;
- and a portfolio $\theta_i = (\theta_{i1}, ..., \theta_{in}) \in \mathbf{R}^n_+$ of shares in the different firms.

We adopt the convention that there is a total of one share outstanding in each firm, so θ_{ij} is the fraction of the j-th firm owned by consumer i. Each firm j is characterized by

- a production set Y_j.

The $(m+n)$-tuple $\mathcal{E}^p = (\{(X_i, e_i, U_i, \theta_i)\}_{i=1}^m, \{Y_j\}_{j=1}^n)$ is called an **economy with production**.

1.1 Equilibrium

An **allocation** for $\mathcal{E}^p$ is an ordered pair $(x, y) = ((x_1, ..., x_m), (y_1, ..., y_n))$ such that $x_i \in X_i$ for $i = 1, ..., m$ and $y_j \in Y_j$ for $j = 1, ..., n$. An allocation (x, y) is **attainable** if

$$\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} e_i + \sum_{j=1}^{n} y_j.$$

A (**Walrasian**) **equilibrium** for $\mathcal{E}^p$ consists of an attainable allocation (x^*, y^*) and a price vector $p^* \neq 0$ such that, for every $i = 1, ..., m$, x_i^* maximizes U_i on the budget set

$$B_i(p^*, \pi^*) = \{x_i \in X_i : p^* \cdot x_i \leq p^* \cdot e_i + \sum_{j=1}^{n} \theta_{ij} \pi_j^*\},$$

and, for every $j = 1, ..., n$ and any production plan $y_j \in Y_j$,

$$p^* \cdot y_j \leq p^* \cdot y_j^* \equiv \pi_j^*.$$

We call (x, y) a **Walras allocation** if (x, y, p) is a Walrasian equilibrium, for some price vector $p \neq 0$.

1.2 Welfare theorems for production economies

Theorem 1 *Let (x^*, y^*, p^*) be a Walrasian equilibrium for $\mathcal{E}^p$ and suppose that the utility function U_i is locally non-satiable at x_i^* for $i = 1, ..., m$. Then (x^*, y^*) is (strongly) Pareto-efficient.*

A **quasi-equilibrium** for $\mathcal{E}^p$ consists of an attainable allocation (x^*, y^*) and a price vector $p^* \neq 0$ such that, for every i and $x_i \in X_i$,

$$U_i(x_i) > U_i(x_i^*) \Longrightarrow p^* \cdot x_i \geq p^* \cdot x_i^*,$$

and for, every j and $y_j \in Y_j$,

$$p^* \cdot y_j \leq p^* \cdot y_j^*.$$

Theorem 2 *Consider an economy with production $\mathcal{E}^p$ and suppose that Y_j is convex for every j and U_i is quasi-concave and locally non-satiable for every i. Then for every Pareto-efficient allocation (x^*, y^*) there is a price vector $p^* \neq 0$ such that (x^*, y^*, p^*) is a quasi-equilibrium.*

A quasi-equilibrium for $\mathcal{E}^p$ is called an **equilibrium with lump sum transfers for $\mathcal{E}^p$** if

$$U_i(x_i) > U_i(x_i^*) \Longrightarrow p^* \cdot x_i > p^* \cdot x_i^*.$$

This condition implies that, for every i, the bundle x_i^* is optimal in the budget set

$$\{x_i \in X_i |\ p^* \cdot x_i \leq p^* \cdot e_i + \sum_{j=1}^{n} \theta_{ij} p^* \cdot y_j^* + t_i\},$$

where the number t_i is the lump sum transfer received by agent i and the transfers satisfy

$$\sum_{i=1}^{m} t_i = 0.$$

Theorem 3 *Let (x^*, y^*, p^*) be a quasi-equilibrium for $\mathcal{E}^p$ and suppose the X_i is convex and U_i is continuous for all i. If for every i there exists $x_i \in X_i$ such that $p^* \cdot x_i < p^* \cdot x_i^*$ then (x^*, y^*, p^*) is an equilibrium with lump sum transfers.*

