

Optimal Security Design

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How should new securities be designed? Traditional theories have little to say on this: the literature on capital structure and general equilibrium theories with incomplete markets takes the securities firms issue as exogenous. This article explicitly incorporates the transaction costs of issuing securities and develops a model in which the instruments that are traded are chosen optimally and the economy's market structure is endogenous. Among other things, it is shown that the firm's income stream should be split so that in every state all payoffs are allocated to the security held by the group that values it most.

It is widely acknowledged that there has been a significant increase in the amount of financial innovation in recent years [see, for example, Miller (1986)]. A vast number of new securities with novel features have been introduced. These include not only corporate securities, such as zero coupon bonds and floating rate bonds, but also other types of securities, such as financial futures, options on indexes, and money market funds, to name but a few. An important question concerns how such securities should be optimally designed; in other words, how should the payoffs to a security be allocated across states of nature in order to maximize the amount

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the issuer receives? Unfortunately, with a few exceptions [see, for example, Duffie and Jackson (1986) and Silber (1981) on the design of futures contracts and Williams (1986) on corporate securities as optimal contracts when there is asymmetric information], traditional theories have little to say on this issue. In this article our aim is to develop a benchmark model of security design by investigating the simple case where only firms issue financial claims.

The literature on firms' capital structure decisions assumes that a firm can issue only debt and equity. It does not consider the more basic question of whether debt and equity are the best securities the firm can issue. The result of Modigliani and Miller (1958) and of subsequent authors [such as Stiglitz (1969, 1974), Baron (1974, 1976), and Hellwig (1981)] that capital structure is irrelevant when markets are complete (assuming no taxes or frictions) suggests that the form of securities issued in this case is also unimportant. Although the result that capital structure is important when markets are incomplete indicates that the form of securities may also matter, it throws little light on their optimal design.

In order to develop a theory of optimal security design, it is clearly necessary to develop a framework in which markets are incomplete. Traditional general equilibrium theories with incomplete markets, such as Diamond (1967), Radner (1972), and Hart (1975), have taken those markets which are open and those which are closed as exogenous. The usual justification for considering incomplete markets is the existence of transaction costs of some sort. However, the relationship between these transaction costs and the securities that are the optimal ones for agents to issue is not considered.

The purpose of this article is to incorporate explicitly the transaction costs of issuing securities in order to determine their optimal design. We suppose that firms bear fixed costs of issuing securities and allocate payoffs to securities across states. Thus, the securities that are traded are chosen optimally and the incomplete-market structure is endogenous. The model is intended to be a simple one that can act as a benchmark. It is an abstraction that gives insight rather than a realistic description of what we observe.

Two important issues must be addressed when constructing this type of model. The first is precisely who can issue securities and what the costs of doing this are. We assume that firms can issue one security for some fixed cost and a second security for some additional amount. The payoffs to these securities are generated by the assets of the firm. Individual investors are unable to issue securities. In particular, we assume that individuals cannot short-sell firms' securities. If individuals could do this without cost, then there could not exist an equilibrium in which firms issue costly securities. The reason is that the short sellers are effectively able to expand the supply of a firm's security more cheaply than firms can. However, in order for a firm to be willing to issue securities, it must be able to recoup the cost of doing so. If firms face competition from short sellers who face no

costs, this will be impossible. The short selling of one firm's securities by another is also excluded, as this leads to nonexistence in the same way.

The justification for these assumptions is that short selling is fundamentally different from issuing securities backed by real assets. The existence of tangible and visible assets reduces the moral-hazard problems associated with the exchange of promises of future payment. In reality, it appears that short selling is very costly and is only rarely done. For most of the analysis, it is assumed that the costs are such that there is no short selling; however, at one point this assumption is relaxed to illustrate what happens when limited short selling is possible at a low cost.

The second issue concerns the knowledge that agents have about the prices of those securities which in principle could be issued by firms but which in equilibrium are not. We assume that both the firms and the consumers are aware of these prices. The trading process that we have in mind is analogous to the Lloyd's of London insurance market, where a price can be obtained for a policy insuring any risk whether or not such policies are actively traded. Our approach is therefore different from that taken by Hart (1979), who assumes that there is an asymmetry between firms and producers: Firms know the prices that would be obtained for securities that are not issued; consumers do not.

We obtain five main results:

1. The constrained efficiency (and existence) of equilibrium is demonstrated.

2. In our model, the number of securities that are issued in equilibrium can exceed the number of states and yet markets are incomplete in the sense that first-best risk sharing is not achieved. When technologies are convex in an appropriate sense, we show that an upper bound on the number of securities issued in equilibrium is $|J| \cdot |I|^2$, where $|J|$ is the number of different types of firms and $|I|$ is the number of different types of consumers.

3. As a point of reference, we consider examples where firms are constrained to use debt and equity but these securities are costly to issue. We show that Modigliani and Miller's (1958) irrelevance result does not hold: the value of firms has to depend on their financial structure to give them an incentive to issue the costly securities. In equilibrium both debt and equity are issued, provided the costs are sufficiently low.

4. If firms are not restricted to issuing debt and equity, but can allocate earnings between two securities in any way they choose, then the optimal securities need not be debt and equity. In fact, under fairly general conditions debt and equity *cannot* be optimal securities. When the firm issues two securities, each one is targeted at a particular clientele. Typically, in any state all the firm's output is allocated to the group that values it most.

5. The role of the no-short-sale assumption is shown to be critical. As argued above, if costless short sales are allowed, then equilibrium fails to

exist. Financial structure must affect firm value in order to provide an incentive for firms to issue securities. Without frictions of some kind or another, there will be arbitrage opportunities that preclude the existence of equilibrium. We illustrate what happens when limited short selling is possible at low cost. In this case equilibrium may or may not exist, depending on the cost of issuing securities.

Following the well-known paper by Miller (1977), a large literature on tax clienteles has arisen [recent examples are Dybvig and Ross (1986), Dammon and Green (1987), and Ross (1987)]. There are a number of parallels between this literature and our article. In the tax clientele literature, individuals with different marginal tax rates value assets differently at the margin. In order to maximize their market value, firms market their securities to particular tax clienteles. In equilibrium, securities are held by the clientele that values them most. Because of the different marginal valuations of different clienteles, tax arbitrage possibilities may arise and equilibrium may not exist unless arbitrage possibilities are limited by constraining short sales, assuming progressive taxation or some other means.

In any model with no short sales and incomplete markets, different investors will typically value assets differently at the margin. In such a setting, firms can increase their market value by issuing securities that take advantage of the different marginal valuations of these different "clienteles." In contrast to the tax clientele perspective, securities are endogenous in our model. Firms design securities to suit particular groups, whereas in the tax clientele literature the form of the securities issued by firms is determined by the tax code. Another distinction is that our clienteles result from differences in preferences and endowments (together with incomplete markets) rather than from tax structure. However, we also need a short-sale constraint to ensure that equilibrium exists, because the different marginal valuations of different investors would present arbitrage opportunities if unbounded short sales were allowed. In addition, in both cases asset supplies determine marginal valuations and hence the form of equilibrium.

Grossman and Stiglitz (1980) discuss a model in which the incentives to trade arise primarily from differences in beliefs. In contrast, in our model the incentives to trade arise from differences in endowments and risk preferences. Nevertheless, in both instances there is an analogous problem concerning the existence of equilibrium. In their model, if endowments are nonstochastic then prices fully reveal traders' information. However, this means that it is not worthwhile acquiring costly information. But if nobody acquires information, it will be worthwhile to do so. Hence, no equilibrium exists. In our model, the gross amount received by firms that issue multiple securities must be strictly greater than the gross amount received by a single-security firm, so that the costs of issuing securities can be recouped. However, if short sales are costless, this creates an arbitrage opportunity. But if no firm issues multiple securities, it will be worthwhile

doing so. Hence, no equilibrium exists unless there are costs of short selling.

The article proceeds as follows. Section 1 gives an informal description of the model; a formally precise definition is contained in the technical Appendix. Section 2 contains an analysis of the general model; the more complex formal proofs are again in the Appendix. The next two sections develop illustrative examples. Some readers may prefer to read these before Section 2. For purposes of comparison, Section 3 shows what happens when there are costs of issuing securities but firms are constrained to issue debt and equity, and Section 4 considers the case where firms are free to allocate their earnings in a particular state to two securities in any way they desire. Section 5 considers the role of short-sale constraints. Finally, Section 6 contains concluding remarks.

1. The Model

The formal definition of the model is given in the Appendix. An intuitive description of its main features is given here.

There are two dates ($t = 0, 1$), and there is a finite set of states of nature ($s \in S$) that occur with probability π_s . All agents have the same informational structure: There is no information at the first date, and the true state is revealed at the second. At each date there is a single consumption good. There is a finite set of types of producers ($j \in J$) and a finite set of types of consumers ($i \in I$). There is a continuum of agents of each type. For simplicity's sake, it is assumed below (except where otherwise stated and in the Appendix) that the measure of each type of producer and each type of consumer is 1.

1.1 Producers

A type of producer is defined by a security set and a cost function. When a firm chooses an element from its security set, it is choosing both its production plan and its financial structure. A producer of type $j \in J$ has a set of production plans Y_j to choose from. A particular production plan is represented by $y_j = (y_j(s))_s \in S$, where $y_j(s)$ denotes the output of the good in state s at date 1 for every $s \in S$.

It is assumed that any production plan can have at most two types of claims issued against it. The analysis can readily be extended to allow for more than two claims, but this makes the exposition more cumbersome and so is not pursued here. The two claims are indexed by $k = 1, 2$. A financial structure is a specification of the claims against the production plan. In the case where the claims are debt and equity, the financial structure corresponds to the mix between debt and equity that the firm uses. Just as there is a continuum of mixes of debt and equity that a firm can adopt, there is a continuum of financial structures that a firm can choose. For each distinct production plan, the corresponding set of possible finan-

cial structures can also be thought of as distinct. Hence, without loss of generality, it is possible to describe a firm's joint choice of production plan and financial structure by its choice of financial structure alone. The financial structures available to the firm are indexed by $e \in E_j$. The index set E_j is referred to as the set of available financial structures. For any financial structure $e \in E_j$ and $k = 1, 2$, let $r^k(e)$ denote the vector of dividends corresponding to the k th claim in the financial structure e . Thus, the properties of different financial structures and different claims are completely described by the dividend functions r^1 and r^2 .

The dividend functions are assumed to satisfy certain natural conditions. Dividends are always nonnegative: $r^1(e) \geq 0$ and $r^2(e) \geq 0$ for all structures $e \in E_j$. In other words, we restrict attention to securities satisfying limited liability. If securities could have negative payoffs, this would be similar to individuals short-selling securities. The reasons for assuming that short sales are costly are discussed in the introduction and in Section 5; similar comments are applicable here. Inactivity is possible; that is, if the producer decides not to operate, then $r^1(e) = r^2(e) = 0$. Dividends are assumed to exhaust the production plan; that is, $r^1(e) + r^2(e) \equiv y_j$. Also, it is possible to issue equity only: $r^1(e) = y_j$ and $r^2(e) = 0$.

The cost function of producers of type $j \in J$ is denoted by C_j . For any financial structure $e \in E_j$, $C_j(e)$ is the cost of operating a firm with a financial structure e . The cost of operating the firm includes the cost of inputs to the production process (i.e., investment) as well as the costs of issuing securities. If a producer decides not to operate a firm and thus issues no securities, no cost is incurred. The null structure corresponding to this action is denoted by $e = 0$, and thus $C_j(0) = 0$.

