# 1 Appendix: The Computation of Fixed Points

In solving for the decision rules of agents in dynamic search-bargaining models, we typically will have to resort to iterative methods. The scalars or functions which define the decision rules of agents are almost invariably of the form

$$v = Tv, \tag{1}$$

where $v$ is the object for which we are seeking a solution and $T$ is a mapping from the set of values which $v$ can possibly take "into" itself. For example, say that $v$ is a scalar element which can take values on the extended real line [i.e., the real line with $+\infty$ and $-\infty$ added], and let $Tv = a + bv$, .with $b \neq 1$. Then $T$ is a linear mapping, and in this case there actually exists a closed form solution for $v$, namely

$$\begin{aligned} v &= a + bv \\ \Rightarrow v &= \frac{a}{1-b}. \end{aligned}$$

The mappings we usually consider in microeconomic applications are not linear, unfortunately. We therefore need more general methods that either serve to guarantee the existence and (possibly) uniqueness of solutions to [1], and/or provide computational methods for solving these implicit functions.

## 1.1 Contraction Mappings

Let $X$ be a metric space which is equipped with a metric $\rho$. Then $\rho(x, y)$ is the *distance* between $x$ and $y$, for $x, y \in X$. The distance function $\rho$ has the following properties:

1. $\rho(x, y) = \rho(y, x)$ (Symmetry)
2. $\rho(x, y) \geq 0$, with $\rho(x, y) = 0$ if and only if $x = y$.
3. $\rho(x, z) \leq \rho(x, z) + \rho(y, z)$ (Triangle Inequality)

We say that the metric space $X$ is *complete* if for every convergent sequence $\{x_n\}$ such that

$$\lim_{n \to \infty} \rho(x_n, x_{n+m}) = 0 \text{ for each } m,$$

there exists an element $\hat{x} \in X$ such that

$$\lim_{n \to \infty} \rho(x_n, \hat{x}) = 0.$$

We call $\hat{x} = \lim_{n \to \infty} x_n$ the *limit point* of the sequence $\{x_n\}$. Completeness requires that this limit point be a member of the space $X$.

**Definition 1** *An operator $T$ which maps $X$ into itself is called a contraction mapping if for some $\psi \in (0, 1)$*

$$\rho(Tx, Ty) \leq \psi \rho(x, y) \text{ for all } x, y \in X.$$

***Example 2*** *Reconsider the linear mapping $Tv = a + bv$, $b \neq 1$. Is this a contraction? Since $\rho(a+bv, a+bv') = \rho(bv, bv')$, the value of $a$ is irrelevant in determining whether $a+bv$ is a contraction. Now since $\rho(bv, bv') \leq \psi \rho(x, y)$ for some $\psi \in (0, 1)$ if and only if $|b| < 1$, this is the property required for $a + bv$ to be a contraction. Thus $Tv = -10000 + .98v$ is a contraction, while $Tv = .2 - 1.3v$ is not.*

It is extremely useful to establish that $T$ is a contraction for at least two practical reasons. Both are apparent from the following theorem.

**Theorem 3** *If $X$ is a complete metric space and $T$ a contraction mapping, then there exists a unique $v$ such that*

$$Tv = v.$$

*Furthermore, for any $x \in X$,*

$$\lim_{n \to \infty} \rho(T^n x, v) = 0,$$

*where $T^1 x = Tx$, $T^2 x = T(T^1 x), ..., T^n x = T(T^{n-1} x)$.*

**Proof.** First we demonstrate uniqueness of the fixed point. If $Tu = u$ and $Tv = v$, then

$$\begin{aligned} \rho(u, v) &= \rho(Tu, Tv) \leq \psi \rho(u, v) \\ \Rightarrow \rho(u, v) &= 0, \end{aligned}$$

so that the fixed point is unique.

For an arbitrary $x \in X$, consider $\rho(T^{n+m}x, T^n x)$. Now

$$\begin{aligned}\rho(T^{n+m}x, T^n x) &\leq \psi^n \rho(T^m x, x) \\ &\leq \psi^n [\rho(T^m x, T^{m-1}x) + ... + \rho(Tx, x)] \\ &\leq \psi^n \rho(Tx, x)[\psi^{m+1} + ... + \psi + 1],\end{aligned}$$

which implies

$$\lim_{n\to\infty} \rho(T^{n+m}x, T^n x) = 0.$$

Since $X$ is complete we know that $v = \lim_{n\to\infty} T^n x$ exists. $T$ is a continuous mapping, since if $\lim_{n\to\infty} \rho(x^n, \hat{x}) = 0$ then $\rho(Tx_n, T\hat{x}) \leq \psi\rho(x_n, \hat{x})$ which has a limiting value of 0. Then

$$Tv = T \lim_{n\to\infty} T^n x = \lim_{n\to\infty} T^{n+1}x = v.$$

■

This theorem demonstrates first that if $T$ is a contraction mapping, there exists a unique solution in the complete metric space $X$. Furthermore, the theorem provides a computational technique to determine the solution, a technique referred to as *successive approximation.* The algorithm is as follows.