2 Efficient production

Suppose that the prices of the inputs and outputs are represented by the vector p. The **profit** generated by a production plan y is defined to be

$$\pi = p \cdot y.$$

Note that taking account of our convention about the signs of inputs and outputs, this definition corresponds to the usual definition:

$$\text{profit} \ = \ \text{revenue} \ - \ \text{costs}.$$

A production plan y is **profit-maximizing** at the price vector p if

$$p \cdot y \geq p \cdot y', \forall y' \in Y.$$

A production plan y is **efficient** if it is impossible to produce more (or use less) of one good without producing less (or using more) of any other good, that is, there does not exist a production plan y' such that

$$y' > y,$$

where $y' > y$ means $y' \geq y$ and $y' \neq y$.

Proposition 4 *If $p \gg 0$ and $y \in Y$ is profit-maximizing, then y is efficient. If the production set Y is convex and $y \in Y$ is an efficient production plan, then y is profit-maximizing for some $p \neq 0$.*

Suppose that a firm has a finite number of plants indexed by $j = 1, ..., n$ and the technology of each plant j is represented by a production set Y_j. The aggregate technology of the firm is represented by the production set Y where

$$Y = \sum_{j=1}^{n} Y_j.$$

Note that we are assuming the various production activities are **independent**, which might not be the case if, for example, some managerial or other functions were performed centrally.

Proposition 5 *Given a price vector p, a production plan $y \in Y$ is profit-maximizing (respectively efficient) in Y if and only if there exist production plans $y_j \in Y_j$ for $j = 1, ..., n$ such that $y = \sum_{j=1}^{n} y_j$ and y_j is profit-maximizing (respectively efficient) in Y_j.*

Proposition 6 *Suppose that $p \gg 0$, $y \in Y$ is profit maximizing, and $y = \sum_{j=1}^{n} y_j$, where $y_j \in Y_j$ for each j. Then y_j is profit-maximizing and efficient in Y_j for each j. Suppose Y is convex, $y \in Y$ is efficient, and $y = \sum_{j=1}^{n} y_j$, where $y_j \in Y_j$ for each j. Then for some price vector p and each j, y_j is profit-maximizing for p in Y_j.*

Note that for the last result we do not need to assume convexity of the individual production sets Y_j, only the convexity of the aggregate set Y.

3 The 2×2 model of production

The 2×2 model of production, so-called because it assumes two produced goods and two factors of production, is a mainstay of the neoclassical theory of growth, development and international trade. There are two **factors**, capital and labor, and two produced goods, labelled $h = 1, 2$. The total endowment of the factors, denoted by K and L, is exogenous. Consumers derive no utility from consumption of capital and labor, so they supply them inelastically as inputs for production. The production technology is represented by two neoclassical production functions

$$Y_h = F_h(K_h, L_h),\ h = 1, 2,$$

where Y_h is the output of good h and K_h and L_h are the inputs of capital and labor for production of good h. The production functions F_h exhibit constant returns to scale:

$$F_h(tK_h, tL_h) = tF_h(K_h, L_h),\ t \geq 0.$$

This allows us to write output per worker y_h as a function capital per worker k_h:

$$y_h \equiv \frac{Y_h}{L_h} = \frac{1}{L_h} F_h(K_h, L_h) = F_h\left(\frac{K_h}{L_h}, \frac{L_h}{L_h}\right) = F_h\left(k_h, 1\right),$$

where $k_h \equiv K_h/L_h$. We denote the output per worker $F_h\left(k_h, 1\right)$ by the function $f_h(k_h)$ and assume that,

> for each $h = 1, 2,\ f_h(\cdot)$ is a C^1, increasing and strictly concave function, satisfying $f_h(0) = 0$ and $f_h'(k_h) \to \infty$ as $k_h \to 0$.

In what follows we normalize the aggregate labor supply $L = 1$—there is no loss of generality in this; it merely amounts to a choice of units—and denote the amount of labor applied to production of good 1 by $0 \leq \lambda \leq 1$.