Producers derive no utility from future consumption. Every producer maximizes his (current) profit, taking as given the market value of firms with different financial structures. Let $v^k(e)$ denote the market value of the k th claim on a firm with financial structure e for $k = 1, 2$, and $e \in E_j$. Then the market value (MV) of a firm with financial structure e is $v^1(e) + v^2(e) \equiv MV(e)$, say. A producer of type j chooses e to solve

$$\max_{e \in E_j} MV(e) - C_j(e)$$

Since the producers' problem is nonconvex, there may not be a unique maximum. Let v_j denote the distribution of financial structures chosen by producers of type j ; that is, for any measurable set $H \subset E_j$, $v_j(H)$ is the measure of producers of type j who choose financial structures $e \in H$.

For simplicity, we have assumed that producers (i.e., the original owners of the firms) consume only at $t = 0$. The objective function of the firm would be the same if the firm had several original owners who consumed at both $t = 0$ and $t = 1$. When a firm is small, its decisions do not affect the set of securities available to the economy as a whole; either the securities issued by the firm are already spanned by other firms' securities or they must be priced so that in equilibrium consumers hold negligible amounts. In either case the only effect of the choice of a change in financial

structure is on the wealth of the firm's owners. So value maximization is consistent with utility maximization and will be unanimously approved by the firm's owners. This was shown by Hart (1979) in a different context.

1.2 Consumers

Let Ω denote the set of nonnegative consumption bundles. For any consumption bundle $x \in \Omega$, it is useful to write $x = (x^0, x^1)$, where x^0 denotes consumption at date 0 and where $x^1 = (x^1(s))_{s \in S}$ is the vector of consumption levels in different states at date 1. That is, $x^1(s)$ is the consumption level in state s at date 1 for every $s \in S$.

A consumer type $i \in I$ is characterized by a utility function and an endowment of goods. The utility function of type i consumers is denoted by $U_i(x)$. Preferences are assumed to be well-behaved: U_i is strictly increasing and strictly quasi-concave. The consumer's endowment is denoted by $w_i \in \Omega$.

A portfolio is represented by an ordered pair $\alpha = (\alpha^1, \alpha^2)$ of measures on the set $E \equiv \cup_{j \in J} E_j$. For any measurable set $H \subset E$, $\alpha^k(H)$ is the number of units of the k th claim on firms with financial structures $e \in H$ held in the portfolio. Since short sales are not allowed, portfolios are nonnegative: $\alpha^k(H) \geq 0$ for any measurable set $H \subset E$ and any $k = 1, 2$.

Recall that $v^k(e)$ is the price in terms of consumption at date 0 of the k th claim on a firm with financial structure e for any $k = 1, 2$ and $e \in E_j$. For any security price function $v = (v^1, v^2)$, the value of a portfolio $\alpha \in A$ is denoted by $\alpha \cdot v$, where

$$\alpha \cdot v \equiv \sum_{k=1}^2 \int_E v^k d\alpha^k$$

Similarly, the income at date 1 from a portfolio $\alpha \in A$ is denoted by $\alpha \cdot r$, where

$$\alpha \cdot r \equiv \sum_{k=1}^2 \int_E r^k d\alpha^k$$

In this notation the budget constraints of a consumer of type i can be written as follows:

$$\begin{aligned} x^0 &= w_i^0 - \alpha \cdot v \\ x^1 &= w_i^1 + \alpha \cdot r \end{aligned}$$

(The budget constraints are written as equations because U_i is assumed to be strictly increasing.) For any security price function v and any choice of portfolio $\alpha \in A$, the budget constraints define a unique consumption bundle $x = \xi_i(\alpha, v)$. So the consumer's problem is to choose a portfolio α to maximize utility, subject to the requirement that the consumption bundle $\xi_i(\alpha, v)$ implied by α and v is feasible. Formally,

$$\max_{\alpha \in A} U_i[\xi_i(\alpha, v)] \text{ s.t. } \xi_i(\alpha, v) \geq 0$$

1.3 Equilibrium

We define our equilibrium concept in two stages. First we outline a Walrasian type of equilibrium concept, and then we add an extra condition. Our notion of equilibrium is similar to that suggested by Hart (1979), but with one important difference, which is pointed out below.

In Walrasian equilibrium all agents maximize (expected) utility, taking prices for both issued and unissued securities as given, and all markets clear at the prevailing prices. Formally, a Walrasian equilibrium must specify a price function $v = (v^1, v^2)$, a portfolio $\alpha_i = (\alpha_i^1, \alpha_i^2)$ for each type of consumer $i \in I$, and a distribution v_j of financial structures for each type of producer $j \in J$. The equilibrium conditions are:

- (E1) For every type $i \in I$, α_i maximizes the consumers' utility, subject to their budget constraints and to the nonnegativity constraints, given the price function v .
- (E2) For every type $j \in J$, all the mass v_j is concentrated on financial structures that maximize profits.
- (E3) Markets must clear: $\sum_{i \in I} \alpha_i = \sum_{j \in J} \nu_j$

In equilibrium each firm takes the securities issued by other firms as given, and in a large economy a single firm's securities form a negligible part of any investor's portfolio. Thus, the firm expects its securities to be priced according to investors' marginal valuation of a unit of the security. These marginal valuations will be independent of the firm's actions because any individual holds a negligible amount of its securities. This is the sense in which our concept of equilibrium involves price taking by firms. The *rational conjecture condition* says that the price that firms expect to receive if they issue a security is the maximum amount that any individual would be prepared to pay for a very small quantity of it. For any consumer type $i \in I$ with an equilibrium consumption bundle $x_i \in \Omega$, let $p_i(x_i)$ denote the vector of marginal rates of substitution defined by

$$p_i(x_i) = \frac{\frac{\partial U_i(x_i)}{\partial x^s}}{\frac{\partial U_i(x_i)}{\partial x^0}}$$

If we write $p_i(x_i) = (p_{is}(x_i))$, then $p_{is}(x_i)$ is the amount of date 0 consumption that a consumer of type i is willing to give up in exchange for an extra unit of consumption in state s at date 1 for any $s \in S$. Suppose that a security offers a vector of dividends $r^k(e)$ at date 1. Suppose also that every consumer has positive consumption at date 0. For some very small quantity ε of this security, a consumer of type i ought to be willing to pay $\varepsilon p_i(x_i) \cdot r^k(e)$ units of consumption at date 0. In other words, the price that he should pay is $p_i(x_i) \cdot r^k(e)$. The equilibrium price $v^k(e)$ should be the maximum such willingness to pay; that is,

$$v^k(e) = \max_{i \in I} p_i(x_i) \cdot r^k(e)$$

for every claim $k = 1, 2$ and financial structure $e \in E_j$. (If not all consumers have positive consumption at date 0, the maximum should be taken over the set of types that do.)

An *equilibrium* is defined to be a Walrasian equilibrium that satisfies the rational conjecture condition. This condition does not affect the equilibrium allocation in any substantive way. It is automatically satisfied in Walrasian equilibrium for every security that is actually issued. Because of the restrictions on short sales, this is not the case for securities that are not issued. However, if it is not satisfied for unissued securities in some Walrasian equilibrium, then one can satisfy it simply by reducing the value of $v^k(e)$ appropriately without making any other change in the Walrasian equilibrium.

Hart (1979) distinguishes between markets that are open and markets that are closed. If a particular security is not issued by any firm, he assumes that market to be closed. Also, consumers are not allowed to trade in that security and they do not observe a market-clearing price for that security. On the other hand, producers know the price that would clear the market if a small quantity of a nonissued security were to be introduced.

An equilibrium as we have defined it differs from Hart's in one important respect. As far as producers are concerned, the two concepts are the same; it really makes no difference whether producers see prices quoted for every possible security or have rational conjectures about what prices would be if a new security were issued. For consumers, on the other hand, it makes a difference whether they can trade every security at the quoted price, even if they choose not to trade some of them in equilibrium. In other words, there is a difference between a closed market and a market that is open but inactive because no trade takes place at the prevailing price. The importance of this difference is discussed in Section 2.

2. General Properties of Equilibrium

The first property we note is that the equilibria of our model are all constrained-efficient. An equilibrium is said to be constrained-efficient if a planner who is subject to the same transaction costs as individual agents cannot make everyone better off. In this context the condition "subject to the same transaction, costs" is interpreted to mean that the planner can reallocate securities and consumption only at the first date. (The formal definition of *constrained efficiency* that we use is given in the Appendix.)

Under this interpretation the constrained efficiency of equilibrium is not surprising. Using the budget constraints to solve for second-period consumption, we can express the utility of any agent in terms of first-period consumption and of the portfolio of securities he holds at the end of the period. Then the economy is seen to be isomorphic to an Arrow-Debreu economy in which only first-period consumption and securities are traded. Pareto efficiency in this Arrow-Debreu economy is equivalent to constrained efficiency in the original. So our result can be seen as a special

case or application of the First Fundamental Theorem of welfare economics. Thus, we have the following result:

Theorem: 1. *Every equilibrium is constrained-efficient.*

Proof. See the Appendix. ■

This result is not surprising when seen in the right perspective. Nevertheless, it stands in sharp contrast to the claims of Hart (1980) and Makowski (1980); both authors argue in the context of a product-differentiation model formally similar to ours that "equilibrium" need not be efficient. The reason for this difference of opinion can be traced to their definition of equilibrium. It is important to understand the difference between their concept and ours because there are substantive modeling issues involved. Both Hart (1980) and Makowski (1980) use a concept of equilibrium introduced in Hart (1979). As outlined in Section 1, Hart's concept is as follows. Prices are quoted for securities that are actually issued in equilibrium. If a security is not issued, there is no market for it; no price is quoted, and consumers simply assume that it cannot be bought. On the other hand, firms must make conjectures about the price at which a security could be sold if it were to be introduced; otherwise, they could not make an optimal decision about which security to offer. These conjectures are assumed to be rational; that is, the conjectured price equals that maximum price at which the economy would absorb a small amount of the security. The rest of the definition is standard. Let us call Hart's concept a conjectural *equilibrium* to distinguish it from our *Walrasian equilibrium*.

This sketch of conjectural equilibrium allows us to see the essential cause of market failure in the Hart-Makowski model and why it is absent in ours. There is a complementarity between products in their models that may lead to a pecuniary externality. This externality can easily be explained by an example. Suppose that nuts and bolts can be produced only by different firms. Nuts and bolts have value only if consumed together. If nuts are not produced, the marginal value of a bolt is zero, so bolts will not be produced either. The same argument shows that nuts will not be produced if bolts are not produced. Thus, both types of firms are maximizing profits at zero output given the behavior of the other and their own (rational) conjectures. Yet this conjectural equilibrium is not efficient, since a coordinated increase in the output of both products could make everyone better off.

The same kind of phenomenon clearly could occur in our model if we analyzed it by using the conjectural equilibrium concept. Think of an economy with two types of firms, each of which has positive output in one state only (thus, each type of firm can issue essentially one kind of security), and suppose that consumers have indifference curves for consumption in these two states that do not intersect the axes (for example, a Cobb-Douglas utility function). These consumers will not value consumption in one of

these states unless they have positive consumption in the other. Thus, we may have a conjectural equilibrium in which neither type of firm produces output and issues securities. This is an equilibrium for firms because they rightly conjecture that their securities would be worthless if issued in isolation. Consumers are in equilibrium, on the other hand, because the markets for both securities are assumed to be closed if the supply of the securities is zero. The reason why this cannot happen in our model is that prices are quoted for every security whether it is actually issued or not. If the prices of both securities were zero, firms would be maximizing profit at zero output, but consumers would not be maximizing utility if they thought they could buy at the prevailing price and they still demanded none. So markets will not clear at a zero output level. In the Hart-Makowski version of the model, on the other hand, the securities markets are closed to consumers. They do not need to worry about clearing these markets.