*Table A.*1
*Method of Successive Approximation*

Begin by setting $C > 0$, $k = 0$, and $v_0$.

**1.** Given $v_k$, compute $v_{k+1} = Tv_k$
**2.** Compute $D_k = \rho(v_{k+1}, v_k)$
**3.** If $D_k \leq C$, $\hat{v} = v_{k+1}$
If $D_k > C$, repeat steps $(1) - (3)$.

In Table A.1 we have written the stopping rule for the algorithm in terms of the absolute distance between the iterate $v_{k+1}$and the iterate $v_k$. From the contraction mapping theorem, we know that this distance monotonically declines in the number of iterations, $k$. There are many possible stopping rules to use however in deciding when we are "close enough" to the true value of the fixed point to terminate the iterative procedure. When $T$ is a contraction, it is also possible to set a stopping rule that has the property that the error after $N(\varepsilon, v_0) + 1$ iterations is no larger than $\varepsilon$. when starting from the initial value $v_0$. To find $N(\varepsilon, v_0)$ requires the following result.

**Theorem 4** *If $T$ is a contraction mapping, then*

$$\rho(T^n v_0, v) \leq (1-\psi)^{-1} \rho(T^n v_0, T^{n+1} v_0), \ \forall v_0 \in X, \tag{2}$$

*where $\psi$ is the modulus of the operator $T$.*

**Proof.** Since $\lim_{n\to\infty} \rho(T^n v_0, v) = 0$, we have $\lim_{m\to\infty} \rho(T^n v_0 T^{n+m} v_0) = \rho(T^n v_0, v)$. Now

$$\begin{aligned}
\lim_{m\to\infty} \rho(T^n v_0, T^{n+m} v_0) &\leq \rho(T^n v_0, T^{n+1} v_0) + \rho(T^{n+1} v_0, T^{n+2} v_0) + ... \\
&\leq (1 + \psi + ...) \rho(T^n v_0, T^{n+1} v_0) \\
&\leq (1-\psi)^{-1} \rho(T^n v_0, T^{n+1} v_0)
\end{aligned}$$

■

We can use this result to set the number of iterations of $T$ we will compute in the following manner. Say that we are willing to tolerate a discrepancy between the computed value of the fixed point, $T^n v_0$, and the true value, $v$, of $\varepsilon > 0$. Then given any starting point $v_0$, we will stop the iterative procedure after iteration $N(\varepsilon, v_0) + 1$, where

$$\begin{aligned}
\varepsilon(1-\psi) &\leq \rho(T^{N(\varepsilon,v_0)} v_0, T^{N(\varepsilon,v_0)+1} v_0) \\
\varepsilon(1-\psi) &> \rho(T^{N(\varepsilon,v_0)-1} v_0, T^{N(\varepsilon,v_0)} v_0).
\end{aligned}$$

No matter what stopping rule one uses, when $T$ is a contraction with modulus $\psi$, it is always possible to use [2] to bound the size of the approximation error.

In many cases, it is not possible to demonstrate that a particular mapping is a contraction, such as was the case in some of the linear functions we saw in the examples. Even if $T$ is a contraction, it may be the case that the method of successive approximation converges slowly or "unevenly." In all these cases, often it is still possible to use a "modified" method of successive approximation to solve for the fixed point Of course, if $T$ is not a contraction, existence and uniqueness of a fixed point $v$ is not guaranteed. For the present, we assume that we have established the existence of a unique equilibrium, so the problem is only the computation of it.

Let us now assume that $T$ is not necessarily a contraction mapping, but that there exists a unique fixed point of $T$ which is denoted by $v$. Define another map $L$ as follows:

$$Lv = \zeta Tv + (1 - \zeta)v, \quad \zeta \in [0, 1]. \tag{3}$$

Note the following.