3.1 Efficient production

Suppose that the prices of produced goods are exogenously given p_1 and p_2. Think of a small open economy in the prices of produced goods are determined by international trade. Since any amount of the produced goods can be bought or sold at the prices (p_1, p_2), the total quantity of goods available for consumption is a function of the value of goods produced. In other words, efficient production must maximize the value of output. The efficient allocation of factors is determined by the following problem:

$$\begin{array}{ll} \max & \lambda p_1 f_1(k_1) + (1-\lambda)\, p_2 f_2(k_2) \\ \text{s.t.} & \lambda k_1 + (1-\lambda)\, k_2 \leq K \\ & k_1, k_2 \geq 0,\ 0 \leq \lambda \leq 1. \end{array}$$

We assume that this problem has an interior solution $(k_1^*, k_2^*, \lambda^*)$ for the given values of (p_1, p_2):

$$k_1^*, k_2^* > 0,\ 0 < \lambda^* < 1.$$

Then the concavity of the objective function and the convexity of the feasible set implies that the following first-order conditions are necessary and sufficient:

$$\begin{gathered} p_1 f_1'(k_1^*) = \mu = p_2 f_2'(k_2^*) \\ p_1 f_1(k_1^*) - p_2 f_2(k_2^*) = \mu \left(k_1^* - k_2^*\right). \end{gathered}$$

Using the first condition to eliminate the Lagrange multiplier μ from the second, the conditions can be be rewritten as

$$\begin{gathered} p_1 f_1'(k_1^*) = p_2 f_2'(k_2^*) \\ p_1 \left(f_1(k_1^*) - f_1'(k_1^*)k_1^*\right) = p_2 \left(f_2(k_2^*) - f_2'(k_2^*)k_2^*\right). \end{gathered}$$

The first condition requires the equality of the marginal revenue product of capital in each activity; the second requires the equality of the marginal revenue product of labor in each activity.

Dividing the first equation by the second we get

$$\frac{f_1'(k_1^*)}{\omega_1(k_1^*)} = \frac{f_2'(k_2^*)}{\omega_2\left(k_2^*\right)}, \tag{1}$$

where $\omega_h(k_h) \equiv f_h(k_h) - f_h'(k_h)k_h$ for $h = 1, 2$. Notice that $f_h'(k_h)$ is decreasing in k_h because the marginal product of capital is decreasing and $\omega_h(k_h)$ is increasing in k_h because

$$\omega_h'(k_k) = f_h'(k_h) - f_h'(k_h) - f_h''(k_h)k_h = -f_h''(k_h)k_h > 0.$$

This means that the left and right sides of the equation are decreasing in k_1 and k_2 respectively. So the solutions to this equation have the form $k_2 = \phi(k_1)$, where ϕ is increasing.

Definition 7 *We say that good* 1 *is capital intensive (and good* 2 *is labor intensive) if, for any pair* k_1^* *and* k_2^* *satisfying (1) we have* $k_1^* > k_2^*$.

In what follows we always assume that good 1 is capital intensive. Note that the first-order conditions only involve the capital-labor ratios and that once the capital labor ratios are determined, the optimal allocation of labor is determined by the constraint $\lambda k_1^* + (1 - \lambda)\, k_2^* = 1$. This recursive structure is crucial for many of the special properties of the 2 × 2 model.

3.2 Equilibrium

Now we define a competitive equilibrium for this economy. We can ignore consumers, because their consumption will not affect the prices $p = (p_1, p_2)$ and their supply of factors is inelastic. So we only need to find prices at which the factor markets clear. We assume that each good is produced by a large number of competitive firms. Since they produce subject to constant returns to scale, there are no profits to distribute.

An allocation for this economy is a 4-tuple $(k_1, \ell_1, k_2, \ell_2) \in \mathbf{R}^4_+$ where k_h is the capital-labor ratio of firms producing good h and ℓ_h is the demand for labor from firms producing good h. An allocation $(k_1, \ell_1, k_2, \ell_2)$ is attainable if

$$\sum_{h=1}^{2} k_h \ell_h = K$$

and

$$\sum_{h=1}^{2} \ell_h = 1.$$

An equilibrium consists of an attainable allocation $(k_1^*, \ell_1^*, k_2^*, \ell_2^*)$ and factor prices (w^*, r^*) for capital and labor, respectively, such that for each good $h = 1, 2$, (k_h^*, ℓ_h^*) maximizes profits