At some level it probably seems natural to assume that a market does not exist if the corresponding good or security is not being produced. This kind of reasoning is encouraged by a tendency to equate markets with a gathering of people for the purchase and sale of commodities. But when we speak of a market existing in a Walrasian model, this is not what we have in mind. The existence of a market in this context is just another kind of equilibrium condition. A market “exists” if and only if a market-clearing price prevails; that is, buyers and sellers can agree on a common price at which they all think they can trade any amount that is small relative to the economy as a whole. It is not immediately obvious why the existence of a market in this sense should depend on a nonzero amount of the corresponding commodity being traded. For example, at Lloyd’s of London it is possible to obtain a price for insuring any risk even though no policy of that type is actively traded. Similarly, investment banks will quote prices for tailor-made securities.

There may be reasons why a market does not exist in this sense, but these reasons are not apparent in the Hart-Makowski model. There are no transaction costs; as soon as a firm decides to produce a positive amount of some commodity, a market, that is to say, a market-clearing price, springs costlessly into existence. Nor is it clear that there are other obstacles to the free flow of information implied by a market-clearing price. After all, producers correctly conjecture the prices at which they can sell a non-existent product even without the aid of a market. So it is not clear why, in equilibrium, there should not be a market-clearing price for every commodity. To say that some of these conditions will not be satisfied because the corresponding markets do not “exist” is tautologous. The existence of a market *means* that an equilibrium condition is satisfied, nothing more.

The next result concerns the existence of equilibrium.

Theorem 2. *Under the maintained assumptions the equilibrium set is nonempty*

Proof. See the Appendix. ■

This theorem does not require comment except to note the role of the no-short-sales assumption in guaranteeing the existence of equilibrium. This assumption is needed in order to prevent the equalization of marginal rates of substitution everywhere. As we noted in the introduction, some divergence of marginal valuations between different consumer types is necessary in order to give firms an incentive to create costly securities.

The next question to be addressed is the number of securities needed in equilibrium. Without essential loss of generality, it can be assumed that there is only one type of firm: $|J| = 1$. It will be clear that the argument generalizes easily. Initially, suppose that the number of securities is finite. Let Y be a convex, compact set of output vectors and let

$$E = \{(\rho_1, \rho_2) \in R_+^s \times R_+^s \mid \rho_1 + \rho_2 = y \in Y\}$$

Let \bar{E} denote the set of nontrivial splits; that is, $\bar{E} = \{e \in E \mid r^k(e) \neq 0, k = 1, 2\}$. Assume also that costs are *convex* in the following sense: $(\lambda_1 + \lambda_2) C(e_3) \leq \lambda_1 C(e_1) + \lambda_2 C(e_3)$ for any $\lambda_1, \lambda_2, \geq 0$ and $e_1, e_2, e_3 \in E$ such that $(\lambda_1 + \lambda_2)r(e_3) = \lambda_1 r(e_1) + \lambda_2 r(e_2)$.

This definition of convexity is quite general in that it allows costs to vary with financial structure as well as with the number of securities. For example, financial structures with a large proportion of debt may be more costly to issue than financial structures with a small amount of debt. Thus, the issuing costs for an expensive two-security financial structure could be several times greater than the issuing costs for a relatively cheap two-security financial structure. However, a case of particular interest, which is considered below in some detail, is that in which the costs of issuing securities depend only on the number issued and are independent of the qualitative properties of the securities.

Without loss of generality, consider a pair of consumer types i_1 and i_2 who jointly own firms with financial structures e_1 and e_2 in E . We can assume, again without loss of generality, that consumer types i_k own all of the k th security. Let λ_1 denote the measure of firms with structure e_1 owned by these consumers, and let λ_2 denote the measure of firms with structure e_2 owned by these consumers. By convexity of Y there exists a financial structure $e_3 \in \bar{E}$ such that

$$(\lambda_1 + \lambda_2) r^i(e_3) = \lambda_1 r^i(e_1) + \lambda_2 r^i(e_2)$$

for $i = 1, 2$. Thus, we can replace the firms that have financial structures e_1 and e_2 with an equal measure of firms that have financial structure e_3 and leave consumers equally well off. The convexity assumption on C ensures that firms are equally well off choosing the financial structure e_3 :

$$\begin{aligned} \sum_{k=1}^2 v^k(e_3) - C(e_3) &\geq \sum_{k=1}^2 p_k(x_k) \cdot r^k(e_3) - C(e_3) \\ &\geq (\lambda_1 + \lambda_2)^{-1} \sum_{b=1}^2 \lambda_b \left\{ \sum_{k=1}^2 p_k(x_k) \cdot r^k(e_b) - C(e_b) \right\} \end{aligned}$$

$$\geq (\lambda_1 + \lambda_2)^{-1} \sum_{b=1}^2 \lambda_b \left\{ \sum_{a=1}^2 v^a(e_b) - C(e_b) \right\}$$

Since $\sum_h v^h(e_1) - C(e_1) = \sum_h v^h(e_2) - C(e_2)$, the preceding calculation shows that e_3 is optimal. In this way we have shown that for every pair (i_1, i_2) of consumer types, there need be only one (nontrivial) financial structure.

We may also need firms with trivial financial structures (i.e., firms that issue only one security). But making the appropriate convexity assumptions, there will need to be only one security (i.e., one trivial financial structure) per consumer type. Then the total number of securities in equilibrium (or in any constrained-efficient allocation) need not exceed $|I| \cdot |I - 1| + |I| = |I|^2$. The extension to $|J|$ types of firms is immediate.

Theorem 3. *With convex technologies the number of securities needed in equilibrium is less than or equal to $|J| \cdot |I|^2$.*

Note that in “proving” Theorem 3 we have assumed that the number of securities is finite to start with. Since any finite measure can be approximated arbitrarily closely by a measure with finite support, the argument can be extended to the general case simply by “taking limits.”

It is interesting to compare Theorem 3 with results obtained in standard stock market economies [Hart (1977)]. When each firm issues only a single security (equity), the number of securities needed is $|J| \cdot |I|$. For each type of firm $j \in J$ each consumer type gets its own tailor-made company (production plan), and this is clearly the best that one can do. We allow firms to vary production plans and financial structures, whereas Hart allows them only to vary production plans. The $|J| \cdot |I|$ securities in Hart’s model correspond to the $|J| \cdot |I|$ firms with trivial (one-security) financial structures in our model. In addition, however, there are potentially $|J| \cdot |I| \cdot |I| - 1$ cases where a pair of investors effectively choose a production plan and split it optimally between them. Theorem 3 gives only an upper bound; the actual number of securities might be less if, for example, the number of states was very small.

Finally, we consider the form that optimal securities take. Until now we have combined production costs with the costs of issuing securities and have described them both by the cost function $\{C_j\}$. In the last part of this section we assume that production costs are convex and that the costs of issuing securities are fixed. More precisely, we assume that a firm issuing one security incurs a cost c_1 and that a firm issuing two securities incurs a cost $c_1 + c_2$. These costs are independent of the qualitative properties of the securities issued. When these conditions are satisfied, we say that the costs of issuing securities are *fixed costs*. Under these conditions there is no loss of generality in assuming that securities take a special form. A security is *extreme* if, in any state of nature, it promises either the entire

product of the firm or nothing. These are the only securities that need to be considered when issuing costs are fixed.

Theorem 4. *Suppose that issuing costs are fixed costs. Then for any equilibrium there exists another equilibrium in which the consumption of every agent and the production of every firm are the same and in which only extreme securities are issued.*

Proof. Consider a fixed, but arbitrary, firm with output vector y . Let X be the set of extreme points of the set $R \equiv \{0 \leq r \leq y\}$. Then $x \in X$ implies $x(s) \in \{0, y(s)\}$ for any state $s \in S$. Any nontrivial financial structure consistent with the output vector y can be identified with a vector $r \in R$: if $e \in \bar{E}$ and $r^1(e) + r^2(e) = y$, then put $r_1(e) = r$ and $r^2(e) = y - r$. For any $r \in R$ there exists a set $\{x_k\} \subset X$ and a set of numbers $\{\alpha_k\}$ such that $\alpha_k > 0$, $\sum_k \alpha_k = 1$, and $r = \sum_k \alpha_k x_k$. Suppose that a measure $m > 0$ of firms choose a production plan y and issue securities with payoffs r and $y - r$, respectively. For each k let a measure $a_k m$ choose a production plan y and issue securities with payoffs x_k and $y - x_k$, respectively.

By assumption, the cost to the firm has not changed. The value of the firm is also unchanged. From the definition of equilibrium the value of a firm issuing $(x_k, y - x_k)$ cannot be greater than the value of one issuing $(r, y - r)$. However, the aggregate value of the firms cannot be less; otherwise, consumers would never have purchased the original securities. Thus, firms are maximizing profits with the new financial structures.

Since the new securities span the old, consumers can obtain their existing consumption bundles by holding appropriate amounts of the new securities. Market clearing must hold since each unit of old securities corresponds to a unique convex combination of extreme securities. Finally, the cost to consumers has not changed since, by the definition of equilibrium, the cost of each consumer's portfolio cannot decrease and yet the aggregate value must be the same. ■

We have proved the theorem for the case where firms issue two securities. It can be seen that the result also applies where firms are permitted to issue more than two securities.

In Section 4 we give examples which illustrate that the optimal securities involve splitting the firm's income stream so that in every state all payoffs are allocated to the security held by the group that values it most. Before doing this, we provide a point of reference by considering the case where firms can issue only standard securities.

3. Examples with Debt and Equity

This section develops illustrations of the case where firms are restricted to issuing debt and equity. This situation can arise if the legal system is such that debt and equity are the only types of securities that can be used. The defining characteristic of debt is that the *par payment* (i.e., the prom-

ised payment) is the same in all states. In states where the output of the firm is below the par payment, the debt holders receive the entire output. If the firm's output is above the par payment in all states, the debt is *safe*; otherwise, it is *risky*.

In all the examples it is assumed that there is one type of producer and that the measure of firms is 1. The set of output vectors that each firm can produce is $Y = \{0, \bar{y}\}$. If a firm does not operate so that its output vector is 0, its costs are also 0. If a firm operates and produces the output vector \bar{y} , its costs are c_1 if it issues one security and $c_1 + c_2$ if it issues two securities; in other words, the marginal cost of issuing the second security is c_2 . Example 1a is a specific illustration where c_1 is assumed to be zero and c_2 is strictly positive. Example 1b is a more general version of 1a where c_1 and c_2 are both positive.