**Proposition 5** *If $T$ possesses a unique fixed point $v$, $v$ is also the unique fixed point of $L$.*

**Proof.** First we show that $v$ is a fixed point of $L$. Since

$$v = Tv,$$

then

$$Lv = \zeta Tv + (1 - \zeta)v = v,$$

so $v$ is also a fixed point of $L$. Uniqueness is easily established by noting that if $u$ was another fixed point of $L$, then

$$\begin{aligned} u &= \zeta Tu + (1 - \zeta)u \\ &\Rightarrow \zeta u = \zeta Tu \\ &\Rightarrow u = Tu, \end{aligned}$$

which is a contradiction. Thus $L$ has the same unique fixed point as $T$. ■

Note that the scalar parameter $\zeta$ doesn't have to belong to the interval $[0, 1]$ for this argument to go through; in fact it can be anything.[1] The reason we have restricted it to the unit interval is because of the nature of the fixed point problems we typically confront in solving stationary search models. In most cases, we want to "dampen" the oscillations that occur between iterations of the value function. Let $v_n$ denote the iterated value after $n$ iterations (starting from some initial point $v_0$) Then the value of $v_{n+1}$ using the map $L$ is

$$v_{n+1} = \zeta Tv_n + (1 - \zeta)v_n,$$

[1] See Judd (1997, Chapter ??) for a discussion of this point.

which is a convex combination (when $\zeta \in [0,1]$) of $Tv_n$ and the original value $v_n$. When $\zeta = 1$, we have the "classic" successive approximation algorithm described in Table A.1. When $\zeta = 0$, the algorithm remains forever stuck at the initial value $v_0$, which is obviously an undesirable situation. Therefore, in setting $\zeta$ the trade-offs are between instability and speed (conditional on convergence) for high values of $\zeta$ versus slow but steady convergence for low values of $\zeta$. A value of $\zeta$ in the neighborhood of .3 often works well for the types of fixed point problems considered in this monograph.

Bear in mind that when using a dampening factor $\zeta < 1$, the stopping rule used in deciding when to stop the iteration sequence should be modified accordingly. Clearly, when $\zeta$ is low there will be relatively small changes between the iterates $v_n$ and $v_{n+1}$ for purely artificial reasons. If one is using a criterion of the form $|v_n - v_{n+1}| < \varepsilon$ to decide when to stop with the operator $T$, one might one to use $|v_n - v_{n+1}| < 10^{-2}\varepsilon$ when using the operator $L$ with $\zeta = .3$. For lower values of $\zeta$, one would want to use even smaller values than $10^{-2}\varepsilon$ in the stopping rule.

Table A.2 contains an example of the method of successive approximation and its "modified" form for the linear mappings we considered above. All iteration sequences begin from the starting value $v_0 = 0$. Column 1 contains the iteration sequence for the mapping $Tv = -10 + .8v$. We know that this function is a contraction mapping, and therefore should converge to its unique fixed point ($v = Tv$) of -50 from any starting value. This is in fact what we observe, with convergence to the fifth decimal point by iteration 70.

Columns 2 through 4 contain approximation sequences for the mapping $Tv = -10 - 1.2v$. While there is a unique fixed point for this map (equal to -10/2.2), the map itself is not a contraction. We see that starting from the point $v_0 = 0$, when $\zeta = 1$ (column 2) the algorithm diverges. This is not the case in columns 3 and 4, where $\zeta$ was set to .2 and .8, respectively. The best performance in this case was for $\zeta = .2$, in this case convergence was both rapid and "smooth." Convergence was also obtained for $\zeta = .8$, but was noticeably slower. Of course, which $\zeta$ works best in any specific problem will depend on the nature of the map and the starting value, and cannot generally be determined except through trial and error.

*Table A.2*
*Illustration of Method of Successive Approximation*
($v_0 = 0$)

| | $Tv$ or $Lv$ | | | |
|---|---|---|---|---|
| **Iteration** | $-10 + .8v$ | $-10 - 1.2v$ | $\varsigma(-10 - 1.2v) + (1 - \varsigma)v$ | |
| | | | $\varsigma = .2$ | $\varsigma = .8$ |
| **10** | -44.63129 | 23.59880 | -4.53167 | -4.25323 |
| **20** | -49.42354 | 169.71636 | -4.54541 | -4.52667 |
| **30** | -49.93810 | 1074.4378 | -4.54545 | -4.54425 |
| **40** | -49.99335 | ⋮ | | -4.54538 |
| **50** | -49.99929 | | | -4.54545 |
| **60** | -49.99992 | | | |
| **70** | -49.99999 | | | |