$$p_h f_h(k_h)\ell_k - r^* k_h \ell_h - w^* \ell_h.$$

The necessary and sufficient conditions for profit maximization at an interior solution

$$p_h f_h'(k_h^*) = r^*$$

and

$$p_h f_h(k_h^*) - p_h f_h'(k_h^*) k_h^* = w^*,$$

are the same as the sufficient conditions for efficiency. Hence there is an equivalence between an efficient allocation and an equilibrium. If $(k_1^*, k_2^*, \lambda^*)$ is an interior efficient allocation then we can define an equilibrium $\{(k_1^*, k_2^*, \ell_1^*, \ell_2^*), (w^*, r^*)\}$ by putting $\ell_1^* = \lambda^*$ and $\ell_2^* = 1 - \lambda^*$ and putting

$$r^* = p_1 f_h'(k_h^*)$$

and

$$w^* = p_h f_h(k_h^*) - p_h f_h'(k_h^*) k_h^*.$$

Conversely, if $\{(k_1^*, k_2^*, \ell_1^*, \ell_2^*), (w^*, r^*)\}$ is an interior equilibrium, then $(k_1^*, k_2^*, \lambda^*)$ is an efficient allocation, where $\lambda^* = \ell_1^*$.

Several classical results of trade theory are easily derived from this framework. The **Rybcszynski Theorem** states that, assuming that good 1 is capital intensive and the equilibria are interior, an increase in the endowment of capital will increase the production of the capital-intensive good (good 1) and reduce the production of the labor-intensive good (good 2). This follows directly from the fact that, as long as production of both goods is positive and both factors are used in both sectors, the capital labor ratios are independent of the endowment of capital. Hence, the only way to absorb an increase in capital is to increase the production of good 1 and, since this means increasing the labor applied to good 1, the production of good 2 must decrease.

The **Stolper-Samuelson Theorem** states that, assuming that good 1 is capital intensive and the equilibria are interior, an increase in the price of the capital-intensive good (good 1) will increase the price of the factor used intensively in that good (capital) and lower the price of the other factor (labor). An increase in p_1 will increase the production of good 1 (this follows from the fact that the production possibility curve is concave towards the origin, which in turn is implied by the concavity of the technology and the capital intensity condition). We have seen that any movement along the efficient production locus will change k_1^* and k_2^* in the same direction. Thus, the constraint $\lambda^* k_1^* + (1 - \lambda^*) k_2^* = K$ implies that

$$\lambda^* dk_1^* + (1 - \lambda^*) dk_2^* + (k_1^* - k_2^*)\, d\lambda^* = 0.$$

Since $(k_1^* - k_2^*)\, d\lambda^* > 0$ and dk_1^* and dk_2^* have the same sign, we must have $dk_1^* < 0$ and $dk_2^* < 0$. The factor prices satisfy

$$(w^*, r^*) = p_2 \left(\omega_2(k_2^*), f_2'(k_2^*)\right),$$

so w^* will fall and r^* will increase as k_2^* decreases.

Finally, if we think of equilibrium in a two-country world, the **Factor Price Equalization Theorem** tells us that, assuming interior equilibria and the factor intensity assumption, the factor prices (w^*, r^*) must be the same in every country. Clearly, the product prices p will be equalized by free international trade. The capital-labor ratio is a function of the product prices, so as long as both goods are produced in each country, the capital-labor ratios for each good must be equalized. From the first-order conditions the factor prices are uniquely determined by the output prices and the capital-labor ratios, so factor prices are also equalized across countries.

4 Linear models of production

The material in these notes is based on David Gale's *Theory of Linear Economic Models*, published by the Rand Corporation (1960) and University of Chicago Press (1989).

4.1 The simple, linear, production model

There are assumed to be m activities and n goods. The **linear production model** is represented by an $m \times n$ matrix

$$\mathbf{A} = [a_{ij}]_{m\times n} .$$

The real number a_{ij} represents the net output of good j when activity i is operated at unit intensity. A negative number $a_{ij} < 0$ indicates that good j is a net input to activity i and a positive number $a_{ij} > 0$ indicates that good j is a net output from activity i. Each activity in the linear production model can have many inputs and many outputs.