There are two types of consumers, and the measure of each group is 1. Consumers have a von Neumann-Morgenstern utility function \hat{U}_i . In this case, for any consumption bundle $x \in \Omega$, $U_i(x)$ is the expected utility of x , where

$$U_i(x) = \sum_{s \in S} \pi_s \hat{U}_i(x^0, x^1(s))$$

Example 1a

As mentioned above, in this illustration $c_1 = 0$ and $c_2 > 0$. There are two states ($s = 1, 2$) that are equally likely, so $\pi_s = 0.5$. It is assumed that the output vector \bar{y} that each firm can produce when it operates is

$$\bar{y}(1) = 1 \quad \bar{y}(2) = 2$$

The two types of consumers ($i = a, n$) have von Neumann-Morgenstern utility functions

$$\hat{U}_a = x^0 + V(x^1) \quad \text{with } V' > 0, V'' < 0$$

$$\hat{U}_n = x^0 + x^1$$

Hence, group a is risk-averse in date 1 consumption and group n is risk-neutral. When a specific functional form is used for V , it is assumed that

$$V(x^1) = 2 \ln(1 + x^1)$$

It is helpful to define the marginal utility of consumption of type i in state s as

$$\mu_{is}(x^1) = \frac{p_{is}(x^1)}{\pi_s}$$

so that

$$\mu_{ns} = 1$$

Since $c_1 = 0$, all firms operate and issue at least one security. First,

consider the case where c_2 is sufficiently large that firms issue only one security, which must be equity:

$$r^1(s) = \bar{y}(s) \quad \text{for all } s$$

It follows from the definition of equilibrium that the price that firms expect to receive if they issue a security is the maximum amount that any individual would be prepared to pay for a very small quantity of it. Hence, the value of the one-security firms' equity is

$$v^1 = \max_{i=1,2} \sum \pi_s \mu_{is} \bar{y}(s) = \sum \pi_s \mu_s \bar{y}(s)$$

where the term μ_s is used to denote the marginal utility of consumption that is relevant for determining market values. Now $\mu_s = \mu_{as} > 1$ for all s if endowments are such that only the risk-averse group holds the security, and $\mu_s = \mu_{ns} = 1$ for all s if they are such that the risk-neutral group holds the security. Consider the case where both groups hold the security, so that $\mu_s = \mu_{ns} = 1$ for all s and $v^1 = 1.5$. The situation in which the supply of the security is sufficiently small that only the risk-averse group holds it can be similarly analyzed.

It can straightforwardly be shown that the risk-averse group's demand for the firms' equity is

$$a^{1*} = 0.629$$

(where a^{1*} is used here and below to denote the demand of the risk-averse group when $v^1 = 1.5$). The risk-neutral group holds the remaining 0.371 of the security. The total value of a firm is $v^1 = 1.5$. The marginal utilities of consumption for the risk-averse group are

$$\mu_{a1} = 1.228 \quad \mu_{a2} = 0.886$$

Next, suppose that a firm issues debt so that the payoffs to its securities are as shown in Table 1. Given that all other firms are issuing equity which is priced with $\mu_s = \mu_{ns} = 1$ for all s , it follows that the risk-averse group will value this firm's debt the most because $\sum \pi_s \mu_{as} = 1.057 > \sum \pi_s \mu_{ns} = 1.000$. Since the risk-averse group holds all of this debt, its price is determined by the group's marginal utilities of consumption μ_{as} and is thus 1.057. This firm's equity is valued the most by the risk-neutral group since $\mu_{n2} = 1.000 > \mu_{a2} = 0.886$. Because the risk-neutral group holds all of this equity, its price is 0.5. Thus, the total value of such a firm is $1.557 - c_2$.

It can be seen that the capital structure shown in Table 1 is in fact the optimal one for a firm issuing debt and equity in this situation. The cost of issuing a second security is independent of its payoffs. Since the debt is held by the risk-averse group, its value would increase by only $\pi_2 \mu_{a2} = 0.443$ for each unit of output that the par payment on the debt is increased above 1. This is less than the $\pi_2 \mu_{n2} = 0.5$ that the risk-neutral group would be prepared to pay if the payoff were allocated to the equity it holds. Thus, such a change would lower the value of the firm. Similarly, a reduction in

Table 1
Payoffs to debt and equity securities in Example 1

Claim	Payoff		Price
	State 1	State 2	
r^d	1	1	1.057
r^e	0	1	0.500

Suppose that a firm issues debt (claim r^d) and equity (claim r^e) when all other firms issue just one security. The risk-averse group values the debt most because it allows this group to smooth its consumption. The risk-averse group bids the debt's price above its expected payoff of 1 to $(0.5)(1)(1.228 + 0.886) = 1.057$, where 0.5 is the probability of the two states, 1 is the debt's payoff in both states, and 1.228 and 0.886 are the group's marginal utilities of consumption in the two states. At this price the risk-neutral group would like to short-sell the debt but is constrained by the short-sale restriction. Similarly, the firm's equity is valued most by the risk-neutral group, which bids its price to $(0.5)(1) = 0.5$, and the short-sale constraint binds on the risk-averse group.

the par payment on the debt will also reduce the value of the firm since the risk-averse group values these marginal payoffs more than the risk-neutral group does. A similar argument holds in all two-state examples: it is always optimal for a firm to issue the maximum amount of risk-free debt possible. It can easily be seen that this is a special feature of the two-state case; with three or more states, firms can find it optimal to issue either risky or risk-free debt, depending on the parameter values.

Thus, for $c_2 > 0.057$ only equity will be issued by firms, and for $c_2 < 0.057$ some proportion of the firms will issue both debt and equity; therefore, three types of securities will exist in total: the one-security firms' equity and the two-security firms' debt and equity. The risk-neutral group of consumers holds the equity of both the one- and two-security firms. The risk-averse group holds the equity of the one-security firms and the debt of the two-security firms. The prices of both types of equity claims are equal to their expected return, and the price of the debt is such that firms are indifferent between issuing one or two securities. The debt is priced above its expected return in order to compensate firms for the cost that they bear in issuing it; for example, if $c_2 = 0.05$, then the price of the debt is 1.05, the price of the equity of a two-security firm is 0.5, and the price of the equity of a one-security firm is 1.5. The proportion of firms issuing two securities is 0.111, and the proportion issuing one is 0.889. The risk-averse group holds 0.556 of the one-security firms' equity. As $c_2 \rightarrow 0$, more firms issue debt. When $c_2 = 0$, the debt's price is equal to its expected return and the risk-averse group holds just debt, while the risk-neutral group holds the rest of the securities. There is *full risk sharing* in the sense that all firms allocate their output between their security holders efficiently (i.e., either marginal utilities of consumption at $t = 1$ are equated or all output goes to the group with the highest marginal utility; in the case here, marginal utilities are equated).

Example 1b

Consider next what happens when c_1 and c_2 are both positive. (To allow the above results to be placed in context, here we shall analyze a gener-

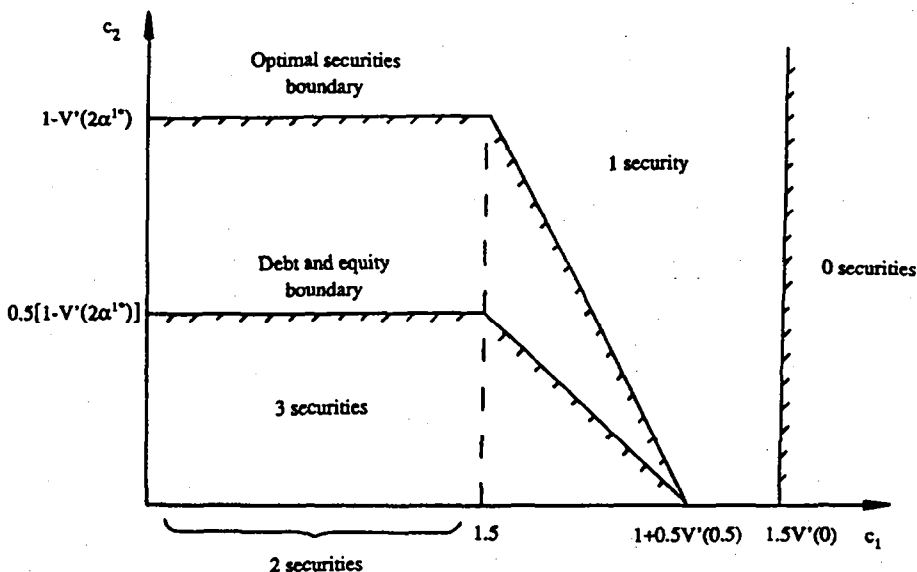


Figure 1
The relationship between the issuing costs and the total number of securities issued in Examples 1 and 2

These examples have two equally likely states and one firm type with output (1, 2) in the two states. The symbol c_1 represents the cost of a firm issuing one security, and c_2 represents the cost of a firm issuing a second security. The marginal utility of consumption of risk-averse and risk-neutral investors is $V(\cdot)$ and 1, respectively. The number of securities in the figure refers to the total number of securities issued by all firms (e.g., three if some firms issue two contingent claims and others issue only equity). The debt-and-equity boundary is the locus of costs in Example 1b at which the solution switches from all firms issuing equity to some firms issuing equity and others issuing both debt and equity. The optimal securities boundary is the locus of costs in Example 2b at which the solution switches from all firms issuing equity to some firms issuing equity and others issuing two optimally designed securities.

alization of Example 1a.) Now the only restrictions on the utility function are that $V'(1) = 1$ and $V'(0)$ is finite [i.e., $V(x^1) = 2 \ln(1 + x^1)$ is a special case of this]; otherwise, the details are the same as in Example 1a.

Figure 1 illustrates the relationship between the issuing costs and the number of securities issued. Example 1a corresponds to the c_2 axis where $c_1 = 0$. For $c_2 > 0.5[1 - V'(2a^{1*})]$ ($= 0.057$ in Example 1a), there are only one-security firms, and so the total number of securities issued is 1. For $c_2 < 0.5[1 - V'(2a^{1*})]$, some firms issue debt and equity and the remainder only equity, so that three securities are issued in total. The line labeled "Debt-and-equity boundary" represents the remainder of the boundary between the regions where one and three securities are issued. (The line labeled "Optimal securities boundary" is discussed below in Section 4.)

It is not worthwhile for any firms to operate and issue a security unless $c_1 < 1.5 V'(0)$. For values of c_1 above this level, no investment is undertaken and no securities are issued. As c_1 falls below $1.5 V'(0)$, more firms operate and issue one security. Initially, these are held entirely by the risk-averse group since $V'(0) > 1$, and so $\mu_{as} > \mu_{ns} = 1$ for all s . Eventually there comes a value $c_1 = 1 + 0.5 V'(0.5)$ where group a 's demand is 0.5, so that $\mu_{a2} =$

1. For $c_1 \leq 1 + 0.5V'(0.5)$, $\mu_{a2} \leq \mu_{n2} = 1$. For sufficiently small c_2 , it becomes worthwhile for some firms to issue debt and equity since the risk-neutral group will be prepared to pay $\mu_{n2} = 1$ ($\geq \mu_{a2}$) per unit of expected income in state 2 and will hold the two-security firm's equity. The debt and one-security firms' equity are held by the risk-averse group. As c_2 rises, the boundary also rises, because $\mu_{n2} - \mu_{a2}$ must be sufficiently large to allow two-security firms to recoup their additional costs. Along the boundary, v^2 falls and a^2 increases. It is this that causes $\mu_{n2} - \mu_{a2}$ to rise.

When $c_1 = 1.5$ it is profitable for one-security firms to operate and be held by the risk-neutral group. At this point there is *full investment* in the sense that all firms operate and issue at least one security. Since the risk-neutral group holds at least some of the one-security firms' equity, it is priced at 1.5. There is no further scope for a fall in the price of the one-security firms' equity, and so the boundary is horizontal as c_1 falls from 1.5 to 0.

The only region where two types of securities are issued in total is along the c_1 axis where $c_2 = 0$ between 0 and 1.5. Here all firms operate, and since the marginal cost of issuing a second security is zero, they issue enough debt for the risk-averse group to smooth completely its consumption across states. Thus, in this region there is full investment and full risk sharing.

Another feature of this example is worth noting. Even though there are only two states, there can nevertheless be three securities in existence without there being complete markets in the sense that full risk sharing is possible. The reason for this is that there are no short sales. Thus, when there are costs of issuing securities and when there are constraints on short sales, the question of whether or not the number of securities is greater or less than the number of states is of less significance than when these factors are ignored.

The standard Modigliani-Miller theory, adapted to include corporate taxes and bankruptcy costs, suggests that firms choose their capital structure to trade off the tax advantage of debt against the cost of going bankrupt. Jensen and Meckling (1976), among others, have argued that a weakness of the theory is that it predicts that before the existence of the corporate income tax, debt would not be used at all since bankruptcy was costly. Of course, debt was widely used before the introduction of the tax. They argue that this demonstrates the importance of other factors, such as asymmetric information, in determining firms' capital structures. The examples above illustrate that, together with having no short sales, the costs of issuing securities are also a form of friction which are consistent with the existence of interior optimal debt ratios even when there is no tax advantage to debt. (The no-short-sale constraint may represent a cost, possibly arising from asymmetric information.)

Myers (1984) has stressed that conventional theories of capital structure are inconsistent with the observation that similar firms in the same industry often have significantly different debt/equity ratios. It can be seen that the

illustrative examples above have this feature. In equilibrium, firms of the same type are indifferent between having an optimally levered capital structure and being unlevered because of nonconvexities. In more complex examples with more types of consumers and firms, it is possible for there to be many equally profitable optimal levels of leverage for firms of the same type. Allen (1987) also suggests a theory that is consistent with similar firms having different debt ratios. However, in Allen's model the asymmetric nature of the equilibrium results from the strategic interaction of firms in an imperfectly competitive product market. In contrast, here the asymmetric nature of equilibrium arises because it provides better risk-sharing opportunities for investors.

4. Examples with Optimal Securities

In this section it is assumed that firms are not restricted to issuing debt and equity. Instead, when they issue two securities they are free to allocate the firm's output in a particular state between the two securities in any way they wish. It is shown that the optimal securities are not debt and equity; nevertheless, they do have a particularly simple form.

Before discussing specific examples, we make some general observations about the optimality of debt and equity when the issuing costs are positive. *If consumers have smooth preferences, then it is never optimal to issue debt and equity.* The argument is as follows. If issuing two securities is costly, the firm will issue two securities only if this increases the firm's gross value. A necessary condition for this is the existence of two types of consumers, each of which values one of the optimal securities more highly than the other type does. For example, a firm of type j chooses a financial structure $e \in E_j$ and sells security $r^1(e)$ to type 1 consumers and security $r^2(e)$ to type 2 consumers. Then

$$\begin{aligned} v^1(e) &= p_1(x_1) \cdot r^1(e) > p_2(x_2) \cdot r^1(e) \\ v^2(e) &= p_2(x_2) \cdot r^2(e) > p_1(x_1) \cdot r^2(e) \end{aligned}$$

If either of these inequalities were violated, it would be more profitable for the firm to issue a single security to one of these types. A necessary condition for these inequalities to hold is that for each type there is a state in which the firm has positive output and that this type has a higher marginal valuation than the other type. Then each security must be allocated the entire output in one state at least, so the securities cannot correspond to (risky) debt and equity.

The importance of the smoothness of consumers' preferences is that it ensures that for any $e \in E_j$ and $i \in I$, $v^i(e) \geq p_i(x_i) \cdot r^i(e)$. This condition must hold in any Walrasian equilibrium, even if we do not impose the rational conjecture condition (cf. Section 1). Thus, if there exists a structure $\hat{e} \in E_j$ such that

$$p_1(x_1) \cdot r^1(\hat{e}) + p_2(x_2) \cdot r^2(\hat{e}) > p_1(x_1) \cdot r^1(e) + p_2(x_2) \cdot r^2(e)$$

it immediately follows that $v^1(\hat{e}) + v^2(\hat{e}) > v^1(e) + v^2(e)$, contradicting the optimality of $e \in E_j$. This is the formal justification for the last step in the above argument showing that debt and equity are not optimal. The argument breaks down, however, if preferences are not smooth. Take an example like Example 1a in Section 3, but replace the utility function of type a with

$$\hat{U}_a = x^0 + 2 \min \{x^1, 1\}$$

If the costs of issuing securities are positive but sufficiently small, it will be optimal for firms to issue securities having returns (1, 1) and (0, 1), respectively (i.e., debt and equity).

Example 2a

Consider first an example with the same structure and parameter values as Example 1a. As before, when c_2 is large so that all firms issue only equity, group a 's demand is 0.629, $\mu_{a1} = 1.228$, and $\mu_{a2} = 0.886$. If a firm were to issue debt as its second security, it would obtain $(0.5)(1.228) + (0.5)(0.886) = 1.057$ for it and $(0.5)(1) = 0.5$ for its equity. However, this is not the best it can do. It can clearly increase its receipts by reducing the payment on its r^1 security in state 2 and increasing the payment on the other security. The r^1 security is held by the risk-averse group, which values each unit of expected revenue in state 2 at only 0.886, whereas the r^2 security is held by the risk-neutral group, which values each unit at 1.000. Similarly, the firm should not reduce the payment on the r^1 security in state 1 since it is held by the risk-averse group, which values each unit at 1.228, whereas the risk-neutral group values each unit at only 1.000. This means that the optimal pair of securities for a firm to issue has the payoffs and prices shown in Table 2.

The optimal securities thus have a particularly simple form. All the output in a particular state should be allocated to the security held by the group that values it most. This feature of optimal securities is quite general. It is due to the fact that firms' objective functions are linear, which arises from the competitive nature of the model. It does not depend on the existence only of two states and of two groups, one risk-neutral and one risk-averse.

The total value of a firm that issues two securities when all other firms issue one security is $1.614 - c_2$. Hence, firms will start issuing debt and equity at the critical cost $c_2^* = 0.114$. This contrasts with the case where firms issue debt and equity where the critical level $c_2^* = 0.057$ is much lower. Thus, optimal securities can permit a strictly better allocation of risk for a given (positive) issue cost than can debt and equity.

The claims issued by two-security firms in this example are, in effect, pure contingent claims. This feature arises from the fact that the number of states of nature is the same as the maximum number of securities that a firm can issue. If the number of states exceeds the number of securities, some optimal claims will have positive payoffs in more than one state. To illustrate this, let us consider an example similar to that above but with

Table 2
Payoffs to optimal securities in Example 2

Claim	Payoff		Price
	State 1	State 2	
r^1	1	0	0.614
r^2	0	2	1.000

The optimal securities involve giving all the firm's payoffs to the security that is held by the group with the highest marginal utility of consumption (MUC) in that state. In state 1 the risk-averse group has an MUC of 1.228, whereas the risk-neutral group has an MUC of 1. Thus, claim r^1 , which is held by the risk-averse group, has a payoff of 1; and r^2 , which is held by the risk-neutral group, has a payoff of 0. In state 2 the MUCs are 0.886 and 1, respectively, so all the output is allocated to the r^2 claim and none to the r^1 claim. The gross value of the firm rises from 1.057, when the debt and equity are issued, to 1.614.

three equally probable states of nature in which firms' outputs are $\mathbf{y} = (1, 2, 3)$. In the case where c_2 is large, so that only one-security firms exist, the marginal utilities of consumption of the risk-averse group in the three states are $\mu_a = (1.374, 1.046, 0.845)$, respectively. A two-security firm would maximize its value by issuing a claim that paid (1, 2, 0) for the risk-averse group and a claim with payoffs (0, 0, 3) for the risk-neutral group. It is also possible to construct examples where all the payoffs in state 2 go to the risk-neutral group; for example, if $\mathbf{y} = (1, 2.5, 3)$, then $\mu_a = (1.408, 0.975, 0.885)$. The only case in which it is optimal for both securities to have a positive payoff in state 2 is when both groups have the same marginal utility of consumption in that state; for example, if $\mathbf{y} = (1, 2.3, 3)$, then $\mu_a = (1.394, 1.000, 0.868)$.

Example 2b

Figure 1 illustrates what happens when c_1 and c_2 are both positive. The line labeled "Optimal securities boundary" is now the boundary between the region with three securities and the region with one security. The explanation of the form of the boundary is similar to that given in Example 1b. The difference is that with optimal securities it is possible to allocate output more efficiently to those who value it most; with debt the problem is that firms cannot give less in state 2 than in state 1. As a result, the optimal securities boundary is shifted out relative to the debt-and-equity boundary. The two boundaries coincide only when they intersect with the c_1 axis at $1 + 0.5 V'(0.5)$. In that case both boundaries are determined by the condition that the amount of a one-security firm's equity held by the risk-averse group is such that $\mu_{a1} = \mu_{n1} = 1$. This is unaffected by the form of the second security. It is only when $c_2 > 0$ and it is necessary that $\mu_{a1} = \mu_{n1} + c_2 > 1$ that the form of the second security makes a difference.

Example 3

The only region in Figure 1 in which two securities in total are issued is when $c_2 = 0$ and c_1 is between 0 and 1.5. However, it is possible to construct examples where the total number of securities is 2 even when c_1 and c_2 are strictly positive. To illustrate this, let us consider another case with

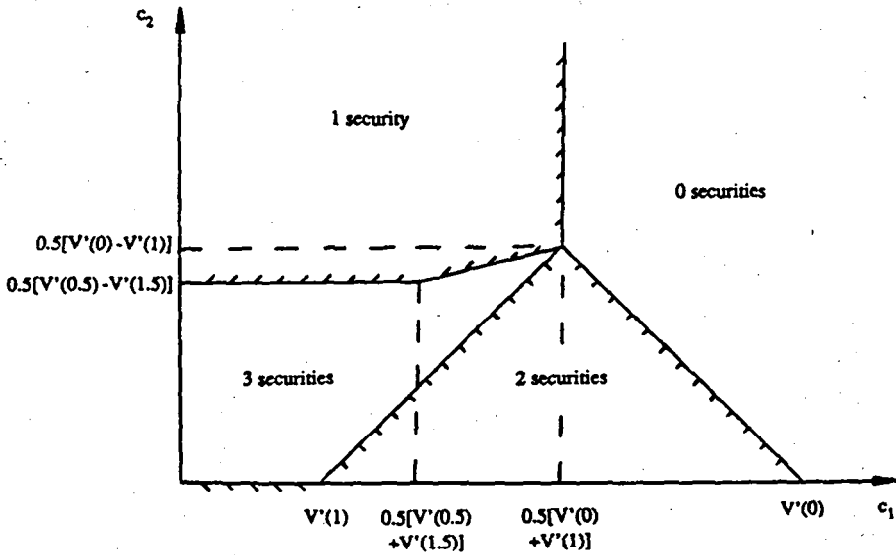


Figure 2a
The relationship between the issuing costs and the total number of securities issued in Example 3
 The example has two equally likely states and one firm type with output (1, 1) in the two states. There are two groups of consumers who differ by the state in which they receive their endowment (the utility function is common). As in Figure 1, c_1 represents the cost of a firm issuing one security, and c_2 represents the cost of a firm issuing a second security. The number of securities in the figure refers to the total number of securities issued by all firms (e.g., three if some firms issue two contingent claims and others issue only equity).

two groups of consumers and one type of producer. Both groups have the same (risk-averse) utility function but differ in their endowment in the two (equally likely) states at $t = 1$. Group 1 receives 1 unit of output in state 1 and 0 units in state 2. For group 2, the reverse is true. When a firm operates, its output vector \mathbf{y} is

$$\mathbf{y}(1) = \mathbf{y}(2) = \mathbf{1}$$

Hence, the example is symmetric and it is this, as opposed to the risk neutrality assumed before, which simplifies it. The measure of consumers of each type and of producers is 1. The utility functions of consumers are the same as those for group a in Examples 1 b and 2 b .