In the **simple linear production model** each activity has only one output and there is precisely one activity that produces each good. Clearly, the number of activities is the same as the number of goods and we can number the activities so that activity i produces a net output of good i. It is convenient to change the notation slightly. Let $a_{ij} \geq 0$ be the amount of good j that is required to produce one unit of good i and let

$$\mathbf{A} = [a_{ij}]_{n\times n}$$

denote the **consumption matrix** of the simple linear production model. If the vector $\mathbf{x} = (x_1, ..., x_n) \geq 0$ denotes the **intensity vector** and $\mathbf{y} = (y_1, ..., y_n) \geq 0$ the bill of goods or **net output**, then the net output can be defined by

$$\mathbf{y} = \mathbf{x} - \mathbf{xA} = \mathbf{x}(\mathbf{I} - \mathbf{A}) .$$

Definition 8 *The consumption matrix* $\mathbf{A}$ *of the simple linear production model is said to be* ***productive*** *if there exists an intensity vector* $\bar{\mathbf{x}} \geq \mathbf{0}$ *such that* $\bar{\mathbf{x}} \gg \bar{\mathbf{x}}\mathbf{A}$.

Theorem 9 *If the consumption matrix* $\mathbf{A}$ *is productive, then* $\mathbf{y} = \mathbf{x}(\mathbf{I} - \mathbf{A})$ *has a solution* $\mathbf{x} \geq \mathbf{0}$ *for any* $\mathbf{y} \geq \mathbf{0}$*.*

Lemma 10 *If the consumption matrix* $\mathbf{A}$ *is productive and* $\mathbf{x} \geq \mathbf{xA}$*, then* $\mathbf{x} \geq \mathbf{0}$*.*

Proof. If the claim is not true, there exists an intensity vector $\mathbf{x} \geq \mathbf{xA}$ such that $x_i < 0$ for some activity i. But the definition of productivity implies the existence of an intensity vector such that $\bar{\mathbf{x}} \gg \bar{\mathbf{x}}\mathbf{A} \geq \mathbf{0}$. Let

$$\theta = \max_{i=1,\dots,n} \left\{ \frac{-x_i}{\bar{x}_i} \right\} = \frac{-x_1}{\bar{x}_1},$$

say. Let

$$\mathbf{x}' = \mathbf{x} + \theta\bar{\mathbf{x}} \geq \mathbf{0},$$

with $x_1' = 0$. But

$$\begin{aligned} \mathbf{x}' = \mathbf{x} + \theta\bar{\mathbf{x}} &\gg \mathbf{xA} + \theta\bar{\mathbf{x}}\mathbf{A} \\ &= \mathbf{x}'\mathbf{A} \geq \mathbf{0}, \end{aligned}$$

so

$$x_1' > \mathbf{x}' \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix} \geq \mathbf{0},$$

a contradiction. ■

Corollary 11 *If the consumption matrix* $\mathbf{A}$ *is productive then the matrix* $\mathbf{I} - \mathbf{A}$ *has full rank.*

Proof. From the lemma,

$$\begin{aligned} \mathbf{x}(\mathbf{I} - \mathbf{A}) &= \mathbf{0} \\ \Longrightarrow -\mathbf{x}(\mathbf{I} - \mathbf{A}) &= \mathbf{0} \\ \Longrightarrow \mathbf{x} \geq \mathbf{0}, &-\mathbf{x} \geq \mathbf{0} \\ \Longrightarrow \mathbf{x} &= \mathbf{0}. \end{aligned}$$

This implies that $(\mathbf{I} - \mathbf{A})$ is invertible. ■

The proof of Theorem 9 now follows easily. For any $\mathbf{y} \geq \mathbf{0}$ Corollary 11 tell us there is a unique $\mathbf{x}$ such that $\mathbf{x}(\mathbf{I} - \mathbf{A}) = \mathbf{y}$, and then since $\mathbf{y} \geq \mathbf{0}$ Lemma 10 tells us that $\mathbf{x} \geq \mathbf{0}$. This completes the proof.