Figure 2a illustrates the relationship between the issuing costs and the total number of securities issued. When just one security is issued, then it must be equity with a payoff of 1 in both states. Thus, it will not be worth issuing just one security unless $c_1 \leq 0.5[V'(1) + V'(0)]$. Again, the best that a two-security firm can do involves splitting itself in such a way that all the output in a particular state is allocated to the group of security holders that values it most. Thus, the optimal securities have the payoffs shown in Table 3. The prices shown are the gross values, given that the two groups consume only their endowments. It will be worth issuing at least two securities whenever $c_1 + c_2 \leq V'(0)$. It follows that no securities

Table 3
Payoffs to optimal securities in Example 3

Claim	Payoff		Price
	State 1	State 2	
r^1	1	0	$0.5V'(0)$
r^2	0	1	$0.5V'(0)$

Again, the best that a two-security firm can do involves splitting itself in such a way that all the output in a particular state is allocated to the group that values it most. Claim r^1 is held by the group that is endowed with output only in state 2. When investors consume only their endowments, they bid claim r^1 's price to $(0.5)(1)V'(0) = 0.5V'(0)$, where 0.5 and 1 are the state 1 probability and payoff, respectively, and $V'(0)$ is the group's marginal utility of consumption in state 1. Similarly, claim r^2 is held by the group that is endowed with output only in state 1.

at all will be issued to the right of the boundary with the kink at $(0.5[V'(1) + V'(0)], 0.5[V'(0) - V'(1)])$.

For c_2 below the kink and to the left of this boundary, firms will find it worthwhile to issue two securities, as shown in Figure 2a. In this region there is full risk sharing but not full investment: all the firms that operate allocate their output between their security holders efficiently, but not all firms that could operate do so. As c_1 falls, more firms operate. When $c_2 = 0$, the point where all firms invest is reached when $c_1 = V'(1)$. To the left of this, all firms still issue only two securities and there is both full risk sharing and full investment. For small values of c_2 , those firms which issue two securities must do at least as well as those which issue one. For sufficiently small values of c_1 , firms are indifferent between issuing one and two securities. But for large values it is not worthwhile issuing just one: firms can do strictly better by issuing two. The points at which all firms issue two securities but are indifferent between this and issuing only one gives the boundary between the regions where two and three securities are issued in total.

All firms find it optimal to issue only one security above and to the left of the kink at $(0.5[V'(1) + V'(0)], 0.5[V'(0) - V'(1)])$. The boundary between this and the region where three securities are issued in total occurs at the points where all firms issue one security but are indifferent between this and issuing two. Thus, when $c_1 = 0$ and all firms issue at least one security that is held equally by the two groups, it occurs at the point where $c_2 = 0.5[V'(0.5) - V'(1.5)]$. In the case shown in Figure 2a, $V''' > 0$ and the point is below $0.5[V'(0) - V'(1)]$; otherwise, it would be above. When c_1 is sufficiently small, all firms operate and the boundary is horizontal. However, when $c_1 > 0.5[V'(0.5) + V'(1.5)]$ the boundary slopes up as c_1 increases and the amount of equity held in the one-security region falls.

Figure 2b shows how the efficiency of investment and risk sharing is affected by the issuing costs. As can be seen, (1) provided that the marginal cost of issuing the first security is sufficiently low, there is full investment efficiency, and (2) provided that the marginal cost of issuing the second security is sufficiently low, there is full risk-sharing efficiency. When both are high, there is a compromise between the two. The only case where

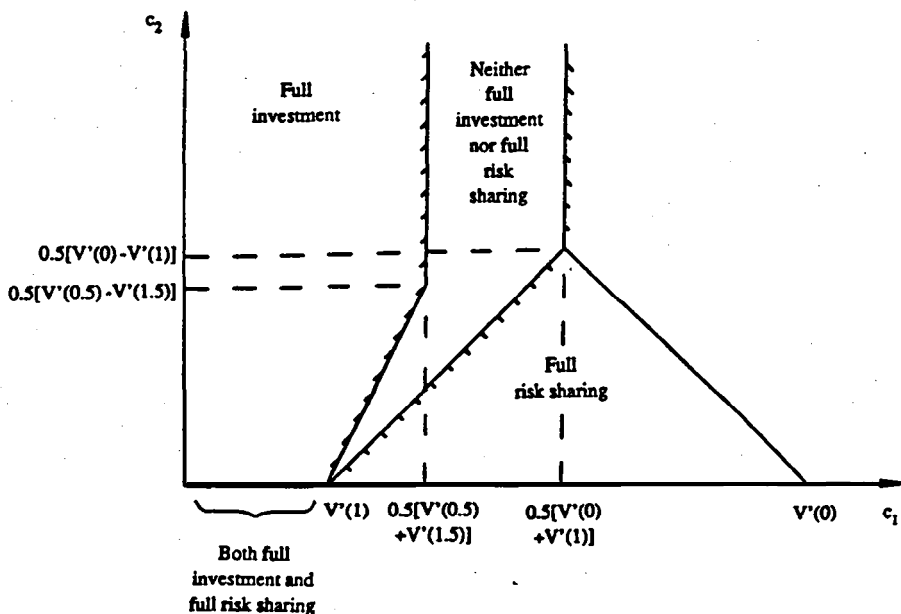


Figure 2b
The relationship between issuing costs, risk sharing, and investment in Example 3
 See the legend for Figure 2a. In the region labeled "Full risk sharing," firms allocate their output to investors efficiently, and in the region labeled "Full investment," all firms operate.

there is full investment and full risk sharing, so that the allocation is fully efficient, is when $c_2 = 0$ and c_1 is sufficiently small.

In summary, this section has developed simple examples to illustrate the optimal design of securities. It has been shown that debt and equity are not optimal securities even in cases where one group is risk-neutral and one is risk-averse. Instead, it is optimal to split the firm in a particularly simple way. The firm should be marketed to the pair of groups of security holders that leads to the highest valuation of the firm. The output in each state should be allocated to the group that values it most highly, with the other getting zero in that state. Only in states where both groups value consumption equally is it optimal to have both security holders receiving positive payoffs. However, Theorem 4 in Section 2 shows that even in this situation there always exists an equivalent equilibrium where securities are extreme in the sense that all output is allocated to one or the other of the groups of security holders. Theorem 4 also demonstrates that the result that optimal securities should be extreme holds in much more complex environments than the simple two-group, two-state example illustrated here.

Of course, in reality firms do issue debt and equity. However, as pointed out in the introduction, the model is meant to be an abstraction that gives insight: the result demonstrates that the basic principle of security design is that the firm should be split in such a way that in any state all the payoffs are allocated to the group that values them most. We have not taken into

account all the relevant institutional details. By extending the model to allow for richer institutional and tax environments, more directly descriptive results should be obtainable. For example, we have not incorporated the tax deductibility of interest or possibilities for intermediation. What we might actually expect to obtain in a model incorporating these factors is firms issuing debt to get its tax advantages and intermediaries repackaging it to get any risk-sharing advantages.

5. The Role of Short-Sale Constraints

With issuing costs, financial structure must matter in order to provide an incentive to firms to issue securities. This potentially creates arbitrage possibilities that must be ruled out by some sort of friction if the existence of equilibrium is to be assured. In every case above it is assumed that short sales are so costly that they are not undertaken. In practice, the costs of undertaking short selling (including both the direct costs and those resulting from the way in which short sales are taxed) are large and the actual amount that is done by nonmembers of exchanges is small [see, for example, Pollack (1986)]. The purpose of this section is to consider the case where there are costs but these are not sufficient to rule out all short sales.

Suppose that there are just fixed costs of short selling; there will still be a nonexistence problem, as described in the introduction. It is again possible to construct a perfectly hedged portfolio and earn a profit equal to c_2 for each firm that is short-sold. By taking a sufficiently large position, it will always be possible to make a profit that is larger than the fixed costs. Hence, some marginal cost of short selling is necessary if an equilibrium is to exist with short sales. A constant marginal cost is not sufficient since this either rules out short sales completely if it is greater than c_2 per firm or fails to prevent the nonexistence problem if it is less than this. Hence, the case of interest is where there is a nonlinear cost of short selling.

There are two obvious possibilities: The costs depend either on the number of shares short-sold or on the value of the shares short-sold. Given the arbitrariness of assuming any number of shares in a firm, it is perhaps easier to adopt the latter approach here. In particular it is assumed that the costs of short selling, σ , are an increasing convex function of the total value of securities, Z , that is short-sold. For purposes of illustration, it is convenient to assume that

$$\sigma = Z^2$$

In the equilibrium of Example 2a, the short-sale constraints bind on both the risk-neutral and the risk-averse groups. The risk-neutral group would of course like to short-sell the r^1 security with payoffs (1, 0) since its price is greater than its expected return. However, the risk-averse group would also like to short-sell the r^2 security with payoffs (0, 2). This is because by combining this with a long position in the one-security firm's equity, they can effectively create the r^1 security at a price equal to its

expected return. Thus, what happens when there are nonlinear cost and when only limited short selling is possible is that the n 's expand the supply of the r^1 security and the a 's expand the supply of the r^2 security.

Consider what happens at the critical value $c_2^* = 0.114$, where introducing a two-security firm at prices $v^1 = 1.000$ and $v^2 = 0.614$ becomes worthwhile. As soon as some firm issues r^1 and r^2 , the two groups will short-sell them, thereby expanding the supplies of both. The price of r^1 in particular must then fall in order to equate demand and supply. But at a lower price it is not worthwhile for any firm to issue two securities. Hence, similar to the case with unlimited short sales, no equilibrium exists. In order for an equilibrium to exist, c_2 must be sufficiently small that at the price where demand equals supply for r^1 it is nevertheless worthwhile for a firm to issue two securities. For all values of c_2 below the critical level $c_2^* = 0.076$, where this becomes feasible, an equilibrium exists. At this point the n 's short-sell an amount 0.115 of r^1 and the a 's short-sell an amount 0.038 of r^2 . However, between c_2^* and c_2^* no equilibrium exists.

The possibility of short selling effectively introduces another way for securities to be issued. Since in the example considered in this section the costs of short selling are quadratic, it is always worthwhile for individuals to issue a limited amount of securities. For small amounts, they have a cost advantage over firms in expanding the supply of the security. At least for low values of c_2 , this expanded low-cost supply of the security leads to a better allocation of risk than would occur without short selling. The problem is that for intermediate values of c_2 , the competition between firms and individuals can lead to the nonexistence of equilibrium.

6. Concluding Remarks

This paper develops a framework for considering the question of optimal security design. Since this is of interest only when markets are incomplete, a model is used in which the absence of full risk-sharing possibilities is due to the existence of transaction costs for issuing securities. In constructing any model of this type, two questions must be addressed. The first concerns the form of friction that must be introduced so that equilibrium can exist. The second involves the way in which the markets for unissued securities are represented. The approaches taken above of introducing costs of short sales and of assuming that both the firms and the consumers know the prices of unissued securities are not the only alternatives for resolving these questions. Their main advantage is that they provide a simple and tractable benchmark.

Within this framework many issues remain to be addressed. Among other things; an important question that we hope to pursue in future research concerns the optimal design of securities in multiperiod models. In addition to implications for capital structure, such analyses should also have implications for dividend policy. The role of the assumptions concerning short sales and unissued securities also points to the importance of research

aimed at providing a more detailed understanding of the microstructure of markets.

We have avoided some difficult questions concerning the existence of equilibrium by assuming no short sales. In the context of the tax clientele literature, Schaefer (1982) argues that short-sale constraints will be necessary for the existence of equilibrium if there exist differentially taxed assets that have perfectly correlated pretax and after-tax payoffs. However, Dammon and Green (1987) have shown that if firms' payoffs have unique risk, then equilibrium can exist without short-sale constraints. In ongoing research we are considering extensions of the approach used in this article—extensions in which short sales are not limited by the assumption that they are too costly and in which firms' payoffs have unique risk.