Corollary 12 *A consumption matrix* $\mathbf{A}$ *is productive if and only if* $(\mathbf{I}-\mathbf{A})^{-1} \geq \mathbf{0}$.

Proof. The i-th row of $(\mathbf{I}-\mathbf{A})^{-1}$ is a vector $\mathbf{x}_i$ such that $\mathbf{x}_i(\mathbf{I}-\mathbf{A}) = \mathbf{e}_i$, where $\mathbf{e}_i = (0, ..., 0, 1, 0, ..., 0) \geq 0$. Then, as we have just seen, $\mathbf{x}_i \geq \mathbf{0}$.

Conversely, $(\mathbf{I}-\mathbf{A})^{-1} \geq \mathbf{0}$ implies that $\mathbf{x} = \mathbf{e}(\mathbf{I}-\mathbf{A})^{-1} \geq \mathbf{0}$, where $\mathbf{e} = (1, ..., 1)$. So $\mathbf{x}(\mathbf{I}-\mathbf{A}) = \mathbf{e} \gg \mathbf{0}$, which proves that $\mathbf{A}$ is productive. ■

Note: The theory can be extended to deal with **semi-productive** technologies. A consumption matrix is said to be semi-productive if there exists an intensity vector $\bar{\mathbf{x}} \geq \mathbf{0}$ such that $\bar{\mathbf{x}} > \bar{\mathbf{x}}\mathbf{A}$. Partition the activities into two sets, those that can support a positive net output (**productive activities**) and those than cannot (**non-productive activities**). Then it can be shown that if i is a productive activity and j is non-productive, $a_{ij} = 0$. In other words, the non-producible goods are not needed as inputs to the producible goods. One can, in a sense, ignore the non-productive activities (non-producible goods) and apply the theory to the productive activities (producible goods).

4.2 The Leontief model

The simple linear production model describes the production of commodities by means of commodities. There are no primary factors of production. The Leontief model allows for labor inputs in addition to the use of goods as inputs. More precisely, the **simple Leontief model** consists of a simple linear production model, represented by the consumption matrix $\mathbf{A}$, together with a vector of labor inputs $\mathbf{a}_0 = (a_{10}, ..., a_{n0}) \gg \mathbf{0}$. Without essential loss of generality, we assume the labor supply equal one unit. This is just a normalization; because of constant returns to scale, an increase in the labor supply will just increase the scale of production without changing anything else.

Theorem 13 *Suppose that the consumption matrix of the simple Leontief model is productive. Then there exists a vector* $\mathbf{p}$*, unique up to a scalar multiple, such that the profit of each activity is zero.*

Proof. The zero-profit condition is equivalent to

$$(\mathbf{I}-\mathbf{A})\,\mathbf{p} = \mathbf{a}_0.$$

Then

$$\mathbf{p} = (\mathbf{I} - \mathbf{A})^{-1} \mathbf{a}_0 \geq \mathbf{0},$$

because $(\mathbf{I} - \mathbf{A})^{-1}$ is non-negative by Corollary 12. ■

A **general Leontief model** satisfies all the properties of the simple Leontief model, except that for every good j there is a set of activities S_j that can produce good j. An intensity vector $\mathbf{x}$ is an element of $\mathbf{R}_+^m$ and for any intensity vector $\mathbf{x}$ the net ouput vector $\mathbf{y} = (y_1, ..., y_n)$ is defined by putting

$$y_j = \sum_{i \in S_j} x_i - \sum_{i=1}^{m} x_i a_{ij}$$

for each good $j = 1, ..., n$. A feasible intensity vector $\mathbf{x}$ must satisfy

$$\sum_{i=1}^{m} x_i a_{i0} \leq 1,$$

of course. The set of feasible net output vectors satisfying this feasibility constraint is denoted by Y and called the **net output space**.

Theorem 14 ***(Non-Substitution)*** *If a general Leontief system is productive, then there exists a set of activities $i_1, ..., i_n$ where $i_j \in S_j$ for $j = 1, ..., n$, such that the simple Leontief model formed by these activities has the same output space as the original model.*

Let $\mathcal{Y}$ denote the set of net outputs $\mathbf{y}$ (not necessarily non-negative) such that for some intensity vector $\mathbf{x} \geq \mathbf{0}$

$$y_j \leq \sum_{i \in S_j} x_i - \sum_{i=1}^{m} x_i a_{ij}, \text{ for } j = 1, ..., n,$$

and

$$\mathbf{x}\mathbf{a}_0 \leq 1.$$

It is clear that $\mathcal{Y}$ is convex, closed and non-empty. Since $\mathbf{A}$ is productive, there exists a net output vector $\bar{\mathbf{y}} \gg \mathbf{0}$. By scaling if necessary, we can ensure that $\bar{\mathbf{y}}$ belongs to the boundary of $\mathcal{Y}$ and choose the intensity vector $\bar{\mathbf{x}} \geq \mathbf{0}$ so that $\bar{\mathbf{x}}$ minimizes the labor input $\bar{\mathbf{x}}\mathbf{a}_0$ required to produce $\bar{\mathbf{y}}$.

Claim 15 *There exists a price vector $\mathbf{p} \neq 0$ such that $\mathbf{p} \cdot \bar{\mathbf{y}} \geq \mathbf{p} \cdot \mathbf{y}$ for any $\mathbf{y} \in \mathcal{Y}$.*

The claim follows directly from the Minkowski Lemma.

For each activity $i = 1, ..., m$ let $\mathbf{y}_i$ denote the net ouput vector (not necessarily non-negative) that is produced by activity i using all the available labor. That is, for any good j and activity $i \in S_j$, define $\mathbf{y}_i$ by putting

$$y_{ij} = \frac{1}{a_{0i}} (1 - a_{ij})$$

and

$$y_{ik} = -\frac{a_{ik}}{a_{0i}}, \text{ for } k \neq j.$$

Claim 16 *For some non-negative numbers* $\delta_1, ..., \delta_m$

$$\bar{\mathbf{y}} = \sum_{i=1}^{m} \delta_i \mathbf{y}_i.$$

In fact, we can choose $\delta_i = \bar{x}_i a_{0i}$ for each i and this proves

$$\sum_{i=1}^{m} \delta_i = \sum_{i=1}^{m} \bar{x}_i a_{0i} = \bar{\mathbf{x}}\mathbf{a}_0 = 1.$$

Claim 17 *For each activity* $i = 1, ..., m$, $\mathbf{p} \cdot \mathbf{y}_i \leq \mathbf{p} \cdot \bar{\mathbf{y}}$ *and* $\mathbf{p} \cdot \mathbf{y}_i < \mathbf{p} \cdot \bar{\mathbf{y}}$ *implies* $\delta_i = 0$.

Proof. Consider the problem

$$\begin{array}{rl} \max & \sum_{i=1}^{m} \delta_i \mathbf{p} \cdot \mathbf{y}_i \\ \text{s.t.} & \delta_i \geq 0, \ \forall i, \text{ and } \sum_{i=1}^{m} \delta_i = 1. \end{array}$$

The Kuhn-Tucker conditions for this problem imply that there exists a multiplier $\lambda > 0$ such that

$$\mathbf{p} \cdot \mathbf{y}_i \leq \lambda,$$

for $i = 1, ..., m$, and

$$\delta_i (\mathbf{p} \cdot \mathbf{y}_i - \lambda) = 0.$$

The claim follows immediately from the Kuhn-Tucker conditions once we realize $\lambda = \mathbf{p} \cdot \bar{\mathbf{y}}$. ■

As a corollary, we can use the last claim to show that the activities used to produce $\bar{\mathbf{y}}$ are optimal. Normalize the price vector $\mathbf{p}$ so that $\mathbf{p} \cdot \bar{\mathbf{y}} = 1$. Then for each $i = 1, ..., m$ we have $\mathbf{p} \cdot \mathbf{y}_i \leq 1$ and $\mathbf{p} \cdot \mathbf{y}_i < 1$ implies $\delta_i = 0$. This is equivalent to saying that, for each $j = 1, ..., n$ and $i \in S_j$, $p_j - \mathbf{p} \cdot \mathbf{a}_i - a_{i0} \leq 0$ and $p_j - \mathbf{p} \cdot \mathbf{a}_i - a_{i0} < 0$ implies $\delta_i = 0$. That is, at this price vector $\mathbf{p}$ the activities used ($\delta_i > 0$) are earning zero profits and the loss making activities are not used ($\delta_i = 0$).