Appendix

Formal definition of the model

There are two dates ($t = 0, 1$) and a finite set of states of nature ($s \in S$). All agents have the same informational structure: There is no information at the first date, and the true state is revealed at the second. At each date there is a single consumption good. Since commodities are distinguished by the date and state in which they are delivered, the commodity space is $\mathbf{R}^{S|+1}$. There is a finite set of types of consumers ($i \in I$) and a finite set of types of producers ($j \in J$). To justify the assumption of perfect competition, we assume that there is a continuum of agents of each type. For each $i \in I$, let $m_i > 0$ denote the measure of consumers of type i ; for each $j \in J$, let $n_j > 0$ denote the measure of producers of type j .

Producers. A type of producer is defined by a security set and a cost function. The security set of type $j \in J$ is denoted by E_j , which is simply an index set. Any production plan can have at most two types of claims issued against it; for any $e \in E_j$, these two claims are indexed by $k = 1, 2$. The properties of the different securities are described by two dividend functions (r^1, r^2) defined on the global security set $E = \cup_{j \in J} E_j$; thus, $r^k(e)$ denotes the vector of dividends corresponding to the k th claim in financial structure e . The dividend functions define the production plan, or output vector, associated with each financial structure: $y_j(e) \equiv r^1(e) + r^2(e)$ for any $e \in E_j$. Let $Y_j = \{y_j(e) \mid e \in E_j\}$ denote the set of feasible output vectors for firms of type $j \in J$. Securities are assumed to satisfy the following conditions:

- A1a. E_j is a compact metric space for every $j \in J$.
- A1b. For all $j \in J$ there exists $e \in E_j$ such that $r^1(e) = r^2(e) = 0$.
- A1c. For all $j \in J$ and any $y \in Y_j$ there exists $e \in E_j$ such that $r^1(e) = y$ and $r^2(e) = 0$.
- A2. $r^k: E \rightarrow \mathbf{R}^{S|}$ is a continuous function.

Since E is an index set, assumptions (A1a) and (A2) are technical conveniences. Assumption (A1b) represents the possibility of issuing no secu-

rities (i.e., the producer decides not to produce any output). Assumption (A1c) says that it is possible to issue a single security (equity).

The cost function of producers of type $j \in J$ is denoted by $C_j: E_j \rightarrow \mathbf{R}_+$. $C_j(e)$ is the cost, in units of output at date 0, of operating a firm with financial structure $e \in E_j$. Without essential loss of generality, we may assume that these costs include the cost of inputs to the production process (i.e., investment). It is natural to assume that

$$\text{A3a. } C_j(e) = 0 \text{ if } e \in E_j \text{ and } r^1(e) = r^2(e) = 0.$$

$$\text{A3b. } C_j: E_j \rightarrow \mathbf{R}_+ \text{ is lower-semicontinuous.}$$

Every producer is assumed to maximize his profit, taking as given the market value of the different kinds of firms he can operate. Let $MV(e)$ denote the value of a firm with structure $e \in E$, measured in units of consumption goods at date 0. Then a producer of type $j \in J$ chooses $e \in E_j$ to maximize $MV(e) - C_j(e)$. The producer takes prices (and hence MV) as given; he does not need to concern himself with the actions of other agents. Not all producers of the same type will choose the same financial structure, because the objective function is not quasi-concave. The equilibrium choices of the j th type are represented by a measure v_j defined on E_j . A measure like v_j is called a distribution; for any measurable set $H \subset E_j$, $v_j(H)$ is the measure of producers of type j who choose securities of type $e \in H$. A distribution v_j is admissible if $v_j(E_j) = n_j$ (i.e., if all producers of type j are accounted for). N_j denotes the set of admissible distributions of producers of type $j \in J$. Then for each $j \in J$ we can define the optimal security correspondence ψ_j by putting

$$\psi_j(MV) = \operatorname{argmax}_{v \in N_j} \int_{E_j} (MV - C_j) dv$$

for any market-value function MV . Under price-taking behavior, individual profit maximization is equivalent to aggregate profit maximization.

Consumers. Every consumer is characterized by a consumption set, a portfolio set, a utility function, and an endowment. All consumers have the same consumption set $\Omega = \mathbf{R}_+^{s_1+s_2}$ with generic element $x \in \Omega$. It is convenient to partition the consumption set $\Omega = \Omega^0 \times \Omega^1$ and to partition consumption bundles $x = (x^0, x^1) \in \Omega^0 \times \Omega^1$. ($x^0 \in \mathbf{R}_+$ denotes consumption at date 0, and $x^1 \in \mathbf{R}_+^{s_1}$ denotes consumption at date 1.) The utility function of consumers of type $i \in I$ is denoted by $U_i: \Omega \rightarrow \mathbf{R}$; the endowment is denoted by $w_i \in \Omega$.

A portfolio is a nonnegative, vector-valued measure $\alpha = (a^1, a^2)$ defined on E . For any measurable set $H \subset E$ and any $k = 1, 2$, $a^k(H)$ is the number of units of the k th claim on a firm with structure $e \in E$ held in the portfolio. No short sales are allowed, so α has values in \mathbf{R}_+^2 . All consumers have the same set of admissible portfolios denoted by A .

A consumer of type $i \in I$ is characterized by the array (Ω, A, U_i, w_i) . We assume that each type satisfies the following properties:

A4a. $U_i: \Omega \rightarrow \mathbf{R}$ is strictly quasi-concave and strictly increasing for every $i \in I$.

A4b. U_i is C^1 on the interior of Ω and continuous at the boundary for every $i \in I$.

Assumption (A4) allows us to define a function $p_i: \text{int } \Omega \rightarrow \mathbf{R}^{s1}$ for each $i \in I$ by putting

$$p_i(x) = \frac{\partial U_i / \partial x^1}{\partial U_i / \partial x^0}$$

for every $x \in \text{int } \Omega$. Since U_i is strictly increasing, $p_i(x) \gg 0$ for every $x \in \text{int } \Omega$.

A security price function is a function $v \equiv (v^1, v^2)$ defined on E to \mathbf{R}_+^2 . For any $e \in E$ and $k = 1, 2$, $v^k(e)$ is the price, measured in units of consumption of date 0, of the k th claim in the financial structure e . We restrict attention to continuous price functions. For any admissible portfolio $\alpha \in A$, the value of the portfolio is defined by $\alpha \cdot v$, where

$$\alpha \cdot v = \sum_{k=1}^2 \int_E v^k d\alpha^k$$

Similarly, the total income at date 1 from a portfolio $\alpha \in A$ is denoted by $\alpha \cdot r$, where $r = (r^1, r^2)$ and

$$\alpha \cdot r = \sum_{k=1}^2 \int_E r^k d\alpha^k$$

In this notation we can write the budget constraints for a consumer of type $i \in I$ as follows:

$$\begin{aligned} x^0 &= w_i^0 - \alpha \cdot v \\ x^1 &= w_i^1 + \alpha \cdot r \end{aligned}$$

(The budget constraints are equations because U_i is strictly increasing.) Using these equations, we can write the consumption bundle x as a function of α and v . So we write $x = \xi_i(\alpha, v)$ for every $i \in I$. Then we define the budget set for type $i \in I$ to be

$$\beta_i(v) = \{\alpha \in A \mid \xi_i(\alpha, v) \in \Omega\}$$

for every price function v . The optimal portfolio correspondence is defined by putting

$$\phi_i(v) = \underset{\alpha \in \beta_i(v)}{\text{argmax}} U_i(\xi_i(\alpha, v))$$

for any price function v .

Equilibrium. We define our concept of equilibrium in two stages. First we outline a Walrasian type of concept; then we add an extra condition suggested by Hart (1979). In Walrasian equilibrium all agents maximize utility, taking prices as given, and all markets clear at the prevailing prices.

An equilibrium must specify a portfolio α_i for each type of consumer $i \in I$, a distribution v_j for each type of producer $j \in J$, and a continuous price function: $v: E \rightarrow \mathbb{R}_+^2$. We define a Walrasian equilibrium to be an array $\langle (\alpha_i)_{i \in I}, (v_j)_{j \in J}, v \rangle$ satisfying the following properties:

$$\alpha_i \in \phi_i(v) \quad \text{for every } i \in I \quad (E1)$$

$$v_j \in \psi_j(v^1 + v^2) \quad \text{for every } j \in J \quad (E2)$$

$$\sum_{i \in I} \alpha_i = \sum_{j \in J} v_j \quad (E3)$$

Properties (E1) and (E2) require utility maximization on the part of consumers and producers, respectively; (E3) requires market clearing.

In a large economy the securities issued by any producer are negligible relative to the size of the economy as a whole. Without loss of generality, we can assume that if a new security is issued, it will be widely held and each consumer will hold a negligible amount of it. In that case a security is valued according to the marginal rates of substitution of those who hold it. The *rational conjecture condition* can be written as follows:

$$v^k(e) = \max \{p_i(\xi_i(\alpha_i, v)) \cdot r^k(e)\} \quad \text{for every } e \in E \text{ and } k = 1, 2 \quad (E4)$$

The maximum in condition (E4) is taken over $\{i \in I \mid \xi_i^0(\alpha_i, v) > 0\}$. An equilibrium is defined to be a Walrasian equilibrium that satisfies (E4).

Condition (E4) does not affect the equilibrium allocation in any substantive way. (E4) is automatically satisfied in Walrasian equilibrium for every security that is actually issued. If (E4) is not satisfied in some Walrasian equilibrium, one can satisfy it simply by reducing the value of $v^k(e)$ appropriately without making any other change in the Walrasian equilibrium.

Efficiency. The appropriate concept of efficiency is constrained efficiency. We ask whether a central planner can make everyone better off by using only the markets and technologies available to private individuals. (Under our assumptions there is no difference between making everyone better off and making some people better off and no one worse off.) The allocation at date 1 is determined by trades in securities at date 0. Therefore, an equilibrium is constrained-efficient if it is impossible to make everyone better off by means of transfers of goods and securities at date 0. Formally, an equilibrium $\langle (\alpha_i)_{i \in I}, (v_j)_{j \in J}, v \rangle$ is constrained-efficient if there does not exist an obtainable allocation $\langle (\hat{\alpha}_i)_{i \in I}, (\hat{v}_j)_{j \in J} \rangle$ and transfers $(\tau_b)_{b \in I \cup J}$ satisfying these conditions:

(a) $\hat{\alpha}_i \in A$, $\hat{v}_j \in N_j$, and $\tau_b \in \mathbb{R}$ for every $i \in I$, $j \in J$, and $b \in I \cup J$.

(b) $\sum_{i \in I} m_i \hat{\alpha}_i = \sum_{j \in J} \hat{v}_j$ and $\sum_{i \in I} m_i \tau_i + \sum_{j \in J} n_j \tau_j = 0$.

(c) $\int_{E_j} (v^1 + v^2 - C_j) dv_j < \int_{E_j} (v^1 + v^2 - C_j) d\hat{v}_j + \tau_j$ for every $j \in J$.

(d) $U_i[\xi_i(\alpha_i, v)] < U_i[\xi_i(\hat{\alpha}_i, v) + (\tau_i, 0)]$ for every $i \in I$.

[The fact that we use the equilibrium price function v in the definition of constrained efficiency is immaterial. It simply allows us to use the notation introduced earlier.]

Proof of Theorems 1 and 2

Theorem 1. Every equilibrium is constrained-efficient.