Claim 18 *Without loss of generality, we can choose the numbers* $\delta_1, ..., \delta_m$ *so that* $\delta_i > 0$ *for exactly* n *activities.*

This is a simple corollary of the well known Caratheodory Theorem. To prove the claim, notice that all the activities with positive values of δ_i lie in the hyperplane H defined by

$$H = \{\mathbf{y} : \mathbf{p} \cdot \mathbf{y} = \mathbf{p} \cdot \bar{\mathbf{y}}\},$$

which is isomorphic to $\mathbf{R}^{n-1}$. So Caratheodory's Theorem implies that $\bar{\mathbf{y}}$ is a convex combination of at most n of the vectors $\{\mathbf{y}_i\}$.

We can select a set of basic activities $\mathcal{B} = \{i_1, ..., i_n\}$ such that $i_j \in S_j$ and let $\hat{\mathbf{A}}$ denote the simple Leontief model consisting of the n basic activities. Then $\hat{A}$ is productive (because it can produce $\bar{\mathbf{y}}$) and hence can produce any net output $\mathbf{y} \in Y$. It remains to show that the amount of labor used is less than or equal to 1. But this follows from the optimality of each activity. In fact, if $\mathbf{y}$ is produced using the activities in $\mathcal{B}$ then the amount of labor used is $\mathbf{p} \cdot \mathbf{y}$. If the same net output vector is produced using the activities $i = 1, ..., m$ at intensity $\mathbf{x} \geq \mathbf{0}$ then the amount of labor used is

$$\sum_{i=1}^{m} x_i a_{i0} \geq \sum_{j=1}^{n} p_j \sum_{i \in S_j} x_i - \sum_{i=1}^{m} x_i \mathbf{p} \cdot \mathbf{a}_i = \mathbf{p} \cdot \mathbf{y},$$

so if $\mathbf{y}$ is in the output space of the generalized Leontief model it must be in the output space of the simple Leontief model.

This completes the proof of the Non-Substitution Theorem.

5 Appendix: Properties of production sets

A **production plan** or vector is an element $y = (y_1, ..., y_\ell) \in \mathbf{R}^\ell$ where y_h represents the net output of commodity h. Thus $y_h < 0$ represents a net input of $-y_h$ units of commodity h to the production process. A **production set** $Y \subset \mathbf{R}^\ell$ is the set of all production vectors that are technologically feasible for an economy or a firm. The production set Y is sometimes described by a **transformation function** $F(\cdot)$:

$$Y = \{y \in \mathbf{R}^\ell : F(y) \leq 0\}.$$

It is usually assumed that $F(y) = 0$ if and only if y belongs to the boundary of Y. Production sets are often assumed to have some of the following properties:

Y is nonempty. There is at least one feasible vector $y \in Y$.

Y is closed. If y^q is feasible and $y^q \to y^0$ as $q \to \infty$ then y^0 is also feasible.

No free lunch. There are no inputs without outputs.

$$Y \cap \mathbf{R}^\ell_+ \subseteq \{0\}.$$

Possibility of inactivity.

$$0 \in Y.$$

Free disposal.

$$Y - \mathbf{R}^\ell_+ \subseteq Y.$$

Non-increasing returns to scale.

$$\alpha y \in Y, \forall y \in Y, \forall \alpha \in [0, 1].$$

Constant returns to scale.

$$\alpha y \in Y, \forall y \in Y, \forall \alpha \geq 0.$$

Additivity.

$$y + y' \in Y, \ \forall y, y' \in Y.$$

Convexity.

$$\alpha y + (1 - \alpha) y' \in Y, \ \forall y, y' \in Y, \forall \alpha \in [0, 1].$$