Proof: Suppose not. Then in the previous notation

$$\xi_i^0(\hat{\alpha}_i, v) + \hat{\alpha}_i \cdot v + \tau_i > w_i^0 \quad (i \in I)$$

and

$$\int_{E_j} [v^1 + v^2 - C_j + \tau_j] d\hat{v}_j > \int_{E_j} [v^1 + v^2 - C_j] dv_j \quad (j \in J) \quad (1)$$

Adding up, we get

$$\sum_{i \in I} m_i \{ \xi_i^0(\hat{\alpha}_i, v) + \hat{\alpha}_i \cdot v + \tau_i \} > \sum_{i \in I} m_i w_i^0 \quad (2)$$

The attainability condition for the new allocation has the form

$$\sum_{i \in I} m_i \xi_i^0(\hat{\alpha}_i, v) + \sum_{j \in J} \hat{v}_j \cdot v = \sum_{i \in I} m_i w_i^0$$

Then substituting from condition (b) above yields

$$\sum_{i \in I} m_i \{ \xi_i^0(\hat{\alpha}_i, v) + \hat{\alpha}_i \cdot v \} = \sum_{i \in I} m_i w_i^0$$

so Equation (2) and condition (b) imply that $\sum_{j \in J} \int_{E_j} \tau_j d\hat{v}_j < 0$. From the equilibrium conditions we know that

$$\int_{E_j} (v^1 + v^2 - C_j) d\hat{v}_j \leq \int_{E_j} (v^1 + v^2 - C_j) dv_j$$

so for some $j \in J$ Equation (1) is violated. ■

Theorem 2. Under the maintained assumptions the equilibrium set is nonempty.

Proof: For each $q = 1, 2, \dots$, let E_j^q be a finite subset of E_j satisfying the same properties as E_j and such that $E_j^q \rightarrow E_j$ as $q \rightarrow \infty$, where convergence of E_j^q is with respect to the Hausdorff metric. We prove existence for the economy with E_j replaced by E_j^q for all $j \in J$, and then the theorem follows by taking limits.

Let $E^q = \cup_{j \in J} E_j^q$ and let A^q denote the set of nonnegative measures on E^q to \mathbb{R}^2 with the property that

$$\alpha^k(\{e\}) \leq \frac{n_j + 1}{\min \{m_i\}}$$

for any $e \in E_j$; $k = 1, 2$; $j \in J$, and $a \in A^q$. Thus, any portfolio a observed in equilibrium must certainly belong to A^q . For each $i \in I$, let $\beta^q(v) = \beta_i(v) \cap A^q$ and define $\phi^q(v)$ by

$$\phi^q(v) = \operatorname{argmax}_{\alpha \in \beta^q(v)} U_i(\xi_i(\alpha, v))$$

for any $e \in E^q$, $k = 1, 2$, and any consumption bundle x_i .

Let

$$V^q = \{v: E^q \rightarrow R^2 \mid v^1(e) + v^2(e) \leq K\}$$

where K is very large. Then

$$\phi^q: V^q \rightarrow A^q$$

is upper hemicontinuous (u.h.c.), nonempty, and convex-valued. Similarly, define ψ^q by putting N^q equal to the set of nonnegative real-valued measures on E^q such that $\nu_j(E_j) = n_j$ for all $j \in J$, and then

$$\psi^q(v) = \operatorname{argmax}_{\nu \in N^q} \sum_{i=1}^2 \int E_j (MV - C_j) d\nu^i$$

for any $v \in V^q$. Then the usual arguments show that

$$\psi^q: V^q \rightarrow M(E^q)$$

is u.h.c., nonempty, and convex-valued. Define a correspondence

$$\zeta^q: V^q \rightarrow R^{E^q} \times R^{E^q}$$

by

$$\zeta^q(v) = \sum_{i \in I} m_i \phi^q(v) - \sum_{j \in J} \hat{\psi}^q(v)$$

for every $v \in V^q$, where $\hat{\psi}^q(v) = \{(v, \nu) \mid \nu \in \psi^q(v)\}$. Then ζ^q inherits the properties of (ψ^q) and (ϕ^q) . Choose K in the definition of V^q so that in any attainable allocation, $p_i(x_i) \cdot r^k(e) < K$.

By a standard fixed-point argument there exists $(z^q, v^q) \in R^{E^q} \times R^{E^q} \times V^q$ such that $z^q \in \zeta^q(v^q)$ and $v^q \cdot z^q \geq V^q \cdot z^q$. If $z^{qk}(e) > 0$, then $v^{qk}(e) = K$, but in that case $a^k(e) = 0$ for any $\alpha \in \phi^q(v^q)$ and any $i \in I$, a contradiction. Conversely, $z^{qk}(e) < 0$ implies that $v^{qk}(e) = 0$, in which case $a^k(e) = (n_j + 1)/\min \{m_i\}$ for some j , where $e \in E_j$ and $\alpha \in \phi^q(v^q)$. (Unless $r^k(e) = 0$; in that case, put $z^{qk}(e) = 0$ without loss of generality.) Again we have a contradiction. So $z^q = 0$.

It is straightforward to check that conditions (a) to (c) of the definition of equilibrium are satisfied (i.e., none of the artificial constraints is binding). To ensure condition (d), simply replace $v^q(e)$ by

$$\min \{v^q(e), \max p_i(\xi_i(\alpha, v^q))\}$$

where $\alpha_i \in \phi^q(v^q)$ and the maximum is taken over $\{i \in I \mid \xi_i(\alpha_i, v^q) > 0\}$. [Note that strict quasi-concavity of U_i implies that $\xi_i(a_i, v^q)$ is unique.] It is easy to check that this does not disturb the other equilibrium conditions for any $v: E^q \rightarrow R_+^q$.

For each E^q we have an equilibrium $\{(\alpha_i^q), (v_j^q), v^q\}$ for the corresponding economy. The measures (α_i^q) and (v_j^q) can be extended to all of E in an obvious way; so can v^q , and condition (d) tells us that this extension is continuous. Since (α_i^q) and (v_j^q) are bounded, there exist weakly convergent subsequences, which we can take to be the original sequence. Along this sequence $\xi_i(\alpha_i^q, v^q)$ is unique for each i and q , so we can choose a further subsequence along which $\xi_i(\alpha_i^q, v^q)$ converges. (Obviously $\{\xi_i(\alpha_i^q, v^q)\}$ is bounded.) Along this sequence v^q converges to a continuous function $v: E \rightarrow R_+^q$. We claim that the limit point $\{(a_i), (v_j), v\}$ is the required equilibrium. By continuity, conditions (c) and (d) are satisfied. Suppose, contrary to what we want to prove, that condition (a) is not satisfied. Then there exists, for some $i \in I$, an $\hat{\alpha}_i \in \beta_i(v)$ that is preferred to a_i . This is impossible unless $\hat{\alpha}_i \cdot v > 0$, so without loss of generality we can assume that $w_i^0 - \hat{\alpha}_i \cdot v > 0$ by continuity. Since $a \cdot r \geq 0$ for any $a \in A^q$, for q sufficiently large we can find $\hat{\alpha}^q \in \beta^q(v^q)$ arbitrarily close to $\hat{\alpha}_i$ in the weak sense. (This requires the facts that E is compact and $v^q \rightarrow v$ uniformly on E .) But for q large enough and $\hat{\alpha}^q$ close enough to $\hat{\alpha}_i$, $\hat{\alpha}^q$ must be preferred to α_i^q , a contradiction. This proves condition (a), and the proof of condition (b) is similar. ■

References

- Allen, F., 1987, "Capital Structure and Imperfect Competition in Product Markets," Working Paper 11-87, Rodney L. White Center, Wharton School, University of Pennsylvania.
- Baron, D.P., 1974, "Default Risk, Home-Made Leverage and the Modigliani-Miller Theorem," *American Economic Review*, 62, 176-182.
- Baron, D.P., 1976, "Default Risk and the Modigliani-Miller Theorem: A Synthesis," *American Economic Review*, 64, 204-212.
- Dammon, R.M., and R.C. Green, 1987, "Tax Arbitrage and the Existence of Equilibrium Prices for Financial Assets," *Journal of Finance*, 42, 1143-1166.
- Diamond, P.A., 1967, "The Role of a Stock Market in a General Equilibrium Model with Technological Uncertainty," *American Economic Review*, 57, 759-766.
- Duffie, D., and M.O. Jackson, 1986, "Optimal Innovation of Futures Contracts," Research Paper 917, Stanford Graduate School of Business.
- Dybvig, P.H., and S.A. Ross, 1986, "Tax Clienteles and Asset Pricing," *Journal of Finance*, 41, 751-762.
- Grossman, S.J., and J.E. Stiglitz, 1980, "On the Impossibility of Informationally Efficient Markets," *American Economic Review*, 70, 393-408.
- Hart, O.D., 1973, "On the Optimality of Equilibrium when Market Structure is Incomplete," *Journal of Economic Theory*, 11, 418-443.
- Hart, O.D., 1977, "Take-Over Bids and Stock Market Equilibrium," *Journal of Economic Theory*, 16, 53-83.
- Hart, O.D., 1979, "On Shareholder Unanimity in Large Stock Market Economies," *Econometrica*, 47, 1057-1084.

- Hart, O.D., 1980, "Perfect Competition and Optimal Product Differentiation," *Journal of Economic Theory*, 22, 279-312.
- Hellwig, M.F., 1981, "Bankruptcy, Limited Liability and the Modigliani-Miller Theorem," *American Economic Review*, 71, 155-170.
- Jensen, M.C., and W.H. Meckling, 1976, "Theory of the Firm: Managerial Behavior, Agency Costs and Ownership Structure," *Journal of Financial Economics*, 3, 305-360.
- Makowski, L., 1980, "Perfect Competition, the Profit Criterion and the Organization of Economic Activity," *Journal of Economic Theory*, 22, 222-242.
- Miller, M.H., 1977, "Debt and Taxes," *Journal of Finance*, 32, 261-275.
- Miller, M.H., 1986, "Financial Innovation: The Last Twenty Years and the Next," *Journal of Financial and Quantitative Analysis*, 21, 459-471.
- Modigliani, F., and M.H. Miller, 1958, "The Cost of Capital, Corporation Finance, and the Theory of Finance," *American Economic Review*, 48, 162-197.
- Myers, S.C., 1984, "The Capital Structure Puzzle," *Journal of Finance*, 39, 575-592.
- Pollack, I.M., 1986, *Short-Sale Regulation of NASDAQ Securities*, NASDAQ pamphlet.
- Radner, R., 1972, "Existence of Equilibrium Plans, Prices and Price Expectations in a Sequence of Markets," *Econometrica*, 40, 289-303.
- Ross, S.A., 1987, "Arbitrage and Martingales with Taxation," *Journal of Political Economy*, 95, 371-393.
- Schaefer, S.A., 1982, "Taxes and Security Market Equilibrium," in W.F. Sharpe and C.M. Cootner (eds.), *Financial Economics: Essays in Honor of Paul Cootner*, Prentice-Hall, Englewood Cliffs, NJ., pp. 159-178.
- Silber, W.L., 1981, "Innovation, Competition and New Contract Design in Futures Markets," *Journal of Futures Markets*, 1, 123-155.
- Stiglitz, J.E., 1969, "A Re-Examination of the Modigliani-Miller Theorem," *American Economic Review*, 59, 784-793.
- Stiglitz, J.E., 1974, "On the Irrelevance of Corporate Financial Policy," *American Economic Review*, 64, 851-866.
- Williams, J.T., 1986, "Bonds, Stocks and Options as Optimal Contracts between Corporate Claimants," mimeo, Graduate School of Business Administration, New York University.